On the $n!$–conjecture[1]

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Abstract We discuss the proof given by M. Haiman of the Macdonald positivity conjecture obtained via the solution to the $n!-$conjecture of Garsia and Haiman. This is obtained from the following remarkable Theorem: the Hilbert scheme of $n-$tuples of points in the plane is equal to the $G-$Hilbert scheme of Ito and Nakamura for the action of the symmetric group on the space of such $n-$tuples.

1 Macdonald polynomials, the positivity conjecture

In 1988 Macdonald defined 2-parameter symmetric functions unifying the theory of Hall-Littlewood and Jack polynomials (cf. [M],[M1]).

We recall a variant of his construction (cf. [H]). For given positive integer n we want to define symmetric functions $\tilde{H}_\lambda(X)$ indexed by partitions of n and with coefficients in $\mathbb{Q}(q,t)$. We use the fact that symmetric functions (in infinitely many variables) are polynomials in the Newton functions $\psi_k := \sum_i x_i^k$.

The $\tilde{H}_\lambda(X)$ are implicitly defined using dominance order and plethystic transformations.

For a symmetric function $f(X)$ the plethystic transformation $f(X) \to f(X[1-q])$ is the unique morphism $\mathbb{Q}[\psi_1,\ldots,\psi_m,\ldots]$ to $\mathbb{Q}(q)[\psi_1,\ldots,\psi_m,\ldots]$ sending $\psi_k \to \psi_k(1-q^k)$.

The dominance order of partitions, is

$$(p_1,p_2,\ldots,p_n) \geq (q_1,q_2,\ldots,q_n), \iff \forall k\ p_1+p_2+\cdots+p_k \geq q_1+q_2+\cdots+q_k$$

Finally for a partition λ its dual λ' is obtained exchanging rows and columns in its Young diagram.

1.1 Theorem Macdonald[2]. *There exist unique symmetric functions $\tilde{H}_\lambda(X)$ satisfying:*

1) $\tilde{H}_\lambda(X[1-q])$ lies in the vector space over $\mathbb{Q}(q,t)$ generated by the Schur functions $S_\mu(X),\ \mu \geq \lambda$.

2) $\tilde{H}_\lambda(X[1-t])$ lies in the vector space over $\mathbb{Q}(q,t)$ generated by the Schur functions $S_\mu(X),\ \mu \geq \lambda'$.

3) In the expansion of $\tilde{H}_\lambda(X)$ through Schur functions the coefficient of S_n is 1.

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[2]I follow here Haiman's approach [H]

Positivity conjecture.

The coefficients of the expansion $\tilde{H}_\lambda(X)$ through Schur functions are polynomials in q, t with coefficients positive integers.

For further discussion and a guide through the literature we refer to [H].

2 $n!$ conjecture

Since the work of Frobenius it has been well understood the connection between symmetric functions and representations of the symmetric group. In particular it is useful to associate to the irreducible representation indexed by a partition λ the Schur function $S_\lambda(X)$. Extending this by linearity one has a linear isomorphism $\chi \to F(\chi)$, (called Frobenius character) between the space of characters of the symmetric group on n letters and the space of symmetric functions of degree n.

With this convention suppose we have a bigraded representation of the symmetric group $V_{i,j}$ and let $\chi_{i,j}$ be the corresponding bigraded character. Then we construct the 2-parameter symmetric function, called its bigraded Frobenius character:

$$\sum_{i,j} q^i t^j F(\chi_{i,j}).$$

So, to prove the positivity conjecture one should construct, for each partition λ, a bigraded representation whose bigraded Frobenius character is the Macdonald polynomial $\tilde{H}_\lambda(X)$.

In 1991 Adriano Garsia and Mark Haiman, inspired by similar constructions for the simpler case of $q-$Kostka polynomials (cf. [GP]), proposed such a construction.

Let $R := \mathbb{C}[x_1, \ldots, x_n; y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables.

A partition λ of n will be always identified to a set of n points in the integral lattice. The n pairs $\lambda := \{(i_h, j_h)\},\ h = 1, \ldots, n$ of numbers give (up to sign) the polynomial:

$$D_\lambda := \det(x_k^{i_h} y_k^{j_h}) \in R, \quad (h, k = 1, \ldots, n).$$

Consider the R module structure on R setting:

$$p.f := p(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}; \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})f$$

set:

$$I_\lambda := \{p \in R |\ p.D_\lambda = 0\}, \quad V_\lambda := \{p.D_\lambda | p \in R\},$$

$V_\lambda \cong R/I_\lambda$ is the space spanned by all the derivatives of the polynomial D_λ.

$n!$ *conjecture* [GH]: $\dim R/I_\lambda = \dim V_\lambda = n!$

We let the symmetric group S_n act on $R := \mathbb{C}[x_1, \ldots, x_n; y_1, \ldots, y_n]$ by the diagonal action (or simultaneously on the x and y). Since D_λ is bihomogeneous and skew symmetric it is clear that $V_\lambda = R/I_\lambda$ is a bigraded representation of the symmetric group so:

Second conjecture [GH]: The bigraded Frobenius character of V_λ is the Macdonald polynomial $\tilde{H}_\lambda(X)$.

Both conjectures have now been proved by Mark Haiman, his final results are in [H2]. It has turned out that the most difficult part of the project has been to establish the $n!-$conjecture.

The proof of the $n!-$conjecture is based on a deep property of the Hilbert scheme of $n-$tuples of points in the plane. This connection has also allowed Haiman to solve other conjectures on diagonal harmonics as we shall explain at the end (cf. [H1],[H3]).

In order to see how the Hilbert scheme enters let us first make an elementary remark.

Define a linear form T_λ on R as:

$$T_\lambda(p) := (p.D_\lambda)(0). \tag{2.1}$$

2.2 Lemma. $I_\lambda := \{p \in R |\ T_\lambda(pq) = 0,\ \forall q \in R\}$.

Proof. A polynomial $p \in R$ is 0 if and only if $(q.p)(0) = 0$ for every $q \in R$. Thus $p.D_\lambda = 0$ if and only if $q.(p.D_\lambda)(0) = 0$, for every q. Since $q.(p.D_\lambda) = (qp).D_\lambda$ we have the claim. $\square$

Remark that, if h, k is a pair of natural integers, external to the partition λ we have, for every i:

$$\frac{\partial}{\partial x_i}^h \frac{\partial}{\partial y_i}^k D_\lambda = 0$$

It follows that I_λ contains the monomials $x_i^h y_i^k$ for every such pair. In other words, set J_λ to be the ideal of $\mathbb{C}[x, y]$ generated by the monomials $x^h y^k$, $(h, k) \notin \lambda$ and $A_\lambda := \mathbb{C}[x, y]/J_\lambda$. A_λ has dimension n and, as basis, the monomials $x^i y^j$, $(i, j) \in \lambda$.

Finally, identifying $R = \mathbb{C}[x, y]^{\otimes n}$ we have that R/I_λ is a quotient of $A_\lambda^{\otimes n}$.

The linear form T_λ factors through $A_\lambda^{\otimes n}$ and it is antisymmetric. Any antisymmetric linear form factors through antisymmetrization $A_\lambda^{\otimes n} \to \wedge^n A_\lambda$. We have $\dim \wedge^n A_\lambda = 1$, since $\dim A_\lambda = n$, hence such a form up to scalar is unique.

Thus the $n!$ conjecture is equivalent to:

Rank conjecture The form $T_\lambda(pq)$ on $A_\lambda^{\otimes n}$ has rank $n!$.

It has turned out to be too difficult to analyze directly the ideal J_λ, but rather one must work more globally on H_n, the Hilbert scheme of all ideals I of codimension n in $\mathbb{C}[x, y]$. H_n comes together with the universal family

$$\mathcal{F} := \{(I, p) \in H_n \times \mathbb{C}^2 |\ I(p) = 0\}. \tag{2.3}$$

The projection map $p : \mathcal{F} \to H_n$ is flat and $F := p_* \mathcal{O}_{\mathcal{F}}$ is the tautological vector bundle $F = \{(I, \mathbb{C}[x,y]/I)\}$, a bundle of algebras of dimension n over H_n.

A fundamental theorem of Fogarty [F] states that the Hilbert scheme is smooth of dimension $2n$, from which it follows easily that it gives a (crepant) resolution of singularities:

$$H_n \xrightarrow{\rho} \mathbb{C}^{2n}/S_n.$$

In the Hilbert scheme the ideals J_λ play a special role. In fact on $\mathbb{C}^2$ and hence on H_n, acts a two dimensional torus $\mathcal{T} := \{(\alpha, \beta)\}$ through $(\alpha,\beta)(x,y) := (\alpha x, \beta y)$. A fixed point is just a bihomogeneous ideal and one easily sees that these are exactly the ideals J_λ (indexed by partitions of n).

One uses always the fact that any $\mathcal{T}$ stable closed subset of H_n contains a fixed point.

Now we can globalize the rank conjecture. Consider the antisymmetrization $T : F^{\otimes n} \to \wedge^n F$ which induces a *form* $T(ab)$ on the bundle $F^{\otimes n}$.

2.4 Lemma. *T has generically rank $n!$. The $n!$ conjecture is equivalent to the statement that the form T has always rank $n!$. In this case $F^{\otimes n}/Ker(T)$ is a bundle of algebras each carrying the regular representation of S_n.*

Proof. In a generic point of the Hilbert scheme, the ideal I defines $n-$distinct points and the algebra $\mathbb{C}[x,y]/I = \oplus_{i=1}^n \mathbb{C} e_i$ with the e_i orthogonal idempotents.

It is easily seen that the kernel of T has, as basis, the elements $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$ where the indices are not all distinct. Its complement is the regular representation with basis $e_{\sigma 1} \otimes e_{\sigma 2} \otimes \cdots \otimes e_{\sigma n}$, $\sigma \in S_n$.

If the rank is constant we must have a bundle of regular representations. The set of points with rank $n!$ is open dense and $\mathcal{T}$ stable, hence if the complement, where rank $< n!$, is non empty it must contain a fixed point J_λ contradicting the $n!$ conjecture. $\square$

Let now $\mathcal{B}_n$ be the sheaf of sections of $F^{\otimes n}$ (a bundle of algebras) and $\mathcal{I}_n$ the sheaf of ideals kernel of T.

We have:

$$Spec(\mathcal{B}_n) := \{(I, p_1, p_2, \ldots, p_n) \in H_n \times (\mathbb{C}^2)^n \mid I(p_i) = 0, \ \forall i = 1, \ldots, n\}. \tag{2.5}$$

Now we can define the subvariety X_n of $Spec(\mathcal{B}_n)$ which is the closure of its open subset where all p_i are distinct.

2.6 Lemma. *X_n is defined by the sheaf of ideals $\mathcal{I}_n$. The commutative diagram:*

$$\begin{array}{ccc} X_n & \longrightarrow & \mathbb{C}^{2n} \\ {\scriptstyle p}\downarrow & & \downarrow \\ H_n & \longrightarrow & \mathbb{C}^{2n}/S_n \end{array} \tag{2.7}$$

identifies X_n to the reduced fiber product.

Sketch of proof. The sheaf of ideals $\mathcal{I}_n$ restricted to the part of H_n consisting of reduced subschemes, defines the set where all the p_i are distinct. One obtains immediately the first statement. In the commutative diagram, by construction, X_n is a subvariety of the reduced fiber product $\overline{X}_n$. On the regular part the fiber product is reduced, so it is enough to show that $\overline{X}_n$ is irreducible. One sees this by induction on n using the fact that the preimage under p, of a subscheme supported in a unique point, is a point. □

2.8 Theorem. *The kernel of T has constant rank if and only if X_n is Cohen Macaulay and Gorenstein. In this case $\mathcal{B}_n/\mathcal{I}_n = p_*(\mathcal{O}(X_n))$.*

X_n is Cohen Macaulay if and only if the morphism $p : X_n \to H_n$ is flat. In this case $\mathcal{B}_n/\mathcal{I}_n = p_*(\mathcal{O}(X_n))$.

Proof. Some parts are fairly straightforward. Let us see how the Gorenstein property plays a role. Assume X_n is Cohen Macaulay and Gorenstein. Take a point $I_\lambda \in H_n$ we have seen that there is a unique point in $p^{-1}(I_\lambda)$. The coordinate ring B_λ of the sheme theoretic fiber $p^{-1}(I_\lambda)$ is the local ring in this point modulo a regular sequence, hence by the Gorenstein assumption it has a 1-dimensional socle (a unique minimal ideal). It must necessarily be S_n stable and hence carry the sign representation, then the kernel of T on B_λ is an ideal which, if non 0, must contain the sign representation. This is clearly absurd so the form $T(ab)$ is non degenerate on the $n!$ dimensional algebra B_λ.

The converse follows a similar line.

3 The $G-$Hilbert scheme

It is quite interesting (and useful) to reinterpret the previous discussion as follows.

If we have that the map $p : X_n \to H_n$ is flat, we also have that each fiber of p is a subscheme of length $n!$ in $\mathbb{C}^{2n}$. From the theory of the Hilbert scheme we have then a classifying map $i : H_n \to H_{n!,2n}$ where $H_{n!,2n}$ is the Hilbert scheme parameterizing subschemes of length $n!$ in $\mathbb{C}^{2n}$. On the other hand, the open part of H_n corresponding to subschemes with $n-$distinct points parametrizes the generic orbits of S_n in $\mathbb{C}^{2n}$.

In general, given a finite group G acting faithfully on an irreducible quasi-projective variety X we have the following construction of Ito and Nakamura [I-N]. Consider the open set X^0 (the union of the generic orbits) over which G acts freely. The set of such orbits X^0/G can be identified with a locally closed subset of the Hilbert scheme $H_{|G|,X}$ of finite subschemes of length $|G|$ in X. One sees easily that this is in fact open in the subscheme of G fixed points of $H_{|G|,X}$.

The closure $\tilde{H}_{G,X}$ of X^0/G in $H_{|G|,X}$ is an irreducible component of the subscheme of

$G-$stable finite schemes for which the coordinate ring carries the regular representation.[3]

By continuity the universal family restricted to $\tilde{H}_{G,X}$ is a flat family of $G-$stable finite schemes for which the coordinate ring carries the regular representation.

For a $G-$variety X this closure of X^0/G will be called the *$G-$Hilbert scheme* and denoted $\tilde{H}_{G,X}$. It comes equipped with a proper birational morphism $\rho : \tilde{H}_{G,X} \to X/G$. It seems to be quite interesting to determine when is $\tilde{H}_{G,X}$ smooth, and hence ρ a canonical resolution of singularities (cf. [BKR]). It is not hard to prove that, in our setting, from the flatness of $p : X_n \to H_n$ follows that H_n is identified to the $G-$equivariant Hilbert scheme of $\mathbb{C}^{2n}$ and X_n to its universal family.

4 X_n is Cohen Macaulay and Gorenstein

The main theorem proved by Haiman in [H2], from which he deduces both the $n!$ and the Macdonald positivity conjectures, is:

4.1 Theorem. *X_n is Cohen Macaulay and Gorenstein.*

The idea of the proof is to use induction on n and the fact that, on the points H_n^0 of H_n which do not define n coincident points, we have a local structure, analitically a product

$$\begin{array}{ccc} X_h \times X_k & \xrightarrow{g} & \mathbb{C}^{2h} \times \mathbb{C}^{2k} \\ {\scriptstyle p_h \times p_k}\downarrow & & \downarrow \\ H_h \times H_k & \longrightarrow & \mathbb{C}^{2h} \times \mathbb{C}^{2k}/S_h \times S_k \end{array}$$

By induction if X_n^0 is the open set of X_n lying over H_n^0, then X_n^0 is Cohen Macaulay and Gorenstein, moreover $Rg_*^i\mathcal{O}(X_n^0) = 0, \ \forall i > 0$.

More precisely one has to use the *flag* Hilbert scheme $H_{n,n-1}$ and the corresponding variety $X_{n,n-1}$ and exploit the birational map $X_{n,n-1} \to X_n$.

Set theoretically $H_{n,n-1}$ is made of pairs of ideals

$$I \subset J \subset \mathbb{C}[x,y] | \ \dim \mathbb{C}[x,y]/I = n, \ \dim \mathbb{C}[x,y]/J = n-1.$$

We have 3 natural morphisms:

$$H_{n,n-1} \xrightarrow{q} H_n, \ H_{n,n-1} \xrightarrow{r} H_{n-1}, \ H_{n,n-1} \xrightarrow{f} \mathbb{C}^2.$$

From a result of Tikhomirov and also Cheah (cf. [C]) we have:

[3]These subschemes are called $G-$clusters in [B,K,R].

4.2 Theorem. *$H_{n,n-1}$ is smooth and $H_{n,n-1} \xrightarrow{q \times f} H_n \times \mathbb{C}^2$ has as image the universal family $\mathcal{F}$ and it is a resolution of the singularities of $\mathcal{F}$.*

Composing r with the morphism $H_{n-1} \xrightarrow{s} \mathbb{C}^{2(n-1)}/S_{n-1}$ we obtain

$$H_{n,n-1} \xrightarrow{sr \times f} \mathbb{C}^{2(n-1)}/S_{n-1} \times \mathbb{C}^2 = \mathbb{C}^{2n}/S_{n-1}$$

At this point define $X_{n,n-1}$ as reduced fiber product:

$$\begin{array}{ccc} X_{n,n-1} & \longrightarrow & X_{n-1} \\ \big\downarrow & & p\big\downarrow \\ H_{n,n-1} & \longrightarrow & H_{n-1} \end{array} \tag{4.3}$$

On the varieties $X_{n,n-1}, X_n, H_{n,n-1}, H_n$ we have standard line bundles. On H_n define $\mathcal{O}_{H_n}(1)$ as the maximal exterior power of the tautological bundle and $\mathcal{O}_{H_n}(k) = \mathcal{O}_{H_n}(1)^k$. On $H_{n,n-1}$ we have the line bundles obtained by pull back through the 2 projections, $H_{n,n-1} \xrightarrow{q} H_n$, $H_{n,n-1} \xrightarrow{r} H_{n-1}$ we set:

$$\mathcal{O}(h,k) := r^*\mathcal{O}_{H_{n-1}}(h) \otimes q^*\mathcal{O}_{H_n}(k).$$

On $X_{n,n-1}, X_n$ define also sheaves $\mathcal{O}(h,k), \mathcal{O}(k)$ by pullback from $H_{n,n-1}, H_n$.

We can now reformulate Theorem 3.1 in a more precise form:

4.4 Theorem. *T(n) X_n is Cohen Macaulay, Gorenstein with dualizing sheaf $\mathcal{O}(-1)$.*
U(n) $X_{n,n-1}$ is Cohen Macaulay and Gorenstein with dualizing sheaf $\mathcal{O}(0,-1)$.

The technique of the proof will be to follow the sequence of implications

$$T(n-1) \implies U(n) \implies T(n).$$

To proceed we must compute some canonical sheaves (cf. [H2], §3.6):

4.5 Theorem. *For $H_{n,n-1}, H_n$ we have as canonical sheaves:*

$$\omega_{H_{n,n-1}} = \mathcal{O}(1,-1), \qquad \omega_{H_n} = \mathcal{O}(0) \quad \textit{structure sheaf.}$$

It is necessary first of all to prove:

4.6 Lemma. *Suppose by induction that $X_{n-1} \to H_{n-1}$ is flat, the diagram*

$$\begin{array}{ccc} X_{n,n-1} & \longrightarrow & X_{n-1} \\ \big\downarrow & & p\big\downarrow \\ H_{n,n-1} & \longrightarrow & H_{n-1} \end{array}$$

is a fiber product.

Proof. The statement claims that, forming the fiber product

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p'"'] & X_{n-1} \times \mathbb{C}^2 \arrow[d, "p"'] \\ H_{n,n-1} \arrow[r] & H_{n-1} \times \mathbb{C}^2 \end{tikzcd}$$

we have that Y is reduced and thus it coincides with $X_{n,n-1}$.

In order to prove it we use the fact that p is flat and finite and so also p' is flat and finite, thus Y is Cohen Macaulay and it suffices to prove that it is reduced in codimension 0, which one obtains restricting to the regular part of the diagram. $\square$

Now prove that $T(n-1) \implies U(n)$.

From the previous Lemma it follows that $X_{n,n-1}$ is Gorenstein, moreover the dualizing sheaf relative to the morphism $X_{n,n-1} \to X_{n-1}$ is the pull back of the dualizing sheaf relative to the morphism $H_{n,n-1} \to H_{n-1}$ which is $\mathcal{O}(1,-1)$ while the dualizing sheaf of X_{n-1} is by induction $\mathcal{O}(-1,0)$ thus the tensor product is $\mathcal{O}(0,-1)$.

Now the implication $U(n) \implies T(n)$.

One has to analyze the morphism $g : X_{n,n-1} \to X_n$. The main Theorem follows from general principles from the proposition:

4.7 Proposition.

$$R^i g_* \mathcal{O}(X_{n,n-1}) = 0, \forall i > 0, \quad g_* \mathcal{O}(X_{n,n-1}) = \mathcal{O}(X_n). \tag{4.8}$$

This proposition is based on a basic Lemma and some geometric considerations.

4.9 Lemma. *Given a proper morphism $g : Y \to X$ between algebraic varieties over $\mathbb{C}$. Suppose we have given m global functions $z_1, \dots, z_m$ on X and let Z be the subvariety of X where they vanish and $U := X - Z$ the complement.*

Assume the following conditions:

(1) *The z_i form a regular sequence in every local ring $\mathcal{O}_{X,P}$, $P \in Z$.*
(2) *The z_i form a regular sequence in every local ring $\mathcal{O}_{Y,Q}$, $Q \in g^{-1}(Z)$.*
(3) *Every fiber of g has dimension $< m-1$.*
(4) *On the open set U the canonical morphism $\mathcal{O}_X \to Rg_*\mathcal{O}_Y$ is an isomorphism .*

then the canonical morphism $\mathcal{O}_X \to Rg_*\mathcal{O}_Y$ is an isomorphism (everywhere).

This Lemma has a fairly simple cohomological proof ([H2], Lemma 3.8.5). The hard point is to apply this Lemma to the morphism g. We choose as sequence z_i the $n-1$ functions $y_1 - y_2, \dots, y_1 - y_n$ and we need to verify the hypotheses (1)-(4).

The most difficult is (1).

In any case both for (1) and (2) one proves the stronger statement that $y_1, y_2, \dots, y_n$ is a regular sequence.

One observes:

For (2) by induction all the local rings are Cohen-Macaulay, one must verify that the codimension of the variety given by the equations $y_1 = y_2 = \cdots = y_n = 0$ is n.

It is enough to do it for $H_{n,n-1}$ since $X_{n,n-1} \to H_{n,n-1}$ is finite.

For (3) it is enough to analyze the morphism $H_{n,n-1} \to H_n$.

The fiber has maximal dimension on the fixed points, i.e. the ideals J_λ and consists of the ideals of dimension 1 of $\mathbb{C}[x,y]/J_\lambda$. By direct inspection one sees that this is a projective space of dimension $d-1$ where d is the number boundary cases of the diagram. For $n > 3$ the inequality is easy while for $n \leq 3$ one must prove the Lemma directly.

To prove (4) we see that on U the morphism is locally isomorphic to a product of two morphisms $g : X_{k,k-1} \times X_{h,h-1} \to X_k \times X_h$. Thus we can proceed by induction.

(1) is the difficult part.

First a reduction:

Presenting H_n and X_n as blow-ups.

First H_n.

Let $R := \mathbb{C}[x_1, \dots, x_n; y_1, \dots, y_n]$ be the polynomial ring in $2n$ variables:

$$S := \mathbb{C}[x_1, \dots, x_n; y_1, \dots, y_n]^{S_n}, \quad A := \{f \in R | \sigma(f) = \epsilon_\sigma f,\ \sigma \in S_n\}$$

the invariants and the alternating elements under the diagonal action of S_n (ϵ_σ denotes the sign of the permutation). Finally let $I := AR$ be the ideal of R generated by A. We add an indeterminate t and consider the two graded algebras:

$$U_n := S \oplus_{i=1}^{\infty} A^i t^i, \quad V_n := R \oplus_{i=1}^{\infty} I^i t^i$$

One acts with S_n on $R[t]$ by the diagonal action on R and acting on t with the sign representation.

4.10 Theorem. *a)* $U = V^{S_n}$.

b) $H_n = Proj(U_n), \quad X_n = Proj(V_n)$.

Sketch of proof.

It is well known that H_n can be described as follows (cf. [N]).

Consider the variety Z of triples (X, Y, v) where X, Y are two $n \times n$ matrices with $XY = YX$, $v \in \mathbb{C}^n$ a vector. Define Z_0 to be the open set of Z where the vectors $X^i Y^j v$ generate the space $\mathbb{C}^n$.

Z_0 is smooth, the group $GL(n, \mathbb{C})$ acts freely on Z_0 and finally $H_n = Z_0 / GL(n, \mathbb{C})$.

On Z many conjectures are open but we know that it is irreducible and that pairs of diagonalizable matrices are dense in Z.

Construct the quotient in two steps: $\tilde{H}_n = Z_0/SL(n,\mathbb{C})$, $\quad H_n = \tilde{H}_n/\mathbb{C}^*$ we see that $Z_0/SL(n,\mathbb{C})$ is an open set of the variety $Z//SL(n,\mathbb{C})$.

By definition $Z//SL(n,\mathbb{C})$ is the spectrum of the ring of invariants and, by classical invariant theory such invariants are generated by:

$$tr(M),\ [M_1v, M_2v, \dots, M_nv]$$

where M, M_i denote monomials in the matrices X, Y and $[M_1v, M_2v, \dots, M_nv]$ denotes the determinant of these vectors.

By the previous remarks we can compute this ring by restricting it to pairs of diagonal matrices and then it is easy to see that it is identified to U_n. Moreover the open set $Z_0/SL(n,\mathbb{C})$ is the part where at least one of the determinants $[M_1v, M_2v, \dots, M_nv]$ is non 0 from which the statement easily follows.

As for X_n one has clearly a commutative diagram:

$$\begin{array}{ccc} Proj(V_n) & \longrightarrow & \mathbb{C}^{2n} \\ {\scriptstyle p}\downarrow & & \downarrow \\ Proj(U_n) & \longrightarrow & \mathbb{C}^{2n}/S_n \end{array} \tag{4.11}$$

comparing it to 2.7 one has the claim.

At this point one has the next reduction:

In order to prove condition (1) it suffices to prove that, for every d the ideal I^d is a free module over the polynomial ring $\mathbb{C}[y_1, \dots, y_n]$.

This last statement will be further reduced to a more combinatorial statement. In order to explain it we must introduce some new objects, the *polygraphs*.

Thus, given a positive integer l let us denote by $[l]$ the segment $[1, 2, \dots, l]$.

Given a function $f : [l] \to [n]$ consider the induced linear map $\pi_f : (\mathbb{C}^2)^n \to (\mathbb{C}^2)^l$, and its graph $W_f \subset (\mathbb{C}^2)^n \times (\mathbb{C}^2)^l$.

The union $Z(n,l) := \cup_f W_f$ as f varies on the set of all functions $f : [l] \to [n]$ is a *polygraph.* Clearly on this polygraph operate various groups, in particular we will use the group S_l, and its subgroups, which permutes functions and graphs.

Call x_i, y_i the coordinates on $(\mathbb{C}^2)^n$ and a_i, b_i those on $(\mathbb{C}^2)^l$.

The coordinate ring of $Z(n,l)$ is a quotient $R(n,l) := \mathbb{C}[x_i, y_i, a_j, b_j]/I(n,l)$.

We consider next $l = nd$ and decompose

$$(\mathbb{C}^2)^n \times (\mathbb{C}^2)^n \times \cdots \times (\mathbb{C}^2)^n = (\mathbb{C}^2)^l$$

the group S_n^d operates permuting separately the coordinates of the factors and on the ring $R(n,l)$). Consider the subspace $R(n,l)^\epsilon$ of antisymmetric elements (with respect to all factors S_n).

Lemma. *There is a canonical isomorphism as R modules of $R(n,l)^\epsilon$ and I^d.*

Proof.

Let $f_0 : [l] \to [n]$ be defined as $f_0(kn+i) = i, \ \forall 1 \le i \le n, 0 \le k < d$ call $W_{f_0} = (\mathbb{C}^2)^n$ the associated graph. The restriction of functions of $R(n,l)$ to such a space is a morphism of R modules and sends $a_{kn+i} \to x_i, \ b_{kn+i} \to y_i$, it is easily seen that it maps $R(n,l)^\epsilon$ surjectively to J^d.

It is enough to prove that it is also injective. In fact an antisymmetric function vanishes on W_f if $f(kn+i) = f(kn+j), 1 \le i \le n, 0 \le k < d, \ i \ne j$. Any other function is in the orbit of f_0 relatively to S_n^d. By antisymmetry a function in $R(n,l)^\epsilon$ is determined completely by its values on W_{f_0}.

So finally let us show that everything will follow from the final key statement:

4.12 Theorem. *$R(n,l)$ is a free module on the polynomial ring $\mathbb{C}[y_1, \dots, y_n]$.*

Assume the last statement. Decomposing $R(n,l)$ into isotypic components with respect to S_n^d we see that $R(n,l)^\epsilon$ is a direct summand as module and so from the freeness of $R(n,l)$ follows also that of I^d isomorphic to $R(n,l)^\epsilon$.

At the end of this long sequence of reductions we have to face Theorem 4.12. This is proved by Haiman with a very long and complex induction using only a very careful bookkeeping and commutative algebra which occupies more than 30 pages of his paper. Rather than try to discuss this highly technical point we prefer to discuss some further developments. Nevertheless one should point out that, from the considerations that we will see in the next paragraph (Theorem 5.1), the appearence of polygraphs and their properties are geometrically natural.

5 Diagonal harmonics

The space of diagonal harmonics D_n can be defined as the subspace of polynomials in $\mathbb{C}[x_1, \dots, x_n; y_1, \dots, y_n]$ solutions of the system of differential equations $P(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i})f = 0$ where P runs over all polynomials without constant term which are symmetric with respect to the diagonal action of S_n.

D_n can also be identified with polynomial ring $\mathbb{C}[x_1, \dots, x_n; y_1, \dots, y_n]$ modulo the ideal generated by all S_n invariant polynomials without constant term.

Garsia and Haiman discovered a series of interesting conjectures mixing algebra, combinatorics and geometry on the space D_n, which is a bigraded, finite dimensional representation of S_n. The simplest of which describe its dimension as a vector space and its structure as representation.

1. $\dim D_n = (n+1)^{n-1}$.

2. As a representation D_n is isomorphic to the permutation representation on the set of *parking functions* tensored by the sign representation.[4]

More precise conjectures on the bigraded character can be found in [GH1], in particular a rather remarkable expression for its bigraded Frobenius character.

One can attack these conjectures using the Lefschetz fixed point formula of Atiyah-Bott (cf. [AB]).

The idea is to prove that the space of diagonal harmonics can be identified with the global sections of the vector bundle $p_*\mathcal{O}_{X_n}$ restricted to the subvariety of H_n consisting of subschemes supported at 0 and then compute its character by localization principles.

This vector bundle is acted upon by the torus T and the Lefschetz fixed point formula of Atiyah-Bott can be applied provided one knows the vanishing of suitable cohomology groups. Finally these vanishing theorems can be deduced by applying the theory of Bridgeland King Reid to X_n (this can be done because of the solution of the $n!-$conjecture). Their theory in our case establishes an equivalence of derived categories *the BKR correspondence* (generalized McKay correspondence) between the derived category of coherent sheaves on H_n and that of S_n-equivariant modules over R.

The announced geometric interpretation of polygraphs and their property is:

5.1 Theorem. *Under the BKR correspondence the polygraph $R(n,l)$ corresponds to the bundle F^l.*

The locus $V(y_1,\dots,y_n)$ in X_n is a complete intersection.

This provides the requested vanishing theorems.

Once all of this is done one obtains finally an explicit formula, in term of Macdonald polynomials of the bigraded character of D_n (cf. [H3], 3.10 and 3.11). In particular the two previous conjectures follow from this formula.

Final comments The theory we have sketched applies to the sum of two copies of the standard reflection representation of S_n. It thus suggests possible generalizations to other reflection groups.

[4] $n-$people choose a spot out of n linearly ordered parking spots. They must reach their chosen spot and, if free park in it, otherwise reach the first free spot and park. The choice made is a *parking function* if, no matter in which order the people arrive they always park.

At the moment it is not very clear what can be generalized and in which form, as it is completely misterious the question of understanding which $G-$Hilbert schemes are smooth. Haiman has made some computations for type B_n.

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