QUANTUM SCHUBERT CELLS AND REPRESENTATIONS AT ROOTS OF 1.

C. De Concini and C. Procesi

February 1994

To the memory of Roger Richardson good mathematician and friend

Introduction One of the several Hopf algebras recently introduced as *Quantum groups* (cf. the address of Drinfeld in the I.C.M. of Berkeley [D] and [Ji],[R-T-F]) is the $q-$analogue of the coordinate ring of a semisimple algebraic group G. It is possible to define in a coherent way a specialization of this algebra $R_q[G]$ when $q=\varepsilon$ is a root of 1 (cf. [D-L]). The resulting algebra $F_\varepsilon[G]$ is a finite module over the coordinate ring Z_0 of G, (which also is central). From the non commutative nature of $R_q[G]$ one inherits a Poisson structure on G and there is a compatibility between this structure and the non commutative algebra $F_\varepsilon[G]$ (which in [D-P] we have axiomatized into the notion of Hamiltonian algebra, here we will not stress this point but only some of its consequences).

We are concerned with the study of the representation theory of $F_\varepsilon[G]$. By general facts it is natural to analyze irreducible representations by first specifying their central character on Z_0. The symmetries appearing in this picture suggest to stratify the spectrum of Z_0 according to the intersections $X_{w_1,w_2} := U^- n_{w_1} B^- \cap U^+ n_{w_2} B^+$ of double cosets.

Define $\tilde{R}_{w_1,w_2}$ as $F_\varepsilon[G]$ restricted to X_{w_1,w_2} modulo its nilpotent radical.

Our main results are:

i) The algebra $\tilde{R}_{w_1,w_2}$ is an Azumaya algebra of rank $\ell^{\frac{1}{2}(\ell(w_1)+\ell(w_2)+\mathrm{rank}(w_1-w_2))}$.

ii) The spectrum of the center of $\tilde{R}_{w_1,w_2}$ is a Galois covering of X_{w_1,w_2} with Galois group the ℓ torsion subgroup of the torus $T/S_{w_1,w_2}$ where $S_{w_1,w_2} := \{t \in T | t^{w_1} = t^{w_2}\}$.

Remark. a) $\ell(w_1)+\ell(w_2)+\mathrm{rank}(w_1-w_2)$ is the dimension of a symplectic leaf in X_{w_1,w_2}.

b) The degree of the Galois covering (equal to the number of irreducible representations with given central character in X_{w_1,w_2}) is $\ell^{n-\mathrm{rank}(w_1-w_2)}$ and $n-\mathrm{rank}(w_1-w_2)$ is the codimension in X_{w_1,w_2} of a symplectic leaf.

Similar results are obtained for the quantized Borel algebras, in particular considering the strata $X_{w,e}$ (e the identity element) of B^+. These results are on one hand a special case of those for G, on the other a necessary step for the general theorem.

partially supported by M.U.R.S.T. 40%

Particular importance have the strata $X_{s_i,e},\ X_{e,s_i}$ where s_i is a simple reflection. In this case the irreducible representations are ℓ dimensional and can be viewed as building blocks (by tensor products) of essentially all other representations.

This result should be compared with the results of Soibelman-Vaksman [S-V] and Levendorski -Soibelman [L-S] in the case of q a positive real number and the ones of Joseph [J] and Hodge-Levasseur [H-L] for a general q not a root of 1. Also Lakshmibai and Reshetikhin [L-R] study quantum Schubert cells but their treatment is not directly connected with our point of view, essentially they study a $q-$analogue of the projective coordinate ring while we take an affine point of view and study the $q-$analogues of the intersection of Schubert varieties with the opposite big cell.

Our techniques consist in studying such algebras as if they were restriction of functions on a Schubert cell. They are partly inspired and based on the work of the previously mentioned authors. The main difference is our stressing of finite dimensional algebras and their representations and Poisson geometry.

The paper is organized as follows: in §1 we recall basic notations and the results and definitions of the theory of Quantum groups and non commutative algebra which are used in this paper. In doing this we have been necessarily somewhat sketchy but we refer to the original papers where the topics here recalled are treated in detail. In §2 we introduce and study certain ideals in the algebras $F_q[G], F_q[B^{\pm}]$ which should be thought as defining some (affine cone over the) *quantum Schubert cell.* In §3 we prove the existence of a birational isomorphism between the coordinate ring of the quantum Schubert cell and another algebra which is a $q-$analogue of the coordinate ring of a suitable root subgroup. For this last algebra we already know the degree by [D-K-P]. In §4 we use the information previously obtained to prove our main theorem. Finally in §5 we discuss the explicit construction of irreducible representations as tensor products of elementary ones.

§1 Notations and recollections

1.1 In this paragraph we wish to recall several well known facts in the theory of semisimple groups over $\mathbb{C}$ (see for example [Bo]).

Consider a finite Cartan matrix $A = (a_{ij})$ of rank n. To A we associate a root system with positive roots Φ^+, simple roots $\Delta := \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, a semisimple Lie algebra $\mathfrak{g}$ and a simply connected semisimple group G all defined over $\mathbb{C}$.

$\mathfrak{g}$ has the decomposition into root spaces $\mathfrak{g} = \mathfrak{t} \oplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{t}$ denotes a fixed Cartan subalgebra.

Let P, P^+ denote the integral, resp. dominant integral, weights. Q the root lattice W the Weyl group, s_i the simple reflections associated to the simple roots α_i (given by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$), ω_i the fundamental weights and $\rho = \sum_{i=1}^n \omega_i$.

In W we have the length function. If $w = s_{i_1}s_{i_2}\dots s_{i_k}$ is a minimal expression we set $k = \ell(w)$.

Given $w \in W$ set $\Phi_w := \{\alpha \in \Phi^+ | w^{-1}(\alpha) \in \Phi^-\} = \Phi^+ \cap w(\Phi^-)$, then $\ell(w)$ equals the cardinality of Φ_w.

If $w = ab$ we say that this is a *reduced decomposition* if $\ell(w) = \ell(a) + \ell(b)$.

In W there is a unique element w_0 of maximal length, it plays a very important role in the theory. It is characterized by the property $w_0(\Phi^+) = \Phi^-$. We have $w_0^2 = 1$ and its length equals the number of positive roots, which will always be denoted by N in this paper.

If $w \in W$ and $w_0 = wa$, this is a reduced decomposition and

$$\Phi^+ = \Phi_w \cup w(\Phi_a), \quad \text{disjoint union.} \tag{1.1.1}$$

$Q \subset P$ is contained in a Euclidean space where the scalar product is denoted by $(\alpha|\beta)$, A is symmetrizable, i.e. there is a diagonal matrix D with entries d_i and DA is the matrix of the scalar products $d_i a_{ij} = (\alpha_i|\alpha_j)$.

Let T, B^+, B^-, U^+, U^- be a maximal torus, the positive, resp. negative Borel subgroups and their unipotent radicals in G (relative to the choices made) and:

$$\mathfrak{t},\ \mathfrak{b}^+ := \mathfrak{t} \oplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha,\ \mathfrak{b}^- := \mathfrak{t} \oplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha,\ \mathfrak{u}^+ := \oplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha,\ \mathfrak{u}^- := \oplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha \tag{1.1.2}$$

be the Lie algebras of these groups.

1.2 We introduce now the fundamental objects of study of this paper namely the quantum function algebra $F_q[G]$, its integral form and its specialization at roots of one.

For all the notations relative to q-numbers, q-factorials etc. the reader can refer, for example, to [D-P].

First of all let us briefly recall the definition of the algebra $F_q[G]$. One starts with the quantized universal enveloping algebra $U_q(\mathfrak{g})$, of Drinfeld-Jimbo, which is generated over the field $\mathbb{C}(q)$ by elements E_i, F_i and $K_i^{\pm 1}$ satisfying suitable q-analogues of Serre relations (see [L]) and which is a Hopf algebra (since this algebra and especially the definition of its comultiplication appear in various slightly different forms in the literature, we will adhere to the definition and notations given in [L]. One should be particularly careful when one uses results from [D-L]).

One then takes $F_q[G]$ as the Hopf algebra consisting of those linear functions f on $U_q(\mathfrak{g})$ satisfying the following properties:

a) There exists a finite codimensional ideal $I \subset U_q(\mathfrak{g})$ such that $I \subset Ker(f)$.

b) For all $i = 1, \dots, r$ there are integers $n_{i1}, \dots, n_{ik}$ such that $f(\prod_{s=1}^k (K_i - q^{n_{is}})) = 0$.

From this definition one sees that, in order to get elements of $F_q[G]$ one can take matrix coefficients of finite dimensional representations on which the spectrum of the K_i's consists of powers of q. Let us denote this class of representations by $\mathcal{C}$. To be more precise, given a representation $V \in \mathcal{C}$, a vector $v \in V$ and a linear form $\varphi \in V^*$ one defines the element $c_{\varphi,v} \in F_q[G]$, by the formula

$$c_{\varphi,v}(x) = \varphi(xv) \qquad \forall x \in U_q(\mathfrak{g}).$$

Furthermore one easily sees that $F_q[G]$ is linearly spanned by matrix coefficients and, by the complete reducibility of finite dimensional representations of $U_q(\mathfrak{g})$ ([L], [Ro]), one gets that one can restrict oneself to take V to be irreducible.

From the theory of the $R-$matrix one has an immediate implication on the commutation rules among the elements $c_{\phi,v}$. Assume that v, w have weights μ_1, μ_2 and that ϕ, ψ have weights ν_1, ν_2 with respect to the action of the elements K_i cf. [L-S]. Then:

$$c_{\phi,v}c_{\psi,w} = q^{-(\mu_1|\mu_2)+(\nu_1|\nu_2)}c_{\psi,w}c_{\phi,v} + \sum c_{\psi_i,w_i}c_{\phi_i,v_i} \tag{1.2.1}$$

where

$$\psi_i \otimes \phi_i = p_i(q)(M_i(E) \otimes M_i(F))\psi \otimes \phi,\ w_i \otimes v_i = p_i'(q)(M_i'(E) \otimes M_i'(F))w \otimes v \tag{1.2.2}$$

where the p_i, p_i' are in $\mathbb{C}(q)$ and M_i, M_i' are monomials of which at least one is not constant.

Let us now recall the classification of the irreducible elements in $\mathcal{C}$.

There is a bijective correspondence between dominant weights and isomorphism classes of irreducible left $U_q(\mathfrak{g})$ modules in $\mathcal{C}$. Given $\lambda \in P^+$ the corresponding module V_λ is characterized as follows. It contains a unique line L_λ such that, if $v \in L_\lambda$, one has $E_i v = 0$ and $K_i v = q^{(\lambda,\alpha_i)}v$ for all $i = 1, \dots, n$. Furthermore Weyl character formula holds for V_λ [L].

Let us now remark that we can consider $F_q[G]$ as a left $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ module. If S denotes the antipode this action is defined by

$$(a \otimes bf)(c) = f(S(a)cb) \qquad \forall a, b, c \in U_q(\mathfrak{g}), f \in F_q[G].$$

We shall often call the action of $U_q(\mathfrak{g}) \otimes 1$ (resp. $1 \otimes U_q(\mathfrak{g})$) the left (resp. right) action of $U_q(\mathfrak{g})$.

The maps $V_\lambda^* \otimes V_\lambda \to F_q[G]$ defined above (by taking matrix coefficients) give an isomorphism of left $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$-modules

$$\delta : \bigoplus_{\lambda \in P^+} V_\lambda^* \otimes V_\lambda \longrightarrow F_q[G] \tag{1.2.3}$$

Let us consider now the generalized braid group $\mathcal{B}$ associated to our root system [Ti]. Let us denote by $t_1, \dots, t_n$ its canonical generators. By $p : \mathcal{B} \to W$ the homomorphism taking t_i to the simple reflection s_i. One can define an action ([L], [L-S]) of $\mathcal{B}$ by algebra automorphisms on $U_q(\mathfrak{g})$ and an action of $\mathcal{B} \times \mathcal{B}$ on $F_q[G]$ by linear isomorphisms. These actions satisfy the following compatibility condition:

$$(h_1, h_2)(a \otimes b\ f) = h_1 a \otimes h_2 b\ ((h_1, h_2)(f))$$

for any $h_1, h_2 \in \mathcal{B}$, $a, b \in U_q(\mathfrak{g})$, $f \in F_q[G]$.

One then easily verifies that the decomposition (1.2.3) is preserved by the $\mathcal{B} \times \mathcal{B}$ action and one gets an action of $\mathcal{B}$ on each irreducible module V_λ compatible with the action of $\mathcal{B}$ on $U_q(\mathfrak{g})$.

Of this action we will systematically use one simple property which we wish to stress.

Let i be a node in the Dynkin diagram, the elements $E_i, F_i, K_i^{\pm 1}$ generate a subalgebra which is the extension to $\mathbb{C}(q)$ of a copy of the algebra $U_{q_i}(sl_2)$, with $q_i = q^{d_i}$. The element $t_i \in \mathcal{B}$ induces on this subalgebra the corresponding Braid automorphism. If we consider a representation V_λ this will decompose under $U_{q_i}(sl_2)$ into irreducible representations which are stable under t_i. Each of this is isomorphic, with the action of t_i to a highest weight representation of $U_{q_i}(sl_2)$. For the representation of highest weight m of the algebra $U_q(sl_2)$, if v_0 is a highest weight vector then $v_m := F^{(m)}v_0$ is a lowest weight vector and moreover (cf. [D-L])

$$t(v_0) = (-q)^m v_m,\ t(v_m) = v_0,\ E^{(m)}v_m = v_0. \tag{1.2.4}$$

We shall often use the previous remarks in the following form:

Lemma. *If v is a vector of weight λ in a representation in $\mathcal{C}$ and $E_i v = 0$ then setting $m = (\lambda|\alpha_i)/d_i,\ v' := F_i^{(m)}v$ we have:*

$$t_i(v) = (-q_i)^m v',\ t_i(v') = v,\ E_i^{(m)}v' = v. \tag{1.2.5}$$

Similarly for a weight vector killed by F_i.

Proof. The hypothesis implies that v is a highest weight vector for a representation of highest weight m of the corresponding $U_q(sl_2)$. □

Given $w \in W$, let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression. One knows that the element $t_w = t_{i_1} \cdots t_{i_r} \in \mathcal{B}$ is independent from the choice of the reduced expression [I-M]. This allows us to define some distinguished elements in $F_q[G]$. Fix $\lambda \in P^+$, consider a lowest weight vector $v_{-\lambda} \in V_{-w_0\lambda}$ ($v_{-\lambda}$ is uniquely defined up to multiplication by a non zero constant). Let $\phi_\lambda \in V^*_{-w_0\lambda} \cong V_\lambda$ be a highest weight vector normalized in such a way that $\phi_\lambda(v_{-\lambda}) = 1$. We then set

$$z_w^\lambda = t_w(\phi^\lambda) \otimes v_{-\lambda} \qquad \zeta_w^\lambda = \phi^\lambda \otimes t_{w^{-1}}(v_{-\lambda}). \tag{1.2.6}$$

(we shall often write z_w (resp. ζ_w) for z_w^ρ (resp. ζ_w^ρ, notice also that $z_e^\lambda = \phi^\lambda \otimes v_{-\lambda}$).

1.3 Let us consider now the three subalgebras U^0, $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ in $U_q(\mathfrak{g})$ generated respectively by the elements K_i, by the elements E_i and K_i and finally by the F_i and K_i. These are Hopf subalgebras and U^0 can be identified with the group algebra of the root lattice Q, where the K_i correspond to the simple roots. The braid group action restricts to U^0 inducing the Weyl group action on Q.

Choose a reduced expression $w_0 = s_{i_1}s_{i_2}\dots s_{i_k}s_{i_{k+1}}\cdots s_{i_N}$, of the longest element w_0 in the Weyl group. From this deduce a convex ordering of the positive roots

$$\beta_1, \beta_2, \dots, \beta_k, \dots, \beta_N$$

and in correspondence root vectors

$$F_{\beta_1}, \dots, F_{\beta_N} \in U_q(\mathfrak{b}^-);\ E_{\beta_1}, \dots, E_{\beta_N} \in U_q(\mathfrak{b}^+)$$

defined as:
(1.3.1)
$$\beta_k := s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\alpha_{i_k}),\ F_{\beta_k} := t_{i_1}t_{i_2}\dots t_{i_{k-1}}(F_{i_k}),\ E_{\beta_k} := t_{i_1}t_{i_2}\dots t_{i_{k-1}}(E_{i_k}).$$

Each F_β is a polynomial in the F_i with coefficients in U^0 and no constant term (similarly for the E_β). One has that the elements:

$$\prod_{i=1}^{N} F_{\beta_i}^{h_i} K_\alpha := F_{\beta_N}^{h_N} \dots F_{\beta_k}^{h_k} \dots F_{\beta_2}^{h_2} F_{\beta_1}^{h_1} K_\alpha, \ \alpha \in Q, \ h_i \geq 0 \tag{1.3.2}$$

are a basis of $U_q(\mathfrak{b}^-)$.

Similarly for the corresponding monomials in $U_q(\mathfrak{b}^+)$. One could also use the automorphisms t_i^{-1} and obtain parallel results (see [L]).

There are basic commutation relations between the elements E_β (or the F_β) which can be found in [L-S] (cf. also [Ri]) and we will often use referring to them as the L-S relations.

We can now define three quotient Hopf algebras of $F_q[G]$. For this consider the restriction of the linear functions in $F_q[G]$ to the subalgebras U^0, $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$. Since these are Hopf subalgebras the kernels of these restrictions are Hopf ideals and we thus get quotient Hopf algebras which we denote by $F[T]$, $F_q[B^+]$ and $F_q[B^-]$. Notice that $F[T]$ is the classical coordinate ring of the torus having Q as character group, while the other two algebras are genuine quantizations of function algebras.

There is a different very important way of presenting the algebras $F_q[B^+]$ and $F_q[B^-]$ by means of the so called Drinfeld pairing (see [Ta]). Remark first of all that one can define (see [D-K-P]) a *simply connected* form $U_q^s(\mathfrak{g})$ by extending the indices α in the K from the root to the weight lattice. Correspondingly we also have algebras $U_q^s(\mathfrak{b}^\pm)$. We have a Hopf algebra pairing between $U_q^s(\mathfrak{b}^-)$ and $U_q(\mathfrak{b}^+)_{op}$ where the subscript *op* reminds us that we have to take the opposite multiplication. Similarly we have a pairing between $U_q(\mathfrak{b}^-)$ and $U_q^s(\mathfrak{b}^+)_{op}$.

Under the canonical pairing we have:

$$(\prod_{i=N}^{1} F_{\beta_i}^{h_i} K_\alpha, \prod_{i=N}^{1} E_{\beta_i}^{h_i} K_\beta) = q^{-(\alpha|\beta)} \prod (h_i)_{q_{\beta_i}^2}!(q_{\beta_i}^{-1} - q_{\beta_i})^{-h_i} \tag{1.3.3}$$

(where $q_\beta = q^{\frac{(\beta,\beta)}{2}}$ and $(h)_q = \frac{q^h-1}{q-1}$) and 0 otherwise.

The algebra $U_q^s(\mathfrak{b}^-)$ under this pairing is identified to $F_q[B^+]$ (similarly for $^-$).

Remark. The $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ action on $F_q[G]$ induces a $U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^+)$ action on $F_q[B^+]$.

Given a left ideal $I \subset U_q(\mathfrak{b}^+)$ and an element $f \in F_q[B^+]$, we have that f is orthogonal in the Drinfeld pairing to the right ideal $S(I)$ if and only if it is killed by I under the left action.

Proof. $(a \otimes 1)f(x) = f(S(a)x) = (f, S(a)x)$. □

1.4 We now set $R = \mathbb{C}[q, q^{-1}]$; in [D-L] one defines a form $R_q[G]$ of $F_q[G]$, i.e. a free R Hopf subalgebra such that the obvious homomorphism $\mathbb{C}(q) \otimes_R R_q[G] \to F_q[G]$ is a isomorphism. We want to recall a few properties of $R_q[G]$. First of all $R_q[G]$ is stable under the $\mathcal{B} \times \mathcal{B}$ action and the elements $z_w^\lambda, \zeta_w^\lambda \in R_q[G]$.

Under the restriction homomorphisms $r_\pm : F_q[G] \to F_q[B^\pm] \cong U_q^s(\mathfrak{b}^\mp)$, the image of $R_q[G]$ equals the subalgebra generated over R by the elements K_λ, $\overline{F}_\beta = (q_\beta - q_\beta^{-1})F_\beta$ (resp. K_λ, $\overline{E}_\beta = (q_\beta - q_\beta^{-1})E_\beta$). Call these algebras $R_q[B^\pm]$.

Furthermore consider the homomorphism

$$\gamma : R_q[G] \xrightarrow{\Delta} R_q[G] \otimes R_q[G] \to R_q[B^+] \otimes R_q[B^-],$$

in [D-L] it is proved that γ is injective and $\gamma(z_e^\lambda) = K_\lambda \otimes K_{-\lambda}$ ($e \in W$ being the identity element), so that $\gamma(z_e^\lambda)$ is invertible in $R_q[B^+] \otimes R_q[B^-]$. Furthermore if we set $d = z_e^\rho$, then the subalgebra $\gamma(R_q[G])[d^{-1}]$ of $R_q[B^+] \otimes R_q[B^-]$ coincides with the one generated by the elements $K_\lambda \otimes K_{-\lambda}$, $\overline{F}_\beta \otimes 1$ and $1 \otimes \overline{E}_\beta$ (see Theorem 4.6 in [D-L]).

The properties of γ allow us to show (cf. 1.2.6):

Proposition. *Let λ, $\mu \in P^+$, $w \in W$ then $z_w^\lambda z_w^\mu = z_w^{\lambda+\mu}$ and $\zeta_w^\lambda \zeta_w^\mu = \zeta_w^{\lambda+\mu}$.*

Proof. We prove the proposition for the z_w^λ's the proof for the ζ_w^λ's being the same.

We proceed by induction on the length of w. If $w = e$ then everything follows from the fact that $\gamma(z_e^\lambda) = K_\lambda \otimes K_{-\lambda}$ and the injectivity of γ. Now write $w = s_i w'$ with $\ell(w') = \ell(w) - 1$. By induction we have $z_{w'}^\lambda z_{w'}^\mu = z_{w'}^{\lambda+\mu}$. Also by [L-S] (see the formula for $\Delta(t_i)$), we have that

$$\begin{aligned} z_w^{\lambda+\mu} &= (t_i, e) z_{w'}^{\lambda+\mu} = (t_i, e)(z_{w'}^\lambda z_{w'}^\mu) \\ &= \sum_{n \geq 0} \frac{q^{\binom{n}{2}}(q - q^{-1})^n}{[n]!} ((t_i, e)(F_i^n \otimes 1 z_{w'}^\lambda))((t_i, e)(E_i^n \otimes 1 z_{w'}^\mu)) \end{aligned}$$

Now $E_i \otimes 1 z_{w'}^\mu = (E_i t_{w'}(\phi^\mu)) \otimes v_{-\mu} = (t_{w'}(t_{w'}^{-1}(E_i)(\phi^\mu))) \otimes v_{-\mu} = 0$ since $t_{w'}^{-1}(E_i)$ is a non commutative polynomial without constant term in $E_1, \dots, E_n$ (cf. 1.3 and [L]). We deduce that

$$z_w^{\lambda+\mu} = ((t_i, e) z_{w'}^\lambda)((t_i, e) z_{w'}^\mu) = z_w^\lambda z_w^\mu$$

as desired. $\square$

1.5 Let us now specialize q to 1, i.e. consider the Hopf algebra $F_1[G] = R_q[G]/(q-1)$. In [D-L] one shows that $F_1[G]$ is isomorphic as a Hopf algebra to the ring of regular functions on the algebraic group G. Furthermore the algebras $R_q[B^+]$ and $R_q[B^-]$ specialize to the ring of functions on two opposite Borel subgroups B^+ and B^- and the homomorphisms $r_\pm$ to the homomorphisms induced by the respective inclusions. The map γ specializes to the algebra homomorphism induced by the morphism $B^+ \times B^- \to G$ sending a pair (b_1, b_2) to $b_1 b_2$. If we then consider the maximal torus $T = B^+ \cap B^-$ then there is a lifting of the quotient $p : \mathcal{B} \to W = N(T)/T$ to $N(T)$ such that the $\mathcal{B} \times \mathcal{B}$ action on $F_1[G]$ is induced, via this lifting, by the action by left and right multiplication (notice that in this case $\mathcal{B} \times \mathcal{B}$ acts by algebra automorphisms).

The fact that $F_1[G]$ is a specialization of $R_q[G]$ gives G the extra structure of a Poisson Algebraic Group (see [D-P]) with Poisson bracket given by

$$\{f_1, f_2\} = \frac{[\overline{f}_1, \overline{f}_2]}{q-1} \quad mod(q-1)$$

for any $f_1, f_2 \in F_1[G]$, with $\overline{f}_1, \overline{f}_2 \in R_q[G]$ being arbitrarily chosen representatives. This Poisson structure is well known. Here let us just recall that a symplectic leaf $\mathcal{O} \subset G$ is contained in exactly one of the sets $X_{w_1,w_2} = B^+w_1B^+ \cap B^-w_2B^-$, $w_1, w_2 \in W$. If this is the case we say that $\mathcal{O}$ is associated to the pair (w_1, w_2) and we have $dim\mathcal{O} = \ell(w_1) + \ell(w_2) + rk(w_1 - w_2)$, where the rank is taken in the reflection representation. In particular in $X_{e,e} = T$ each point is a symplectic leaf. X_{w_1,w_2} is stable under left and right multiplication by T, these transformations are compatible with the Poisson structure and they permute transitively the symplectic leaves in X_{w_1,w_2}.

1.6 We now specialize q to an arbitrary primitive ℓ-th root of unity ε, with the restriction that ℓ is odd and in case G has G_2 components ℓ is prime with 3.

Set $F_\varepsilon[G] = R_q[G]/(q-\varepsilon)$ (resp. $F_\varepsilon[B^\pm] = R_q[B^\pm]/(q-\varepsilon)$). Denote by $r_{\pm,\varepsilon}$ and γ_ε the specializations of $r_\pm$ and γ at ε. One has

Theorem. *There exists injective, $\mathcal{B}\times\mathcal{B}$ equivariant, Hopf algebra homomorphisms called Frobenius homomorphisms, $\mathcal{F} : F_1[G] \to F_\varepsilon[G]$ and $\mathcal{F}_\pm : F_1[B^\pm] \to F_\varepsilon[B^\pm]$ such that:*
1) Their images are central subalgebras over which $F_\varepsilon[G]$ (resp. $F_\varepsilon[B^\pm]$) are finite projective modules of rank ℓ^{dimG} (resp. $\ell^{dimB^\pm}$).
2) The diagrams

$$\begin{array}{ccc} F_1[G] & \xrightarrow{\mathcal{F}} & F_\varepsilon[G] \\ {\scriptstyle r_{\pm,1}}\downarrow & & {\scriptstyle r_{\pm,\varepsilon}}\downarrow \\ F_1[B^\pm] & \xrightarrow{\mathcal{F}_\pm} & F_\varepsilon[B^\pm] \end{array}$$

and

$$\begin{array}{ccc} F_1[G] & \xrightarrow{\mathcal{F}} & F_\varepsilon[G] \\ {\scriptstyle \gamma_1}\downarrow & & {\scriptstyle \gamma_\varepsilon}\downarrow \\ F_1[B^+]\otimes F_1[B^-] & \xrightarrow{\mathcal{F}_+\otimes\mathcal{F}_-} & F_\varepsilon[B^+]\otimes F_\varepsilon[B^-] \end{array}$$

commute.
3) Let $g \in G$. Set $F_\varepsilon[G](g) = F_\varepsilon[G]/\mathcal{F}(\mathfrak{m}_g)F_\varepsilon[G]$, where $\mathfrak{m}_g$ is the maximal ideal of g in $F_1[G]$. Then if $g, h \in X_{w_1,w_2}$, for some pair $(w_1, w_2) \in W \times W$

$$F_\varepsilon[G](g) \cong F_\varepsilon[G](h).$$

4) $\mathcal{F}(z_w^\lambda) = (z_w^\lambda)^\ell$ (here, by abuse of notation we denote an element and its class under specialization by the same symbol).

Proof. The only statement which is not proved in [D-L] is the last which is shown only for $w = e$. To see the general case notice that

$$\mathcal{F}(z_w^\lambda) = \mathcal{F}((t_w, e)(z_e^\lambda)) = (t_w, e)(\mathcal{F}(z_e^\lambda)) = (t_w, e)z_e^{\lambda\ell} = (z_w^\lambda)^\ell$$

by Proposition 1.4. $\square$

1.7 In this paragraph we collect some notions and results of non commutative algebra useful for our work.

Start from the following general concept:

Definition. Given an algebra R over k, $q \in k^*$, we say that an element $a \in R$ is a $q-$element if there exists a basis u_i of R and integers m_i such that, for all i, we have:

$$au_i = q^{m_i} u_i a.$$

The main elementary remark is:

Proposition. *If a is a $q-$element and a non zero divisor in R then a can be inverted in any subalgebra $A \subset R$ containing a. I.e. there is an algebra containing A whose elements are fractions r/a^k.*

Proof. Set $R_i := \{r \in R | ar = q^i ra\}$, by hypothesis $R = \oplus R_i$. One has to show that a is an Öre element, i.e. that given $b \in A$ there is a $c, d \in A$ and non negative integers h, k such that $a^h b = ca$ and $ba^k = ad$. Now if $b \in A$, write $b = \sum_i b_i,\ b_i \in R_i$. If $b = b_i$ for some i we have $ab = q^i ba$. Otherwise we proceed by induction on the number of non zero b_i's. Let m be the maximum index for which $b_i \neq 0$. Write

$$ab = \sum_{i \leq m} ab_i = \sum_{i \leq m} q^i b_i a = q^m ba + \sum_{i \leq m-1} (q^i - q^m) b_i a.$$

Since $b_i a \in R_i$, and clearly the element $b' = \sum_{i \leq m-1}(q^i - q^m) b_i a \in A$, we can apply the inductive assumption to b' and find a $c' \in A$ and a non negative integer h such that $a^h b' = c'a$. We then obtain

$$a^{h+1} b = a^h (q^m ba + b') = (q^m a^h b + c')a$$

as desired. The proof of the remaining property being identical, is left to the reader. $\square$

In any algebra R denote by $(b) := RbR$ the two sided ideal generated by the element b.

Lemma. *If $b \in R$ is a $q-$element then for any positive integer n we have that $b \in \sqrt{(b^n)}$.*

Proof. Decompose $R = \oplus_i R_i$ where $R_i := \{a \in R | ab = q^i ba\}$ (if q is a primitive ℓ root of 1 we sum modulo ℓ). Then $bR_i = R_i b$, so $bR = Rb$ and $(Rb)^n \subset Rb^n$. $\square$

A basic technical problem with which one has often to deal in non commutative algebra is localization of a multiplicative set, which is only possible under special Öre conditions of which we have seen an example. We will not need very complicated localizations, but all our algebras are (left and right) noetherian rings and most of them domains which then have a quotient division algebra. So, when we invert elements in such algebras, we will always think that we do it in this division algebra.

1.8 Recall that an algebra R over a commutative ring A is called Azumaya algebra of rank d (over A) if there is a commutative algebra B faithfully flat over A such that $R \otimes_A B \cong M_d(B)$ (where $M_d(B)$ denotes the algebra of $d \times d$ matrices over B).

We shall use the fact that, if R is an Azumaya algebra over its center Z, S a Z algebra and $\phi : R \to S$ a Z algebra homomorphism then $S = R \otimes_Z C(R)$ where $C(R)$ is the centralizer of $\phi(R)$ in S.

In particular:

1) If both R, S are Azumaya algebras of the same degree any homomorphism $\phi : R \to S$ maps Z in the center C of S and $S = R \otimes_Z C$ (cf. [A-G1,2]).

2) Any ideal I of R is extended from Z i.e. $I = R(I \cap Z)$.

3) If V is a finite dimensional module over an Azumaya algebra R and the center Z of R acts on V as scalars, then the kernel of the action is a maximal ideal $\underline{m}$ and V is a direct sum of copies of the unique irreducible module of R/I.

A general theorem of M.Artin (cf. [A] [Sc]) shows that a ring R is an Azumaya algebra of rank d over its center if and only if it verifies the following two conditions:

1) R satisfies all polynomial identities of $d \times d$ matrices.

2) R has no homomorphism into $h \times h$ matrices over some commutative ring for $h < d$.

Condition 1) is automatically verified if R is contained in a ring of $d \times d$ matrices over a commutative ring or in a finite dimensional central simple algebra S of degree d.

If R is a finitely generated algebra over an algebraically closed field k, condition 2) is equivalent to assume that R has no irreducible representations of dimension $< d$.

Under the same hypotheses, if we assume that R satisfies some polynomial identity and has no nilpotent ideals then the two conditions are equivalent to assume that all irreducible representations of R have dimension d.

Let us now assume that R is a finite projective module over an algebra A contained in the center Z of R and finitely generated over an algebraically closed field k.

For any point P in the spectrum of A (which we think of as a homomorphism of A into k and for which we denote by $k(P)$ the field k as A module) denote by $R(P)$ the algebra $R \otimes_A k(P)$. We think of R as a bundle of algebras over $Spec(A)$ and of the $R(P)$ as the fibers.

Assume next that A is a domain and there is a nilpotent ideal N in R such that R/N is an order in a finite dimensional central simple algebra S of degree d, by hypothesis $A \cap N = 0$ so A is contained in the center Z of R/N and Z is a finite module over A.

Proposition. *If the algebras $R(P)$ are all isomorphic then R/N is an Azumaya algebra over its center Z and the map $Spec(Z) \to Spec(A)$ has fibers of fixed cardinality.*

Proof. The set of points in $Spec(Z)$ where R/N is an Azumaya algebra is the open set of $P \in Spec(Z)$ for which $R/N \otimes_Z k(P) \cong M_d(k)$. The set of points in $Spec(Z)$ where R/N is not an Azumaya algebra is a proper closed subvariety as also proper and closed is the subset of points where the map $\pi : Spec(Z) \to Spec(A)$ is not ètale. Thus there is a non empty open subset O of $Spec(A)$ for which R/N is Azumaya and π is ètale on $\pi^{-1}(O)$. It follows that, for $P \in O$ we have that $R/N(P) \cong M_d(k)^{\oplus n}$ where n is the degree of the map π. In particular we must have that for $P \in O$ the image of N in $R(P)$ must be its nilpotent radical and thus $R/N(P) \cong M_d(k)^{\oplus n}$ is the semisimple quotient of $R(P)$. The claim then follows, in fact by the hypothesis made the algebras $R(P)$ are all isomorphic, from Artin's theorem R/N is Azumaya of degree d.

For any $P \in Spec(A)$ the semisimple part of $R/N(P)$ equals $M_d(k)^{\oplus m}$ where m is the number of points in the fiber of π, hence m is constant. $\square$

§2 Quantum Schubert cells

2.1 Let us give an element $w \in W$ and a dominant weight λ and start defining:

$$v_{-w(\lambda)} := t_w(v_{-\lambda}) \qquad \phi^{w(\lambda)} := t_w(\phi^{\lambda}). \tag{2.1.1}$$

Notice that $v_{-w(\lambda)}$ is, up to scalars, the only vector of weight $-w(\lambda)$ in $V_{-w_0\lambda}$ (similarly for $\phi^{w(\lambda)}$).

In the classical case (q=1) this is one of the extremal vectors, its line thought as a point in projective space lies in the flag variety and is one of the T fixed points center of a corresponding Schubert cell (for B^- in our conventions).

We now define the subspace $V_{w,\lambda} \subseteq V_{-w_0\lambda}$ as follows:

$$V_{w,\lambda} :=< U_q(\mathfrak{b}^-) v_{-w(\lambda)} > . \tag{2.1.2}$$

Proposition. *If $w = s_i w'$ is reduced then $v_{-w(\lambda)} = cE_i^a v_{-w'(\lambda)}$ with a a non negative integer and c a non zero scalar (resp. $\phi^{w(\lambda)} = cF_i^a \phi^{w'(\lambda)}$).*

Proof. Since $v_{-w(\lambda)} = t_i v_{-w'(\lambda)}$ it is enough to apply the representation theory of the quantum $U_q(sl_2)$ generated by $E_i, F_i, K_i^{\pm 1}$ and show that $v_{-w'(\lambda)}$ is a lowest weight vector for this algebra (Lemma 1.2). Thus we need to show that $F_i v_{-w'(\lambda)} = 0$

Now by hypothesis $x = t_{w'}^{-1}(F_i)$ is a polynomial without constant term in the F_j. Thus

$$0 = t_{w'}(x v_{-\lambda}) = F_i v_{-w'(\lambda)}. \tag{2.1.3}$$

The proof in the remaining case is identical. $\square$

2.2

Proposition. *Let $w = s_{i_1} s_{i_2} \dots s_{i_k}$ then:*

$$V_{w,\lambda} :=< E_{i_1}^{h_1} E_{i_2}^{h_2} \dots E_{i_k}^{h_k} v_{-\lambda} > . \tag{2.2.1}$$

Proof. Work by induction.

$$V_{w,\lambda} :=< U_q(\mathfrak{b}^-) v_{-w(\lambda)} >=< U_q(\mathfrak{b}^-) E_{i_1}^a v_{-w'(\lambda)} > . \tag{2.2.2}$$

From the basic commutation relations we have:

$$U_q(\mathfrak{b}^-) E_{i_1} \subset E_{i_1} U_q(\mathfrak{b}^-) + U_q(\mathfrak{b}^-).$$

Thus

$$< U_q(\mathfrak{b}^-) E_{i_1}^a v_{-w'(\lambda)} >\subset< \sum_{b \le a} E_{i_1}^b U_q(\mathfrak{b}^-) v_{-w'(\lambda)} > .$$

Applying induction we have the inclusion:

$$V_{w,\lambda} \subset< E_{i_1}^{h_1} E_{i_2}^{h_2} \dots E_{i_k}^{h_k} v_{-\lambda} > .$$

Conversely by induction and the commutation relations

$$< E_{i_1}^{h_1}E_{i_2}^{h_2}\dots E_{i_k}^{h_k}v_{-\lambda} >\subset< E_{i_1}^{h}U_q(\mathfrak{b}^-)v_{-w'(\lambda)} >\subset< \sum_b U_q(\mathfrak{b}^-)E_{i_1}^{b}v_{-w'(\lambda)} > .$$

We have remarked that $v_{-w'(\lambda)}$ is a lowest weight vector for the $U_{q_i}(sl_2)$ generated by $E_i, F_i, K_i^{\pm 1}$ and $v_{-w(\lambda)}$ is the highest weight vector of the module it generates, we get:

$$< \sum_b U_q(\mathfrak{b}^-)E_{i_1}^{b}v_{-w'(\lambda)} >\subset< \sum_b U_q(\mathfrak{b}^-)F_{i_1}^{b}v_{-w(\lambda)} >\subset V_{w,\lambda}$$

as desired. □

2.3 By the definitions and the representation theory we have

$$V^*_{-w_0\lambda}\otimes v_{-\lambda} = \{x\in F_q[G] | (1\otimes F_i)x = 0,\ K_\alpha x = q^{-(\alpha.\lambda)}x\}.$$

By the formula of the comultiplication in $U_q(\mathfrak{g})$ we get immediately that

$$V^*_{-w_0\lambda}\otimes v_{-\lambda}V^*_{-w_0\mu}\otimes v_{-\mu}\subset V^*_{-w_0(\lambda+\mu)}\otimes v_{-\lambda-\mu}.$$

In fact we have equality by irreducibility and this is the $q-$analogue of Cartan's multiplication.

Let us introduce thus the subalgebra

$$\mathcal{H} = \oplus_\lambda V^*_{-w_0\lambda}\otimes v_{-\lambda}\subset F_q[G]. \tag{2.3.1}$$

We want to study the restriction to $\mathcal{H}$ of the homomorphism $r_+ : F_q[G]\to F_q[B^+]$. We have:

Lemma. *1) The elements z_e^λ are q-elements in $\mathcal{H}$.*

2) The restriction of r_+ to $\mathcal{H}$ extends to a isomorphism between $\mathcal{H}[z_e^{-1}]$ and $F_q[B^+]$.

Proof. Part 1) follows immediately from the commutation relations 1.2.1. In particular z_e is a Öre element and it makes sense to consider part 2).

Using the Drinfeld pairing we identify $F_q[B^+]$ with $U_q^s(\mathfrak{b}^-)$. Under this identification we have that $r_+(z_e) = K_\rho$ so that it is invertible in $U_q^s(\mathfrak{b}^-)$ and hence we can extend r_+ to a homomorphism

$$\tilde{r}_+ : \mathcal{H}[z_e^{-1}]\to F_q[B^+]. \tag{2.3.2}$$

In order to show that (2.3.2) is an isomorphism we need to show that the restriction of r_+ to $\mathcal{H}$ is injective and (2.3.2) is surjective.

Notice that the decomposition (2.3.1) is the decomposition of $\mathcal{H}$ into weight spaces under the action of the algebra $1\otimes U^0\subset U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})$. Since this algebra also acts on $F_q[B^+]$ and r_+ is equivariant, it suffices to show injectivity for each of the spaces $V^*_{-w_0\lambda}\otimes v_{-\lambda}$. This follows immediately since $V_{-w_0\lambda} = U_q(\mathfrak{b}^+)v_{-\lambda}$.

Let us now show the surjectivity of (2.3.2). It is based on the formula:

$$r_+(z_{s_i}) = q^{2d_i}\overline{F}_i K_\rho. \tag{2.3.3}$$

From 1.2.5 and the fact that $(\alpha_i, \rho) = d_i$, we get that $z_{s_i} = -q^{d_i} F_i \otimes 1(z_e)$.

Let now $x = \prod_{i=1}^N E_{\beta_i}^{h_i} K_\alpha \in U_q(\mathfrak{b}^+)$ be a basis element as in (1.3.2). We have $z_{s_i}(x) = -q^{d_i} z_e(S(F_i)x) = 0$ unless $x = E_i K_\alpha$. If this is the case we have

$$z_{s_i}(E_i K_\alpha) = -q^{d_i} z_e(-F_i K_i E_i K_\alpha) = q^{d_i - (\alpha,\rho) + (\alpha_i,\alpha_i) - (\alpha_i,\rho)}.$$

This implies our claim by 1.3.3.

Now since the elements K_λ and F_i generate $U_q^s(\mathfrak{b}^-)$ the surjectivity of (2.3.2) follows. $\square$

Choose a reduced expression $w_0 = s_{i_1} s_{i_2} \dots s_{i_k} s_{i_{k+1}} \dots s_{i_N}$, let us denote by $w_k = s_{i_1} s_{i_2} \dots s_{i_k}$, $a_k = s_{i_{k+1}} \dots s_{i_N}$.

Definition. Set $I_k := \oplus_\lambda V^\perp_{w_k,\lambda} \otimes v_{-\lambda}$.

Set $F[B_k]$ to be the subalgebra of $F_q[B^+] \cong U_q^s(\mathfrak{b}^-)$ generated by the elements

$$F_{\beta_1}, F_{\beta_2}, \dots, F_{\beta_k},\ K_\lambda;\ \lambda \in P.$$

Remark. $I_k \supset I_{k+1}$, $F[B_k] \subset F[B_{k+1}]$.

Proof. The first part follows from Proposition 2.2 the second is clear. $\square$

Notice that, if we fix $w \in W$ we can always find a reduced expression for w_0 such that $w = w_k$ (for $k = \ell(w)$). I_k, $F[B_k]$ depend only on w and not on the choice of a reduced expression. If we want to stress this point of view we shall denote them by I_w, $F[B_w]$.

2.4 Consider now for each i the restriction of the function algebra $F_q[B^+]$ to the subalgebra $U_q(\mathfrak{b}_i) \subset U_q(\mathfrak{b}^+)$ generated by K_i, E_i. By duality this gives a map of $F_q[B^+] \cong U_q^s(\mathfrak{b}^-)$ to the algebra $S_i := \{K_{\omega_i}, F_i\}$. Explicitely this maps $F_j \to 0$, $j \neq i$, $F_i \to F_i$, $K_\lambda \to K_{\omega_i}^{(\lambda|\alpha_i)}$. Similarly we can deduce a mapping to the algebra $\mathcal{T}_i := \{K_{\omega_i}\}$ which can be viewed as function algebra on the subalgebra $\mathcal{T}_i^*$ generated by K_i.

Using the given reduced expression of w_0, consider the map:

(2.4.1)

$$\pi_k : F_q[B^+] \xrightarrow{\Delta^N} F_q[B^+] \otimes F_q[B^+] \dots F_q[B^+] \to \otimes_{s=1}^k S_{i_s} \bigotimes \otimes_{s=k+1}^N \mathcal{T}_{i_s}.$$

We can identify $S_{i_1} \otimes S_{i_2} \otimes \dots S_{i_k} \otimes \mathcal{T}_{i_{k+1}} \otimes \dots \otimes \mathcal{T}_{i_N}$ as an algebra of linear functionals on $U_q(\mathfrak{b}_{i_1}) \otimes U_q(\mathfrak{b}_{i_2}) \otimes \dots U_q(\mathfrak{b}_{i_k}) \otimes \mathcal{T}^*_{i_{k+1}} \otimes \dots \otimes \mathcal{T}^*_{i_N}$ and this map by duality as a restriction of functions

(2.4.2)

$$\pi_k(f)(u_{i_1} \otimes u_{i_2} \otimes \dots u_{i_k} \otimes m_{i_{k+1}} \otimes \dots \otimes m_{i_N}) = f(u_{i_1} u_{i_2} \dots u_{i_k} m_{i_{k+1}} \dots m_{i_N}).$$

In particular if f is the restriction to B^+ of $c_{\phi, v_{-\lambda}}$ we have:

$$f(u_{i_1} u_{i_2} \cdots u_{i_k} m_{i_{k+1}} \cdots m_{i_N}) = \phi(u_{i_1} u_{i_2} \cdots u_{i_k} m_{i_{k+1}} \cdots m_{i_N} v_{-\lambda}),$$

so that, by Proposition 2.2, $f = c_{\phi, v_{-\lambda}} \in Ker(\pi_k)$ if and only if $\phi \in V^\perp_{w_k,\lambda}$.

From the definition of $V_{w_k,\lambda}$ we see that it has a basis of weight vectors of weight less or equal to $-w_k(\lambda)$ thus a sufficient condition for an element ϕ to be in $V^\perp_{w,\lambda}$ is that it is a weight vector of weight not greater or equal to $w_k(\lambda)$.

Definition. We set J_k to be the kernel of π_k.

Proposition. $J_k \cap \mathcal{H} = Ker(\pi_k) \cap \mathcal{H} = \oplus_\lambda V_{w_k,\lambda}^\perp \otimes v_{-\lambda} := I_k$.

Proof. We claim that, if $f \in Ker(\pi_k)$ also $(1 \otimes K_i)f \in Ker(\pi_k)$. Indeed

$$\pi_k((1 \otimes K_i)f)(u_{i_1} \otimes u_{i_2} \otimes \dots u_{i_k} \otimes m_{i_{k+1}} \otimes \dots \otimes m_{i_N}) = f(u_{i_1} u_{i_2} \dots u_{i_k} m_{i_{k+1}} \dots m_{i_N} K_i)$$

and by our definitions and the commutation relations it follows that $u_{i_1} u_{i_2} \dots u_{i_k} m_{i_{k+1}} \dots m_{i_N} K_i$ is again a monomial of the type considered in 2.4.2

This implies that $Ker(\pi_k) \cap \mathcal{H}$ is a direct sum of weight spaces (under the right action of U^0).

Since the decomposition $\mathcal{H} = \oplus_{\lambda \in P^+} V_{-w_0\lambda}^* \otimes v_{-\lambda}$ is the decomposition of $\mathcal{H}$ in weight spaces under the action of $1 \otimes U^0$ the claim follows. □

Corollary. *1) I_k is an ideal in $\mathcal{H}$ and $\mathcal{H}/I_k$ is an integral domain.*

2) $J_k = I_k[z_e^{-1}]$ so it depends only on w and not on the choice of a reduced expression.

Proof. 1) That I_k is an ideal is a consequence of the previous proposition. On the other hand $\mathcal{H}/I_k$ is contained in the image of the map π_k which lies in an algebra which is clearly a twisted polynomial algebra with some elements inverted and hence a domain. 2) is clear. □

Remark. By the above corollary J_k can be denoted by J_w.

Example. J_e is generated by all the elements F_i. The quotient algebra $F_q[B^+]/J_e$ is the commutative (polynomial algebra) generated by K_λ, $\lambda \in P$.

For a simple reflection the ideal J_{s_i} is generated by the elements F_j, $j \neq i$. The algebra $F_q[B^+]/J_{s_i}$ is isomorphic to the subalgebra of $U_q^s(\mathfrak{b}^-)$ generated by F_i, K_λ; $\lambda \in P$.

§3 The birational map

3.1 We have chosen together with a reduced expression of w_0 the elements:

$$\beta_1, \beta_2, \dots, \beta_N; \ E_{\beta_1}, E_{\beta_2}, \dots, E_{\beta_N}, \ F_{\beta_1}, F_{\beta_2}, \dots, F_{\beta_N}.$$

Recall that the ordering of the positive roots is *convex*, in other words if a root β_k is a linear combination with positive coefficients of other roots β_i then these roots cannot be all greater or all less than β_k.

We consider the monomials

$$K_\lambda E_{\beta_1}^{h_1} E_{\beta_2}^{h_2} \dots E_{\beta_N}^{h_N}; \quad E_{\beta_N}^{h_N} E_{\beta_{N-1}}^{h_{N-1}} \dots E_{\beta_1}^{h_1} K_\lambda \tag{3.1.1}$$

we call these monomials respectively the *increasing* and the *decreasing* monomials in the E. Similarly we can do for the F. Recall that the two monomials appearing in 3.1.1 have weight $\beta := \sum_{i=1}^N h_i \beta_i$.

Definition. Let L^i_k, L^d_k be the span of all increasing (resp. decreasing) monomials in the E's with $h_i > 0$ for some $i > k$. Let $\mathcal{U}_k$ be the subalgebra generated by the K_λ's and by the elements $E_{\beta_1} \dots , E_{\beta_k}$.

Notice that $\mathcal{U}_k$ is spanned by the increasing or decreasing monomials 3.1.1 with $h_i = 0$ for all $i > k$.

Lemma. *Any increasing (resp. decreasing) monomial of weight β_k different from $K_\alpha E_{\beta_k}$ lies in L^i_k (resp. in L^d_k).*

More generally, if M_1, M_2 are increasing monomials, and M_1M_2 is of weight β_k and $M_1 \in \mathcal{U}_{k-1}$ has non zero weight then $M_2 \in L^i_k$ (similarly for decreasing).

Proof. This follows from the convexity of the order of the roots. □

In what follows we shall use the following property of the comultiplication which is claimed in [L-S] and proved in the Appendix of [A-J-S]:

$$\Delta(E_{\beta_k}) = E_{\beta_k} \otimes 1 + K_{\beta_k} \otimes E_{\beta_k} + \sum_i a_i \otimes b_i$$

where, if we choose the a_i, b_i to be increasing monomials then $a_i \in \mathcal{U}_{k-1},\ b_i \in L^i_k$ or, if we choose the a_i, b_i to be decreasing monomials then $a_i \in L^d_k,\ b_i \in \mathcal{U}_{k-1}$.

Lemma. *1) L^i_k is the left ideal of $U_q(\mathfrak{b}^+)$ generated by the elements E_{β_j} with $j > k$ (resp. L^d_k is the right ideal generated by the same elements).*

2) $S(E_{\beta_k}) = -K_{-\beta_k} E_{\beta_k} + T$ where $T \in L^i_k \cap L^d_k$.

3) $S(L^i_k) = L^d_k$.

Proof. 1) It is clear that L^i_k is contained in the prescribed left ideal. Conversely consider a product $ME_{\beta_j}, j > k$ where M is an increasing monomial. It is clear from the relations L-S (cf. 1.3) that this is a linear combination of increasing monomials of the form $M' E_{\beta_t}, t \geq j$ hence the claim, (the proof for L^d_k is similar).

2) Recall that for any element a with $\Delta(a) = \sum_j u_j \otimes v_j$ one has $\epsilon(a) = \sum_j S(u_j)v_j$. Apply this to the expression for $\Delta(E_{\beta_k})$ and use the fact that $\epsilon(E_{\beta_k}) = 0$ and part 1).

3) Since $S^2(E_{\beta_j}) = q^{(\beta_j,\beta_j)} E_{\beta_j}$ so that $S^2(L^i_k) = L^i_k, S^2(L^d_k) = L^d_k$ it is enough to show that $S(E_{\beta_j}) \in L^d_k$; this follows immediately from part 2) □

Remark. We can order lexicographically the multiindices $h_N, h_{N-1}, \dots , h_1$. If we fix one of these multiindices $\underline{h}_0$ let us define by $L^d(\underline{h}_0)$ (resp. $L^i(\underline{h}_0)$) the span of all decreasing (resp. increasing) monomials with $\underline{h} \geq \underline{h}_0$ in the lexicographic ordering. Then in general we will have that:

$L^i(\underline{h}_0)$ is a left ideal (resp. $L^d(\underline{h}_0)$ is a right ideal).

3.2

Definition. Set $z_k = z^\rho_{w_k}$.

Lemma. *1) z_k is annihilated under the left action of $U_q(\mathfrak{g})$ on $F_q[G]$ by L_k^i.*

2) $F[B_k]$ is the subspace orthogonal to L_k^d under the Drinfeld pairing. The image of the element z_k in $F_q[B^+] = U_q^s(\mathfrak{b}^-)$ lies in $F[B_k]$.

3) We have

$$z_k = q^{2(\beta_k|\rho)}\overline{F}_{\beta_k} z_{k-1} + P,\ \ P \in F[B_{k-1}]. \tag{3.2.1}$$

Proof. 1) Set $t_{w_k} = t_k$. Clearly $t_k(\phi^\rho)$ is killed by all elements of type $t_k(M)$ where M is a positive monomial in the E's. Since by Lemma 3.1 L_k^i is the left ideal generated by the elements E_{β_j} with $j > k$ and these elements are linear combinations of elements of the previous type, 1) follows.

2) By the very properties of the Drinfeld pairing $F[B_k]$ is the orthogonal to L_k^d since the exponents monomials defining the two spaces lie in complementary sets. From the first part $0 = (a \otimes 1)z_k$ if $a \in L_k^i$, hence $z_k(S(a)u) = 0$ for all u, in particular $(z_k, S(a)) = z_k(S(a)) = 0$ it follows from Lemma 3.1 that z_k is orthogonal to L_k^d hence $z_k \in F[B_k]$.

3) It is similar to the proof of 2.3.3. Set $w = w_k$ and $\overline{w} = w_{k-1}$. We have

$$E_{\beta_k}(t_{\overline{w}}\phi^\rho) = t_{\overline{w}}(E_{i_k}\phi^\rho) = 0.$$

While

$$K_{\beta_k}(t_{\overline{w}}\phi^\rho) = t_{\overline{w}}(K_{i_k}\phi^\rho) = q_{\beta_k}(t_{\overline{w}}\phi^\rho). \tag{3.2.2}$$

We can apply the $U_{q_{\beta_k}}(sl_2)$ generated by $E_{\beta_k}, F_{\beta_k}, K_{\beta_k}^{\pm 1}$ (with corresponding element $t_{\beta_k} = t_{\overline{w}} t_{i_k} t_{\overline{w}}^{-1}$) for which we have seen in proposition 2.1 that $t_{\overline{w}}\phi^\rho$ is a highest weight vector. 3.2.2 implies that the representation that it generates is 2 dimensional with lowest weight vector $t_w(\phi^\rho) = t_{\beta_k} t_{\overline{w}}\phi^\rho$, getting:

$$t_w(\phi^\rho) = -q_{\beta_k} F_{\beta_k}(t_{\overline{w}}\phi^\rho).$$

We have

$$z_k = t_w(\phi^\rho) \otimes v_{-\rho} = -q_{\beta_k}(F_{\beta_k} \otimes 1)z_{k-1},\ \ z_{k-1} = -q_{\beta_k}^{-1}(E_{\beta_k} \otimes 1)z_k$$

(we are using the left action on functions). It follows from 1) that z_k is killed under left action by $L_k^i + L_{k-1}^i E_{\beta_k}$, which is the left ideal spanned by all increasing monomials for which either $h_i \geq 1$, for some $i > k$ or $h_k \geq 2$. It follows that z_k is orthogonal to $S(L_k^i + L_{k-1}^i E_{\beta_k}) = L_k^d + E_{\beta_k} L_{k-1}^d$ hence $z_k \in F_{\beta_k} F[B_{k-1}] + F[B_{k-1}]$. Write now $z_k = F_{\beta_k} u + v,\ u, v \in F[B_{k-1}]$, we want to compute the element u by applying again the Drinfeld pairing.

We have

$$(z_k | E_{\beta_k} M) = (S^{-1}(E_{\beta_k}) \otimes 1 z_k | M).$$

Recall that S^{-1} is S followed by conjugation by $K_{2\rho}$ and $S(E_{\beta_k}) = -K_{-\beta_k} E_{\beta_k} + T$ where $T \in L_k^i$ kills z_k. Thus $S^{-1}(E_{\beta_k}) \otimes 1 z_k = -q^{2(\beta_k|\rho)} z_{k-1}$. It follows that

$$(z_k | E_{\beta_k} M) = -q^{2(\beta_k|\rho)}(z_{k-1} | M).$$

Now, if M is a decreasing monomial in $\mathcal{U}_{k-1}$, we have that

$$(F_{\beta_k}u+v|E_{\beta_k}M)=(q_{\beta_k}-q_{\beta_k}^{-1})^{-1}(u|M),$$

hence $q^{2(\beta_k|\rho)}(z_{k-1}|M)=(q_{\beta_k}-q_{\beta_k}^{-1})^{-1}(u|M)$. Since we already know that $u\in F[B_{k-1}]$ we obtain $u=q^{2(\beta_k|\rho)}(q_{\beta_k}-q_{\beta_k}^{-1})z_{k-1}$ as required. □

In order to continue we need to use all the integral forms of our algebras and maps.

Theorem. *1) The induced map:*

$$\phi_w : F[B_w]\to F_q[B^+]/J_w \tag{3.2.3}$$

is injective.

2) The element z_w is a $q-$element in $F_q[B^+]/J_w$. If $\overline{w}=ws_j$, with $\ell(\overline{w})=\ell(w)-1$ then $z_{ws_j}\in J_{\overline{w}}$. If $w=ab$ is a reduced decomposition the elements z_a,z_w $q-$commute.

3) After inversion (see 1.8) we have:

$$F[B_w][z_w^{-1}]=(F_q[B^+]/J_w)[z_w^{-1}].$$

Proof. Fix a reduced expression $w=s_{i_1}\cdots s_{i_k}$ such that $i_k=j$. Thus $I_w=I_k$, $F[B_w]=F[B_k]$ and so on.

1) We use the integral form $R_q[B_k]$, of $F[B_k]$. That is the subalgebra over R generated by the elements K_λ and $\overline{F}_{\beta_1},\dots,\overline{F}_{\beta_k}$. This is a free R-module with basis the increasing monomials in these generators. It suffices to show that the restriction of the map ϕ_k to $R_q[B_k]$ is injective.

J_k is by Proposition 2.4 the kernel of the map π_k defined in 2.4.1. π_k maps $R_q[B^+]$ in the corresponding tensor product of integral forms and thus it suffices to prove the injectivity after specialization at $q=1$. At $q=1$ the map π_k is induced on functions by a map whose image is a dense subset in $X_{e,w}\subset B^+$ (see 1.5), while the inclusion $R_q[B_k]\to R_q[B^+]$ at $q=1$ corresponds to the projection of B^+ onto the subgroup $B_k=T\prod_{i=1}^k U_{\beta_i}$, where U_β is the root subgroup corresponding to the root β given by the decomposition $B^+=B_k\times\prod_{i=k+1}^N U_{\beta_i}$. In is shown in [D-L], [D-P], that the composed map from $X_{e,w}$ to B_k is birational hence the injectivity follows.

2) The fact that z_k is a $q-$element in $F_q[B^+]/J_k$ is proved in [L-S] and follows from the commutation relations 1.2.1; in fact if $M(E)$ is a non constant monomial in the E_i the element $M(E)t_{w_k}\phi^\rho$ has weight strictly greater than $w_k(\rho)$ and so it is in $V_{w_k,\lambda}^\perp$.

For the same reason $t_{w_k}\phi^\rho\in V_{w_{k-1},\lambda}^\perp$; thus by Proposition 2.4 $z_k\in J_{k-1}$.

For the last part we are claiming that z_i,z_k $q-$commute ($i<k$). By part 1) it is enough to show that they $q-$commute modulo J_k and this follows again by 1.2.1.

3) Let us prove by induction that, for every $i>0$, we have $F_{\beta_{k+i}}z_k^M\in F[B_k]+J_k$ for some M. This is formula 3.2.1 for $i=1$ in view of 2.

Work by decreasing induction assuming that, if $i \geq 2$ we have

$$F_{\beta_{k+i}} z_{k+1}^M = u, \ \bmod\ J_{k+1} \tag{3.2.4}$$

where $u = \sum_{j=0}^m u_j F^j_{\beta_{k+1}}$ and $u_j \in F[B_k]$.

We now claim that, for each j there is a sufficiently high integer t depending on j such that:

$$F^j_{\beta_{k+1}} z_k^t = \sum_{s=0}^h x_s z_{k+1}^s, \ \bmod\ J_{k+1}$$

with $x_s \in F[B_k]$. If $j = 1$ this is just relation 3.2.1, so we can proceed by induction on j. If we multiply $F^j_{\beta_{k+1}}$ by z_k we obtain, using 3.2.1,

$$F^j_{\beta_{k+1}} z_k = aF^{j-1}_{\beta_{k+1}} z_{k+1} + F^{j-1}_{\beta_{k+1}} b$$

with $a, b \in F[B_k]$. Now using the commutation relations between that F_β's one easily gets that $F^{j-1}_{\beta_{k+1}} b = \sum_{r<j} b_r F^r_{\beta_{k+1}}$. Thus using the inductive hypothesis and the fact that z_k and z_{k+1} q-commute mod J_{k+1}, we get the desired conclusion.

From what we have seen it now follows that if we multiply both sides of 3.2.4 by a sufficiently high power of z_k, we obtain

$$F_{\beta_{k+i}} z_{k+1}^M z_k^P = \sum_{j=0}^h v_j z_{k+1}^j, \ \bmod\ J_{k+1}$$

with $v_j \in F[B_k]$. If $M = 0$ we are done, otherwise since reducing modulo J_{k+1} we have a domain, since z_k, z_{k+1} $q-$commute we may divide by the highest power of z_{k+1} and assume that $v_0 \neq 0$. Now reduce modulo z_{k+1} which lies in J_k getting $F_{\beta_{k+i}} z_{k+1}^M z_k^P = v_0$, mod J_k. Since $M > 0$ and $z_{k+1} \in J_k$, we get $v_0 = 0$ mod J_k. This contradicts step 1. $\square$

Remark. From 3.2.1 and 1.3 it follows easily that the ordered monomials in the elements $K_{\omega_1}, \dots, K_{\omega_n}, z_1, \dots, z_k$ are linearly independent.

We work now in the division algebra D of fractions of $F[B_k]$ and use the notation $A < a_1, \dots, a_k >$ for the subalgebra of D generated by a subalgebra A and elements a_i.

Corollary. *1) The algebra $F[B_k]$ is contained in the twisted Laurent polynomial ring generated by $K_{\omega_1}, \dots, K_{\omega_n}, z_1, \dots, z_k$ and their inverses.*

2) Let $a \in F_q[B^+]$, there exists a non negative integer p such that $az_k^p \in F[B_k]$ mod $J_{k+1} + (z_{k+1})$. In particular if $a \in J_k$, $az_k^p \in J_{k+1} + (z_{k+1})$.

Proof. 1) Since $F[B_{k-1}] < z_{k-1}^{-1}, z_k >$ contains $F[B_{k-1}]$, to show that it contains $F[B_k]$ we compute $\overline{F}_{\beta_k}$. By 3.2.1 this is $-q^{-2(\beta_k|\rho)}(z_k - P)z_{k-1}^{-1}$, where $P \in F[B_{k-1}]$ is as in 3.2.1. The rest of our statement follows easily by induction since for $k = 0$ the algebra $F[B_k] = F[T]$ is exactly the Laurent polynomial ring generated by $K_{\omega_1}, \dots, K_{\omega_n}$.

2) From the proof of step 3) in the Theorem we get that our statement holds for the F_β's. Since z_k is a Öre element in $F[B_k]$ the claim follows from this simple fact that we leave to the reader:

If $A \subset B$ are algebras, J an ideal of B, a an Öre element in A then the set

$$C := \{x \in B | \exists p \geq 0,\ xa^p \in A + J\}$$

is a subalgebra.

For the second part use the fact that by step 1 in the Theorem $J_k \cap F[B_k] = 0$. □

Remark. For $w = s_i$ a simple reflection we have that the map ϕ_{s_i} is already an isomorphism (cf. last Remark of 2.4).

§4 Roots of 1

4.1 In this section ε is a primitive ℓ^{th} root of 1, with ℓ as in 1.6.

We have already remarked that the map π_k defined in 2.4.1 induces a map of integral forms:

$$\pi_k : R_q[B^+] \xrightarrow{\Delta^N} R_q[B^+]^{\otimes N} \to \otimes_{s=1}^{k} R_q[B_{i_s}] \otimes_{s=k+1}^{N} R_q[T_{i_s}]. \tag{4.1.1}$$

This map clearly specializes at $q = \varepsilon$ and let us indicate by $\pi_{\varepsilon,k}$ the induced map and by $J_{\varepsilon,k}$ (or $J_{\varepsilon,w}$ if we wish to stress its dependence only on $w = s_{i_1} s_{i_2} \cdots s_{i_k}$) its kernel.

Note that the map $\pi_{\varepsilon,k}$ takes values again in a twisted polynomial algebra which is a domain and so $J_{\varepsilon,k}$ is a (completely) prime ideal.

We are now ready to study the irreducible representations of the algebras $F_\varepsilon[G], F_\varepsilon[B^+]$ introduced in 1.6.

In both cases we have the Frobenius map (see 1.6), i.e. they contain the coordinate ring $\mathbb{C}[G], \mathbb{C}[B^+]$ respectively as Hopf subalgebras in the center and are projective modules of rank $\ell^{dim(G)}$ and $\ell^{dim(B^+)}$ respectively over them.

Let us denote by R, Z_0 one of these two algebras and the corresponding commutative Hopf subalgebra. The spectrum of Z_0 is either the group G or B^+ and we stratify it respectively by strata X_{w_1,w_2} and $X_{e,w} = X_w$, each of these is a union of symplectic leaves permuted by T transitively. It follows that up to the isomorphisms given by T and the ones given by the Hamiltonian flows, the algebras $R(p)$ are constant on these strata (cf. 1.6 3)). Our final task is to study these algebras $R(p)$ for each stratum and prove the results announced in the introduction.

4.2 We start with $F_\varepsilon[B^+]$. In this case the corresponding algebra Z_0 is the coordinate ring of B^+ and its spectrum B^+ is stratified by the locally closed sets X_w.

We shall use the notations of §2. Let $w \in W$. Define Y_w to be the ideal of $Z_0 = \mathbb{C}[B^+]$ formed by the elements vanishing on X_w. Let $Y_{\varepsilon,w}$ be the ideal of the algebra $F_\varepsilon[B^+]$ generated by Y_w.

Lemma. $Y_w = Z_0 \cap J_{\varepsilon,w}$.

Proof. We can assume, as usual that $w = w_k$. We have to understand the restriction of the map $\pi_{\varepsilon,k}$ to $Z_0 = \mathbb{C}[B^+]$. For this remark that Z_0 is a sub Hopf algebra and so the comultiplication is the one induced by the group B^+. The projections to the algebras S_i and $\mathcal{T}_i$ restricted to Z_0 correspond to the inclusions of subgroups B_i^+, T_i which are respectively the positive Borel and the maximal torus of the canonical Sl_2 associated to the node of the Dynkin diagram of index i. The claim follows from the fact that the corresponding product of subgroups is dense in X_w [De]. $\square$

4.3 We come now to the main lemma.

Lemma. $\sqrt{Y_{\varepsilon,w}} = J_{\varepsilon,w}$.

Proof. We can assume as usual that $w = w_k$. From the previous lemma we have that $Y_{\varepsilon,k} \subset J_{\varepsilon,k}$. Since $J_{\varepsilon,k}$ is a prime ideal it is sufficient to show that $J_{\varepsilon,k}$ is in the nilpotent radical of $Y_{\varepsilon,k}$. By induction J_{k+1} is in the nilpotent radical of $Y_{\varepsilon,k+1}$. Now since $F_\varepsilon[B^+]$ is a projective module over Z_0 we have that $F_\varepsilon[B^+]/Y_{\varepsilon,k} = F_\varepsilon[B^+] \otimes_{Z_0} Z_0/Y_k$, which is projective over Z_0/Y_k. Since $z_k^\ell \in Z_0$ (see 1.6) and $z_k \notin J_k$ the element z_k^ℓ will be a non zero divisor in $F_\varepsilon[B^+]/Y_{\varepsilon,k}$. By the Lemma in 1.7 and the fact that z_{k+1} is a q-element mod J_{k+1}, we have that $J_{k+1} + (z_{k+1})$ is in the nilpotent radical of $J_{k+1} + (z_{k+1}^\ell)$. By Corollary 3.2 for any element $c \in J_{\varepsilon,k}$ there is a product $cz_k^{\ell r} \in J_{k+1} + (z_{k+1})$. Thus there is a power $c^m z_k^{\ell rm} \in J_{k+1} + (z_{k+1}^\ell)$. Since $z_{k+1}^\ell \in Y_k$ we have that $c^m z_k^{\ell rm} \in \sqrt{Y_{\varepsilon,k}}$ hence $c^p z_k^{\ell rp} \in Y_{\varepsilon,k}$ for some p . Now z_k^ℓ is a non zero divisor modulo $Y_{\varepsilon,k}$ hence it follows that $c^p \in Y_{\varepsilon,k}$ for some p, hence $J_{\varepsilon,k}$ is in the radical of $Y_{\varepsilon,k}$. $\square$

4.4 We now consider the specializations at $q = 1$ of z_w and ζ_w as functions on the group G. We have

Proposition. *Let D_w (resp. C_w) be the divisor of z_w (resp. ζ_w). Then $D_w \cap \overline{B^-wB^-} = \overline{B^-wB^-} - B^-wB^-$ and $C_w \cap \overline{B^+wB^+} = \overline{B^+wB^+} - B^+wB^+$.*

Proof. We prove the proposition for D_w the proof for C_w being the same. Since D_e is the complement of the big cell B^+B^- and $D_w = wD_e$ by definition, we clearly have that $G - D_w = wB^+B^-$. Thus we have to show that $B^-wB^- = wB^+B^- \cap \overline{B^-wB^-}$. The fact that $B^-wB^- \subset wB^+B^-$ is clear since setting $U_w^- = wU^+w^{-1} \cap U^-$, one has $B^-wB^- = U_w^- wB^- = ww^{-1}U_w^- wB^- \subset wB^+B^-$. To complete the proof it suffices to show that B^-wB^- is relatively closed in wB^+B^-. Now the product map gives an isomorphism of algebraic varieties between wU^+w^{-1} and $U_w^+ \times U_w^-$ where $U_w^+ = wU^+w^{-1} \cap U^+$. Decompose now, again using the product map, wB^+B^- as the product $wU^+w^{-1} \times wB^-$. Substituting we obtain an isomorphism between wB^+B^- and $U_w^+ \times U_w^- \times wB^-$. Under this isomorphism $\{1\} \times U_w^- \times wB^-$ maps isomorphically onto B^-wB^-, so we obtain an isomorphism of wB^+B^- with $U_w^+ \times B^-wB^-$ proving that B^-wB^- is relatively closed in wB^+B^-. $\square$

Let us consider now the algebra $F_\varepsilon[B^+]/J_{\varepsilon,w}$. We are now ready to prove:

Theorem. *1) The algebra $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$ is an Azumaya algebra.*

2) The map $Spec(F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]) \to Spec(F_\varepsilon[B^+])$ is bijective onto $Spec(F_\varepsilon[B_w^+])$.

3) If ℓ is prime with the bad primes (cf. [Bo]) for the root system, the degree of $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$ equals $\ell^{1/2(\ell(w)+\mathrm{rank}(e-w))}$.

Proof. 1) From Lemma 4.3 it follows that, given an irreducible $F_\varepsilon[B^+]$ module, it defines a $F_\varepsilon[B^+]/J_{\varepsilon,w}$ module if and only if its central character in $Spec(Z_0)$ lies in $\overline{X}_w$ and among these the $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$ modules are the ones with central character in X_w since $z_w^\ell \in Z_0$ and its divisor in $\overline{X}_w$ is the complement of X_w (by 1.6 and the previous Proposition). On the other hand we know that the algebras $F_\varepsilon[B^+](g)$, $g \in X_w$ are all isomorphic (Theorem 1.6), hence from Proposition 1.7 1) and 2) follow. From Theorem 3.2 we know that the degree of this algebra equals that of the algebra $F_\varepsilon[B_w]$ which has been computed in [D-K-P] proving 3). □

4.5 From now on we shall assume that ℓ is prime with the bad primes for the root system.

We want now to determine the spectrum of $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$.

For this we start noticing that the algebra $F_\varepsilon[B^+]/J_{\varepsilon,e}$ can be identified with the coordinate ring of the torus T (with character group P). Using this we get a homomorphism $ev_t : F_\varepsilon[B^+] \to k$, $\forall t \in T$ and, since $J_{\varepsilon,e}$ is a Hopf ideal, an action of T on $F_\varepsilon[B^+]$ by algebra automorphisms, defined by $t(f) = ev_t \otimes id \cdot \Delta(f)$ for all $t \in T$, $f \in F_\varepsilon[B^+]$.

The reader can easily verify that, given a matrix coefficient $c_{\phi,v}$ with ϕ a weight vector of weight μ one has $t(c_{\phi,v}) = t^\mu c_{\phi,v}$, so that in particular, $t(z_w^\lambda) = t^{w\lambda} z_w^\lambda$. This clearly implies that the ideal $J_{\varepsilon,w}$ is stable under the action of T so that we get a T-action on $F_\varepsilon[B^+]/J_{\varepsilon,w}$ and, since z_w is a weight vector for T, also a T-action on $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$.

Remark also that by the expression $\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$ and the fact that the elements $\overline{F}_\beta \in J_{\varepsilon,e}$ it follows immediately that the $\overline{F}_\beta$'s are fixed by T.

Having made this considerations we are ready to prove

Theorem. *The spectrum of $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$ is a Galois covering of X_w with Galois group the group $T_{w,\ell}$, of elements of ℓ torsion in T/S_w where $S_w := \{t \in T|\ t^\lambda = 1\ \forall \lambda \in P$ such that $w\lambda = \lambda\}$.*

Proof. Since $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$ is a Azumaya algebra we know that its spectrum equals the spectrum of its center Z_w, so we shall determine Z_w. We start by exhibiting some elements in Z_w.

We have from the $q-$commutation relations 1.2.1 the fact that the z_w^λ are ε-elements in $F_\varepsilon[B^+]/J_{\varepsilon,w}$. Explicitely modulo $J_{\varepsilon,w}$ we have

$$z_{w'}^\mu z_w^\lambda = q^{(w'\mu|w\lambda)-(\mu|\lambda)} z_w^\lambda z_{w'}^\mu$$

and

$$z_e^\lambda z_{w'}^\mu = q^{-(\mu - w'\mu|\lambda)} z_{w'}^\mu z_e^\lambda$$

for all $w' \in W$. Notice now that if we work in $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$ we get, using Proposition 1.4, that, for $\lambda \in P^+$, each z_w^λ is invertible. Take $\lambda \in P$, write $\lambda =$

$\lambda_+ - \lambda_-$ with $\lambda_\pm \in P^+$ and set $z_w^\lambda = z_w^{\lambda_+}(z_w^{\lambda_-})^{-1}$. It is then clear from the definition that the above relations continue to hold even if λ is not dominant.

Let now $P^w \subset P$ denote the sublattice of weights fixed by w. If $\lambda \in P^w$ we set

$$d_w^\lambda = z_w^\lambda z_e^\lambda.$$

It is then clear that the elements d_w^λ are central in $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$ and that the inverse of d_w^λ is $d_w^{-\lambda}$. Furthermore if we set $Z_{0,w} = Z_0/Y_w[z_w^{-\ell}]$, $d_w^{\ell\lambda} \in Z_{0,w}$. Choose a basis $\lambda_1, \dots, \lambda_s$ for P^w. We claim that

$$Z_w = Z_{0,w}[d_w^{\lambda_1}, \dots, d_w^{\lambda_s}] \tag{4.5.1}$$

(of course if $P^w = \{0\}$ this just means that $Z_w = Z_{0,w}$). The elements d_w^λ with $\lambda \in P_\ell^w$ where $P_\ell^w = \{\lambda | \lambda = \sum_{i=1}^s m_i\lambda_i \text{ with } 0 \le m_i \le \ell - 1\}$ clearly span $Z_{0,w}[d_w^{\lambda_1}, \dots, d_w^{\lambda_s}]$. We claim that they are linearly independent on $Z_{0,w}$. Indeed T acts on $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$. Let us restrict our action to the subgroup T_ℓ of ℓ-torsion points in T. It is clear from the above considerations and from the fact that Z_0 is generated by the elements $\overline{F}_\beta$ and $K_{\ell\mu}$that T_ℓ acts trivially on $Z_{0,w}$. On the other hand if $\lambda \in P_\ell^w$ and $t \in T$ we have $t(d_w^\lambda) = t^{2\lambda} d_w^\lambda$. Since P^w is a split direct summand in P and ℓ is odd, the action of T_ℓ factors through an action of $T_{w,\ell}$ and, as λ runs through P_ℓ^w, the characters $t^{2\lambda}$ run through a complete set of distinct characters for $T_{w,\ell}$. This clearly implies our claim and proves that the natural map

$$Spec(Z_{0,w}[d_w^{\lambda_1}, \dots, d_w^{\lambda_s}]) \longrightarrow X_w = Spec(Z_{0,w}),$$

is an unramified Galois covering with structure group $T_{w,\ell}$ (remember that the elements d_w^λ are invertible). On the other hand we know that the algebra $F_\varepsilon[B^+]/J_{\varepsilon,w}[z_w^{-1}]$ is a free module over $Z_{0,w}$ of rank $\ell^{\ell(w)+n}$ and has degree $\ell^{\ell(w)+rk(e-w)} = \ell^{\ell(w)+n-|T_{w,\ell}|}$. It follows immediately that the quotient fields of Z_w and of $Z_{0,w}[d_w^{\lambda_1}, \dots, d_w^{\lambda_s}]$ coincide, so since both algebras are finite as $Z_{0,w}$-modules and $Z_{0,w}[d_w^{\lambda_1}, \dots, d_w^{\lambda_s}]$ is integrally closed, (4.5.1) and the theorem follow. $\square$

We finish this section remarking that all we have stated for the algebras $F_q[B^+]$, $F_\varepsilon[B^+]$ has an analogue for the algebras $F_q[B^-]$, $F_\varepsilon[B^-]$. In particular the coordinate ring $\mathbb{C}[B^-]$ embeds as a central Hopf subalgebra in $F_\varepsilon[B^-]$. We can define, for any $w \in W$ the ideals $J_w^- \in F_q[B^-]$, $J_{\varepsilon,w}^- \subset F_\varepsilon[B^-]$. We can show that the intersection of this second ideal with $\mathbb{C}[B^-]$ is the ideal Y_w^- of function vanishing on the set $X_w^- = B^+wB^+ \cap B^-$, while if we consider the ideal $Y_{\varepsilon,w}^- \subset F_\varepsilon[B^-]$ generated by Y_w^- we have that $\sqrt{Y_{\varepsilon,w}^-} = J_{\varepsilon,w}^-$. In this situation the role of the elements z_w is taken by the elements ζ_w so that in particular ζ_w is a ε-element in $F_\varepsilon[B^-]/J_{\varepsilon,w}^-$ and $F_\varepsilon[B^-]/J_{\varepsilon,w}^-[\zeta_w^{-1}]$ is an Azumaya algebra of degree $\ell^{\frac{1}{2}(\ell(w)+rk(e-w))}$.

4.6 We now pass to the study of the irreducible representations of $F_\varepsilon[G]$.

In this situation we stratify G by the subsets $X_{w_1,w_2} = B^-w_1B^- \cap B^+w_2B^+$ (notice that $X_w = X_{w,e}$ while $X_w^- = X_{e,w}$).

Consider the homomorphism

$$\gamma_\varepsilon : F_\varepsilon[G] \to F_\varepsilon[B^+] \otimes F_\varepsilon[B^-]$$

defined in (1.6). Recall that γ_ε is injective and can be extended to an isomorphism of $F_\varepsilon[G][z_e^{-1}]$ onto the subalgebra $\mathcal{A} \subset F_\varepsilon[B^+] \otimes F_\varepsilon[B^+]$ generated by the elements $\overline{F}_\beta \otimes 1$, $1 \otimes \overline{E}_\beta$, $K_\lambda \otimes K_{-\lambda}$ (here we are using the isomorphisms given by Drinfeld duality). In particular $F_\varepsilon[B^+] \otimes F_\varepsilon[B^-]$ is the twisted Laurent polynomial algebra $\mathcal{A}[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ with $u_i = 1 \otimes z_e^{\omega_i}$.

Definition. For any pair $(w_1, w_2) \in W_1 \times W_2$ we define $J_{\varepsilon,w_1,w_2} \subset F_\varepsilon[G]$ to be the ideal $\gamma_\varepsilon^{-1}(J_{\varepsilon,w_1} \otimes F_\varepsilon[B^-] + F_\varepsilon[B^+] \otimes J^-_{\varepsilon,w_2})$

Remark. If as usual w_0 denotes the longest element in W, the ideal

$$J_{\varepsilon,w_0,e} = \gamma_\varepsilon^{-1}(F_\varepsilon[B^+] \otimes J^-_{\varepsilon,e})$$

equals the kernel of r_ε^+ since the mapping

$$F_\varepsilon[B^+] \to F_\varepsilon[B^+] \otimes F_\varepsilon[B^-]/J^-_{\varepsilon,e} = F_\varepsilon[B^+] \otimes U^0$$

maps $\overline{F}_\beta \to \overline{F}_\beta \otimes 1$, $K_\lambda \to K_\lambda \otimes K_{-\lambda}$ and is clearly injective.

The algebra $F_\varepsilon[G][z_e^{-1}]/J_{\varepsilon,w_1,e}[z_e^{-1}] \cong F_\varepsilon[B^+]/J_{\varepsilon,w_1}$.

We take now as $Z_0 \subset F_\varepsilon[G]$ the image of $\mathbb{C}[G]$ under the Frobenius homomorphism. We set $Y_{w_1,w_2} \subset Z_0$ equal to the ideal of functions vanishing on X_{w_1,w_2} and Y_{ε,w_1,w_2} equal to the ideal in $F_\varepsilon[G]$ generated by Y_{w_1,w_2}. As before we have:

Lemma. $J_{\varepsilon,w_1,w_2} \cap Z_0 = Y_{w_1,w_2}$.

Proof. Using Lemma 4.2, the commutative diagram (see 1.6)

$$\begin{array}{ccc} F_1[G] \cong Z_0 & \xrightarrow{\mathcal{F}} & F_\varepsilon[G] \\ {\scriptstyle\gamma_1}\downarrow & & {\scriptstyle\gamma_\varepsilon}\downarrow \\ F_1[B^+] \otimes F_1[B^-] & \xrightarrow{\mathcal{F}_+ \otimes \mathcal{F}_-} & F_\varepsilon[B^+] \otimes F_\varepsilon[B^-] \end{array}$$

and recalling that the map γ_1 is induced by the product map

$$\gamma_1^* : B^+ \times B^- \to G$$

sending a pair (b_+, b_-) to $b_+ b_-$, we reduce to prove that $X_{w_1,e} X_{e,w_2}$ is a dense subset in X_{w_1,w_2}. The fact that it is contained in X_{w_1,w_2} is clear from the definitions. To show that it is dense it suffices to see, since X_{w_1,w_2} is irreducible, that $dim(X_{w_1,e} X_{e,w_2}) \geq dim(X_{w_1,w_2})$. Now $dim(X_{w_1,w_2}) = \ell(w_1) + \ell(w_2) + n$ while γ_1^* is a principal T bundle onto its image; it follows that $dim(X_{w_1,e} X_{e,w_2}) \geq \ell(w_1) + \ell(w_2) + n$, proving the lemma. $\square$

4.7

Lemma. *The algebra $F_\varepsilon[B^+]/J_{\varepsilon,w_1}\otimes F_\varepsilon[B^-]/J^-_{\varepsilon,w_2}$ is a domain. Furthermore*

$$\sqrt{Y_{\varepsilon,w_1}\otimes F_\varepsilon[B^-]+F_\varepsilon[B^+]\otimes Y^-_{\varepsilon,w_2}}=J_{\varepsilon,w_1}\otimes F_\varepsilon[B^-]+F_\varepsilon[B^+]\otimes J^-_{\varepsilon,w_2}$$

Proof. We have by Corollary 3.2 that the algebra $F_\varepsilon[B^+]/J_{\varepsilon,w_1}$ is a subalgebra in a twisted Laurent polynomial algebra and similarly for $F_\varepsilon[B^-]/J^-_{\varepsilon,w_2}$. Hence their tensor product is also a subalgebra of a twisted Laurent polynomial algebra and hence has no zero divisors. To see the second part of the Lemma notice that by Lemma 4.6 we have that

$$Y_{\varepsilon,w_1}\otimes F_\varepsilon[B^-]+F_\varepsilon[B^+]\otimes Y^-_{\varepsilon,w_2}\subset J_{\varepsilon,w_1}\otimes F_\varepsilon[B^-]+F_\varepsilon[B^+]\otimes J^-_{\varepsilon,w_2},$$

while Lemma 4.3 implies that

$$\sqrt{Y_{\varepsilon,w_1}\otimes F_\varepsilon[B^-]+F_\varepsilon[B^+]\otimes Y^-_{\varepsilon,w_2}}\supset J_{\varepsilon,w_1}\otimes F_\varepsilon[B^-]+F_\varepsilon[B^+]\otimes J^-_{\varepsilon,w_2}.$$

The other inclusion follows immediately from the first part. □

4.8 Let us now consider in $\mathcal{A}$ the ideals $J_{\varepsilon,w_1,w_2}[z_e^{-1}]$ and $Y_{\varepsilon,w_1,w_2}[z_e^{-1}]$. We have

Proposition. $\sqrt{Y_{\varepsilon,w_1,w_2}[z_e^{-1}]}=J_{\varepsilon,w_1,w_2}[z_e^{-1}]$.

Proof. By Lemma 4.7 and the definitions our proposition will follow from the following

Lemma. *Let S be an algebra with an automorphism ϕ, let $S_\phi[x^{\pm1}]$ be the associated twisted polynomial ring. Let I be an ideal in $S_\phi[x^{\pm1}]$ and $\sqrt{I}$ its radical. Then:*

$$S\cap\sqrt{I}=\sqrt{S\cap I}.$$

Proof. The fact that $S\cap\sqrt{I}\subset\sqrt{S\cap I}$ is clear. As for the reverse inclusion, remark that $I\cap S$ is ϕ stable and thus we may reduce modulo $I[x^{\pm1}]$ and assume that $S\cap I=0$. Thus let N be the nilpotent radical of S (clearly ϕ stable), we need to show that $N[x^{\pm1}]$ is nilpotent, but this is clear by the commutation relations between N and x. □

4.9 Let us denote by $\tilde{J}_{\varepsilon,w_1,w_2}$ the ideal $J_{\varepsilon,w_1,w_2}[z_e^{-1}]$ of the algebra $\mathcal{A}$. Also denote by D the subalgebra of $\mathcal{A}$ generated by the elements $\overline{F}_{\beta_i}\otimes 1,\ 1\otimes\overline{E}_{\delta_j},\ K_\lambda\otimes K_{-\lambda}$ where $\beta\in\Phi_{w_1}$, $\delta\in\Phi_{w_2}$ (in fact this algebra is independent of the choice of the reduced expressions). Notice that the algebra $F_\varepsilon[B^+_{w_1}]\otimes F_\varepsilon[B^-_{w_2}]$ of just the twisted Laurent polynomial algebra $D[u_1^{\pm1},\dots,u_n^{\pm1}]$. We have

Proposition. *The algebra D contains the elements $\gamma_\varepsilon(z_{w_1})$ and $\gamma_\varepsilon(\zeta_{w_2})$.*
If we restrict the quotient homomorphism

$$\pi:\mathcal{A}\to\mathcal{A}/\tilde{J}_{\varepsilon,w_1,w_2},$$

to D, we can extend π to a isomorphism between the algebras $D[\gamma_\varepsilon(z_{w_1}\zeta_{w_2})^{-1}]$ and $\mathcal{A}/\tilde{J}_{\varepsilon,w_1,w_2}[\gamma_\varepsilon(z_{w_1}\zeta_{w_2})^{-1}]$

Proof. We start with a general remark. Consider matrix coefficients of the form $c_{\phi,v_{-\lambda}} \in V^*_{-w_0\lambda} \otimes v_{-\lambda}$ (resp. $c_{v^\lambda,\phi} \in v^\lambda \otimes V_{-w_0\lambda}$). For such matrix coefficients one has

$$\gamma_\varepsilon(c_{\phi,v_{-\lambda}}) = c_{\phi,v_{-\lambda}} \otimes z_e^\lambda \qquad \gamma_\varepsilon(c_{\phi^\lambda,v}) = z_e^\lambda \otimes c_{\phi^\lambda,v}.$$

In particular

$$\gamma_\varepsilon(z_{w_1}) = z_{w_1} \otimes z_e^\lambda \qquad \gamma_\varepsilon(\zeta_{w_2}) = z_e^\lambda \otimes \zeta_{w_2}.$$

This clearly implies, by Lemma 3.2 that the elements $\gamma_\varepsilon(z_{w_1})$ and $\gamma_\varepsilon(\zeta_{w_2})$ lie in the algebra $D[u_1^{\pm 1},\dots,u_n^{\pm 1}]$. Since they also lie in $\mathcal{A}$ they must lie in D proving our first claim.

Let us now show the second part. It follows by the above remark that the algebras $F_\varepsilon[B^+]/J_{\varepsilon,w_1} \otimes F_\varepsilon[B^-]/J^-_{\varepsilon,w_2}[\gamma_\varepsilon(z_{w_1}\zeta_{w_2})^{-1}]$ and $F_\varepsilon[B^+_{w_1}] \otimes F_\varepsilon[B^-_{w_2}][\gamma_\varepsilon(z_{w_1}\zeta_{w_2})^{-1}]$ are just the algebras $F_\varepsilon[B^+]/J_{\varepsilon,w_1}[z_{w_1}^{-1}] \otimes F_\varepsilon[B^-]/J^-_{\varepsilon,w_2}[\zeta_{w_2}^{-1}]$ and $F_\varepsilon[B^+_{w_1}][z_{w_1}^{-1}] \otimes F_\varepsilon[B^-_{w_2}][\zeta_{w_2}^{-1}]$ respectively, which are isomorphic. It also follows, since the ideals J_{ε,w_1} and J^-_{ε,w_2} are generated by matrix coefficients of the form considered above that the ideal $J_{\varepsilon,w_1} \otimes F_\varepsilon[B^-] + F_\varepsilon[B^+] \otimes J^-_{\varepsilon,w_2}$ is generated by J_{ε,w_1,w_2} so that

$$F_\varepsilon[B^+]/J_{\varepsilon,w_1} \otimes F_\varepsilon[B^-]J^-_{\varepsilon,w_2} \cong \mathcal{A}/\tilde{J}_{\varepsilon,w_1,w_2}[u_1^{\pm 1},\dots,u_n^{\pm 1}].$$

Putting this together we obtain an isomorphism of twisted Laurent polynomial algebras between $D[\gamma_\varepsilon(z_{w_1}\zeta_{w_2})^{-1}][u_1^{\pm 1},\dots,u_n^{\pm 1}]$ and $\mathcal{A}/\tilde{J}_{\varepsilon,w_1,w_2}[\gamma_\varepsilon(z_{w_1}\zeta_{w_2})^{-1}][u_1^{\pm 1},\dots,u_n^{\pm 1}]$ which coincides with π on D. This clearly implies our proposition. □

4.10

Proposition. *The algebra $F_\varepsilon[G]/\sqrt{Y_{\varepsilon,w_1,w_2}}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$ is an Azumaya algebra of degree $\ell^{1/2(\ell(w_1)+\ell(w_2)+rk(w_1-w_2))}$ over its center.*

Proof. By Proposition 4.4 our algebra is an algebra over the coordinate ring of X_{w_1,w_2}. Moreover by Theorem 1.6 the semisimple parts of the algebras obtained by evaluation of C at the points of X_{w_1,w_2} are all isomorphic.

On the other hand if we invert z_e we obtain, by 4.8 the algebra $\mathcal{A}/\tilde{J}_{\varepsilon,w_1,w_2}$ which by 4.7 is a domain. Since a domain is generically an Azumaya algebra (of degree equal to the degree of the domain), it follows that all irreducible representations of C have the same dimension and hence, since by construction C is semiprime it follows by Artin's theorem and the Nullstellensatz (cf. [Pr]) that C is Azumaya (cf 1.8).

It remains to determine its degree. Let $\mathbb{Z}' := \mathbb{Z}[\frac{1}{2d}]$ where d is the product of the bad primes for our root system. For $w \in W$ let $\Phi_w = \{\beta_1,\beta_2,\dots,\beta_k\}$ be ordered by the convex ordering associated to a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_k}$. Consider the free $\mathbb{Z}'$ module V_w with basis $u_1,u_2,\dots u_k$ and define an antisymmetric bilinear form on V_w by setting

$$a_{i,j} = \langle u_i.u_j\rangle = (\beta_i|\beta_j) \text{ if } i < j.$$

We identify V_w with its dual V_w^* using the given basis. Then the matrix $A_w = (a_{i,j})$ can be thought, as a linear operator from V_w to itself. Also we can consider the matrix $C_w = ((\omega_i|\beta_j))$ as the matrix of a linear map from the module V_w with the basis $u_1, \dots, u_k$ to the module $Q^{\vee'} = Q^\vee \otimes_{\mathbb{Z}} \mathbb{Z}'$ with the basis $\alpha_1^\vee, \dots, \alpha_n^\vee$.

Take now the following antisymmetric matrix

$$S = \begin{pmatrix} A_{w_1} & 0 & -{}^tC_{w_1} \\ 0 & -A_{w_2} & {}^tC_{w_2} \\ C_{w_1} & -C_{w_2} & 0 \end{pmatrix},$$

We shall consider S as a linear map $V_{w_1} \oplus V_{w_2} \oplus P' \to V_{w_1} \oplus V_{w_2} \oplus Q'$ where $P' = P \otimes_{\mathbb{Z}} \mathbb{Z}'$.

By 4.9 the degree of $F_\varepsilon[G]/\sqrt{Y_{\varepsilon,w_1,w_2}}[z_{w_1}^{-1}, \zeta_{w_2}^{-1}]$ equals the degree of the algebra D considered in 4.9 which by [D-K-P] (see also [D-L]) equals the square root of the cardinality of the image of S composed with reduction modℓ. Thus in order to show our claim it suffices to see that $rk\ S = \ell(w_1) + \ell(w_2) + rk(w_1 - w_2)$ and that the image of S is a direct summand.

To see this let us recall that in [D-K-P3] it is proven that the matrix $\begin{pmatrix} A_w & -{}^tC_w \end{pmatrix}$ gives a surjective linear map whose kernel is the $\mathbb{Z}'$ module spanned by the elements $(v_{\omega_i}, (1+w)\omega_i)$, with $v_{\omega_i} = \sum_{r|i_r=i} u_r$ which can be identified with P'. Furthermore under this identification the restriction of C_w to the above kernel is just $1-w$ whose image is a direct summand.

It follows that we can identify the kernel of $\begin{pmatrix} A_{w_1} & 0 & -{}^tC_{w_1} \end{pmatrix}$ with $V_{w_2} \oplus P'$. The restriction of the linear map associated to the matrix

$$\begin{pmatrix} 0 & -A_{w_2} & {}^tC_{w_2} \\ C_{w_1} & -C_{w_2} & 0 \end{pmatrix}$$

to the kernel of $\begin{pmatrix} A_{w_1} & 0 & -{}^tC_{w_1} \end{pmatrix}$ under the previous identification is a linear map $V_{w_2} \oplus P' \to V_{w_2} \oplus Q^{\vee'}$ given by the matrix.

$$T = \begin{pmatrix} -A_{w_2} & {}^tC_{w_2}(1+w_1) \\ -C_{w_2} & 1 - w_1 \end{pmatrix}$$

We need to show that the image of T is a direct summand and its rank equals $l(w_2) + rk(w_1 - w_2)$. This can as well be proven for

$${}^tT = \begin{pmatrix} A_{w_2} & -{}^tC_{w_2} \\ (1+w_1^{-1})C_{w_2} & 1 - w_1^{-1} \end{pmatrix}.$$

Now we know, again by the above result applied this time for w_2, that $\begin{pmatrix} A_{w_2} & -{}^tC_{w_2} \end{pmatrix}$ is surjective, its kernel can be identified with P' and the restriction of the linear map given by $\begin{pmatrix} (1+w_1^{-1})C_{w_2} & 1 - w_1^{-1} \end{pmatrix}$ to this kernel is given by $(1+w_1^{-1})(1 - w_2) + (1 - w_1^{-1})(1 + w_2) = 2(1 - w_1^{-1}w_2)$. Since 2 is invertible in $\mathbb{Z}'$ we deduce that its image is a direct summand whose rank equals $rk(w_1 - w_2)$ as desired. $\square$

Remark. The computation of the degree given a above is contained in [D-K-P3] in the case in which $w_2 = id$ and in fact we use that proof heavily in the above argument. The general case is claimed in [D-L] but the proof given there is incomplete giving only the computation of the rank of the matrix S. We are grateful to M. Costantini for pointing out the gap in [D-L] and providing us with the above argument.

We are now ready to prove the main result of our paper.

Theorem. *1) The algebras $F_\varepsilon[G]/\sqrt{Y_{\varepsilon,w_1,w_2}}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$ and $F_\varepsilon[G]/J_{\varepsilon,w_1,w_2}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$ are isomorphic.*

2) The spectrum of $F_\varepsilon[G]/J_{\varepsilon,w_1,w_2}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$ is a Galois covering of X_{w_1,w_2} with Galois group the group $T_{w_2^{-1}w_1,\ell}$ (see 4.5 for the definition).

Proof. Since by Lemma 4.7 the algebra $F_\varepsilon[G]/J_{\varepsilon,w_1,w_2}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$ is a domain, it is a quotient of $F_\varepsilon[G]/\sqrt{Y_{\varepsilon,w_1,w_2}}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$ so in particular it is a Azumaya algebra. Thus the spectrum of these algebras equals the spectrum of their centers and to show their isomorphism it suffices to show that the spectra of their centers are isomorphic as algebraic varieties. Call these spectra S_1 and S_2 respectively. Both S_1 and S_2 project to X_{w_1,w_2} and we have a commutative diagram

$$\begin{array}{ccc} S_2 & \xrightarrow{\ i\ } & S_1 \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle r}\big\downarrow \\ X_{w_1,w_2} & \xrightarrow{\ id\ } & X_{w_1,w_2}. \end{array}$$

i injects S_2 as an irreducible Zariski closed subvariety of S_1. The maps p and r are finite, and the cardinality of the fibers of r is constant.

Furthermore, since after inverting z_e our two algebras become isomorphic, i is an isomorphism outside the divisor defined by the element Z_e^ℓ. It follows from these considerations that, if we prove that also the cardinality of the fibers of p is constant we shall get the isomorphism of S_2 and S_1.

Notice now that using 4.9 we have that the rank of $F_\varepsilon[G]/J_{\varepsilon,w_1,w_2}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$ as a module over $\mathbb{C}[X_{w_1,w_2}] = Z_0/Y_{w_1,w_2}[z_{w_1}^{-\ell},\zeta_{w_2}^{-\ell}]$ equals the rank of the algebra D as a module over the subalgebra generated by the elements $\overline{F}_{\beta_i}^{\ell}\otimes 1,\ 1\otimes\overline{E}_{\delta_j}^{\ell},\ K_\lambda^\ell\otimes K_{-\lambda}^\ell$ where $\beta\in\Phi_{w_1}$ (resp $\delta\in\Phi_{w_2}$) so it equals $\ell^{\ell(w_1)+\ell(w_2)+n}$. It follows that the map p has degree $\ell^{n-rk(w_1-w_2)} = \ell^{|T_{w_2^{-1}w_1\ell}|}$.

Now, for $\lambda\in P^{w_1^{-1}w_2}$ consider the elements

$$d_\lambda := d_{\lambda,w_1,w_2} := z_{w_1}^{\lambda}\zeta_{w_2}^{w_1(\lambda)}$$

One checks exactly as in 4.5 that these elements are central in $F_\varepsilon[G]/J_{\varepsilon,w_1,w_2}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$. Also, we can repeat word by word the argument in 4.5 and show that the center of $F_\varepsilon[G]/J_{\varepsilon,w_1,w_2}[z_{w_1}^{-1},\zeta_{w_2}^{-1}]$ is generated by $\mathbb{C}[X_{w_1,w_2}]$ and these elements and that S_2 is a Galois covering of X_{w_1,w_2} with Galois group $T_{w_2^{-1}w_1,\ell}$. This in particular implies that the cardinality of the fibers of p is constant and completely proves the theorem. □

Definition. We shall denote by $F_\varepsilon[G]_{w_1,w_2}$ the algebra

$$F_\varepsilon[G]/\sqrt{Y_{\varepsilon,w_1,w_2}}[z_{w_1}^{-1},\zeta_{w_2}^{-1}] \cong F_\varepsilon[G]/J_{\varepsilon,w_1,w_2}[z_{w_1}^{-1},\zeta_{w_2}^{-1}].$$

From the previous proof it follows that, if we denote by Z_{1,w_1,w_2} the algebra generated by the elements $d_\lambda,\ \lambda\in P^{w_1^{-1}w_2}$, the intersection $\mathbb{C}[X_{w_1,w_2}]\cap Z_{1,w_1.w_2}$ equals the algebra generated by the elements $d_{\ell\lambda},\ \lambda\in P^{w_1^{-1}w_2}$. The center of $F_\varepsilon[G]_{w_1,w_2}$ is

$$\mathbb{C}[X_{w_1,w_2}]\otimes_{\mathbb{C}[X_{w_1,w_2}]\cap Z_{1,w_1,w_2}} Z_{1,w_1,w_2}.$$

Let us denote by S_{w_1,w_2} its spectrum. By construction we have a cartesian square

$$\begin{array}{ccc} S_{w_1,w_2} & \xrightarrow{\kappa} & T_{w_2^{-1}w_1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \ell\text{-th power}} \\ X_{w_1,w_2} & \xrightarrow{\tilde{\kappa}} & T_{w_2^{-1}w_1} \end{array} \tag{4.10.1}$$

where all the maps in the diagram are induced by inclusion.

Remark. The torus T acts on each of the varieties in the above commutative diagram. We have $\kappa(tx) = t^2\kappa(x)$ and similarly for $\tilde{\kappa}$. The reader may wish to substitute $T_{w_2^{-1}w_1}$ with its quotient module elements of 2-torsion to get that they are equivariant maps.

4.11 We want to summarize the results of the previous section with more appealing notations. Let us denote the spectrum of $F_\varepsilon[G]$, i.e. the set of equivalence classes of irreducible representations by G_ε and by $K_\varepsilon(G)$ the Grothendieck ring of finite dimensional representations of $F_\varepsilon[G]$, which has as basis the set G_ε. The Frobenius map induces a projection map (central character) and a ring homomorphism:

$$\mathcal{F}^* : G_\varepsilon \to G;\ K_\varepsilon(G) \to K(G).$$

We have then compatible decompositions $G = \cup_{w_1,w_2} X_{w_1,w_2}$, $G_\varepsilon = \cup S_{w_1.w_2}$ so that the diagram:

$$\begin{array}{ccc} S_{w_1,w_2} & \xrightarrow{i} & G_\varepsilon \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathcal{F}^*} \\ X_{w_1,w_2} & \xrightarrow{i} & G \end{array} \tag{4.11.1}$$

commutes (i denotes inclusion).

§5 A semisimplicity property and the construction of irreducible modules.

5.1 In this last section we will discuss the problem of construction of irreducible representations from basic ones. So first of all let us discuss the part $S_{s_i,e}$ of the spectrum with s_i a simple reflection (a similar discussion applies to S_{e,s_i}).

We have seen in 2.4 that for a simple reflection the ideal J_{s_i} is generated by the elements $F_j,\ j \neq i$. The algebra $F_q[B^+]/J_{s_i}$ is isomorphic to the subalgebra of $U_q^s(\mathfrak{b}^-)$ generated by $F_i, K_\lambda;\ \lambda \in P$. On the other hand the computation of 2.3 and the analogue of formula 3.2.1 imply that the image z_i of z_{s_i} in $F_\varepsilon[B^+]$ is $-q^{2d_i}\overline{F}_i K_\varrho$ it follows that the algebra $F_\varepsilon[G]_{s_i,e}$ is canonically isomorphic to the twisted Laurent polynomials $\mathbb{C}[P][F_i, F_i^{-1}]$ where $\mathbb{C}[P]$ is the group ring of the weight lattice and the automorphism is of course induced by the commutation relations $F_i K_\lambda F_i^{-1} = \varepsilon^{(\alpha_i,\lambda)} K_\lambda$. In other words this is the ordinary Laurent polynomial ring in the variables K_{ω_j}, $j \neq i$ over the quantum torus $\mathbb{C}[F_i^{\pm 1}, K_{\omega_i}^{\pm 1}]$ with commutation rtelation $F_i K_{\omega_i} = \varepsilon^{d_i} K_{\omega_i} F_i$ and $\eta = \varepsilon^{d_i}$ is also a primitive ℓ root of 1 since d_i is prime with ℓ by assumption.

Now the irreducible representations of a quantum torus $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, $xy = \eta yx$, which is an Azumaya algebra of rank ℓ, are well known and easy to describe. One fixes two non zero numbers α, β and a vector space with basis vectors $v_1, \dots, v_\ell$ and sets $xv_i = \eta^i \alpha v_i$, $yv_i = \beta v_{i+1}$ $(\ell + 1 = 1)$. Two pairs α, β, α', β' give rise to isomorphic representations if and only if $\alpha = \varepsilon_1 \alpha'$, $\beta = \varepsilon_2 \beta'$ with $\varepsilon_1, \varepsilon_2$ both ℓ roots of 1. The spectrum of this quantum torus is thus the torus of coordinates α^ℓ, β^ℓ which are values of the central character on the two generators x^ℓ, y^ℓ of the center of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$.

5.2 We start working at generic q. We fix two elements $w_1, w_2 \in W$ and we define $J_{w_1,w_2} \subset F_q[G]$ to be the ideal $\gamma^{-1}(J_{w_1} \otimes F_q[B^-] + F_q[B^-] \otimes J_{w_2})$. Take now a simple reflection s_i such that $\ell(s_i w_1) > \ell(w_1)$. Consider the homomorphism

$$\psi : F_q[G] \to F_q[G]/J_{s_i,e} \otimes F_q[G]/J_{w_1,w_2}$$

defined as the composition $(f \otimes g) \cdot \Delta$ where $f : F_q[G] \to F_q[G]/J_{s_i,e}$ (resp. $g : F_q[G] \to F_q[G]/J_{w_1,w_2}$) is the quotient homomorphism.

Lemma. *Let $\lambda \in P^+$. Then:*
1) $\psi(z^\lambda_{s_i w_1}) = z_{s_i}^{w_1(\lambda)} \otimes z^\lambda_{w_1}$.
2) $\psi(\zeta^\lambda_{w_2}) = \zeta^\lambda_e \otimes \zeta^\lambda_{w_2}$.

Proof. 1) Consider in $U_q(\mathfrak{g})$ the subalgebra $\mathcal{L}_i$ generated by E_1, F_i and by the $K_j^{\pm 1}$'s with $j = 1, \dots, n$. It follows from Proposition 2.4 and the definitions that the ideal J_{s_i,s_i} is the kernel of the restriction of elements of $U_q(\mathfrak{g})$ to $\mathcal{L}_i$. Since $J_{s_i,e} \supset J_{s_i,s_i}$, it suffices to show that $\psi'(z^\lambda_{s_i w_1}) = z_{s_i}^{w_1(\lambda)} \otimes z^\lambda_{w_1}$ where

$$\psi' : F_q[G] \to F_q[G]/J_{s_i,s_i} \otimes F_q[G]/J_{w_1,w_2}$$

defined as the composition $(f' \otimes g) \cdot \Delta$, $f' : F_q[G] \to F_q[G]/J_{s_i,s_i}$ being the quotient homomorphism.

For this restrict the representation $V_{-w_0(\lambda)}$ to the subalgebra $\mathcal{L}_i$. It is then easy to see that $E_i v_{-s_i w_1(\lambda)} = 0$ and $K_j v_{-s_i w_1(\lambda)} = q^{-(\alpha_j, s_i w_1(\lambda))} v_{-s_i w_1(\lambda)}$. Set $m = -\frac{(\alpha_i, s_i w_1(\lambda))}{d_i} = \frac{(\alpha_i, s_i w_1(\lambda))}{d_i} \geq 0$. We get that $v_{-s_i w_1(\lambda)}$ is a highest weight vector for a $m+1$ dimensional irreducible $\mathcal{L}_i$-module for which $v_{-w_1(\lambda)}$ is a lowest weight vector. Take the basis $v_0, \dots, v_m$ with $v_j = F^j v_{-s_i w_1(\lambda)}$ for this module and complete it to a basis $v_0, \dots, v_M$ for $V_{-w_0(\lambda)}$), in such a way that the remaining vector also span a $\mathcal{L}_i$ submodule, we clearly have that the matrix cofficient $c_{\phi^{s_i w_1(\lambda)}, v_j} \in J_{s_i,e}$ unless $j \leq m$. Consider now the dual basis $\phi_0, \dots, \phi_M$ to $v_0, \dots, v_M$ and notice that, up to scalars ϕ_m, v_m are multiples of $\phi^{w_1(\lambda)}, v_{-w_1(\lambda)}$ so we renormalize them and assume $\phi_m = \phi^{w_1(\lambda)}$. Also if $j < m$, then $c_{\phi_j, v_{-\lambda}} \in J_{w_1,w_2}$. Since $\Delta(c_{\phi^{s_i w_1(\lambda)}, v_{-\lambda}}) = \sum_{j=1}^M c_{\phi^{s_i w_1(\lambda)}, v_j} \otimes c_{\phi_j, v_{-\lambda}}$ it follows that

$$\psi'(z^\lambda_{s_i w_1}) = \psi'(c_{\phi^{s_i w_1(\lambda)}, v_m} \otimes c_{\phi_m, v_{-\lambda}}) = c_{\phi^{s_i w_1(\lambda)}, v_m} \otimes z^\lambda_{w_1}$$

It follows that in order to show our claim we need to show that $f'(c_{\phi^{s_i w_1(\lambda)}, v_m}) = z_{s_i}^{w_1(\lambda)}$ (notice that this makes sense since if $\mu \in P^+$ is orthogonal to α_i, $z_{s_i}^{\mu} = z_e^{\mu}$ is invertible in $F_q[G]/J_{s_i,s_i}$). It is clear from weight considerations that $f'(c_{\phi^{s_i w_1(\lambda)}, v_m}) = a z_{s_i}^{w_1(\lambda)}$ for some non zero scalar a, we only need to show that this scalar is 1. Now f' is t_i equivariant so that we get $f'(c_{\phi^{s_i w_1(\lambda)}, v_m}) = (t_i, e)(f'(c_{\phi^{w_1(\lambda)}, v_m})$. Since $\phi^{s_i w_1(\lambda)}(v_m) = 1$, we obtain that $f'(c_{\phi^{w_1(\lambda)}, v_m}) = z_e^{w_1(\lambda)}$ (they have the same value on the identity element). This clearly implies our claim.

The easy part 2) is left to the reader. $\square$

We now specialize to ε, our ℓ root of 1. Notice that, since ψ preserves the integral forms, it induces a map

$$\psi_\varepsilon : F_\varepsilon[G] \to F_\varepsilon[G]/J_{\varepsilon, s_i, e} \otimes F_\varepsilon[G]/J_{\varepsilon, w_1, w_2}$$

and the conclusion of our lemma holds after specialization.

For $q = 1$ we get that this map is induced by the multiplication in G. By the Bruhat decomposition the product gives us a morphism:

$$X_{s_i, e} \times X_{w_1, w_2} \to X_{s_i w_1, w_2} \tag{5.2.1}$$

and $X_{s_i,e} X_{w_1,w_2}$ is a dense subset in $X_{s_i w_1, w_2}$. It follows that the map ψ_1 factors through $\mathbb{C}[G]/Y_{s_i w_1, w_2}$ and localizes to give a map

$$\mathbb{C}[G]/Y_{s_i w_1, w_2}[z_{s_i w_1}^{-1}, \zeta_{w_2}^{-1}] \to \mathbb{C}[G]/Y_{s_i, e}[z_{s_i}^{-1}] \otimes \mathbb{C}[G]/Y_{w_1, w_2}[z_{w_1}^{-1}, \zeta_{w_2}^{-1}].$$

Using the compatibility of the Frobenius homomorphism (cf 1.6) and Proposition 1.4, we have that the map ψ_ε factors first through the ideal $Y_{\varepsilon, s_i w_1 . w_2}$, and also through its radical since the algebra $F_\varepsilon[G]/J_{\varepsilon, s_i, e} \otimes F_\varepsilon[G]/J_{\varepsilon, w_1, w_2}$ is contained in the tensor product of twisted Laurent polynomial rings and so it is a domain.

By Theorem 4.10, we can the localize getting a map

$$F_\varepsilon[G]/J_{\varepsilon, s_i w_1, w_2}[z_{s_i w_1}^{-1}, \zeta_{w_2}^{-1}] \to F_\varepsilon[G]/J_{\varepsilon, s_i, e}[z_{s_i}^{-1}] \otimes F_\varepsilon[G]/J_{\varepsilon, w_1, w_2}[z_{w_1}^{-1}, \zeta_{w_2}^{-1}]$$

which by the way it has been constructed is injective.

5.3 We will need a simple geometrical fact. If v is a non zero vector in a Euclidean space V let us denote by r_v the orthogonal reflection relative to the hyperplane orthogonal to v. The map $1 - r_v$ has image $\mathbb{R}v$. Let $v_1, v_2, \dots, v_k \in V$ be non zero vectors.

Lemma. *The image of* $1 - r_{v_1} r_{v_2} \dots r_{v_k}$ *is contained in the span of* $v_1, \dots, v_k$ *its kernel contains the orthogonal to the the span of* $v_1, \dots, v_k$.

Its rank is k *iff the vectors are independent, in this case the kernel equals the orthogonal of the span of* $v_1, \dots, v_k$.

Proof. It is a simple induction from the formula $1 - r_{v_1} r_{v_2} \dots r_{v_k} = 1 - r_{v_2} \dots r_{v_k} + (1 - r_{v_1}) r_{v_2} \dots r_{v_k}$. $\square$

Let $A = \mathbb{Z}[d^{-1}]$ where d is the product of the bad primes, then under the canonical scalar product P_A, Q_A are in perfect duality.

We need two facts from the theory of root systems. First, if w is an element of the Weyl group we can write $w = s_{\beta_1}s_{\beta_2}\dots s_{\beta_k}$ for independent roots β_i (so that from the previous lemma P^w is the orthogonal to these roots).

The second fact is that any maximal independent sets of roots spans a lattice in Q which intersects the root system in a new root system. Choose a basis for this new root system, then this basis can be obtained from the basis Δ of Q by a sequence of steps of the following type. Replace in Δ one of the simple roots with the negative of the longest root take the root system with this set as basis and continue. From the definition of bad primes and this fact it follows that any maximal set of linearly independent roots is a basis over A of Q_A

We need some degree computations. Notice first that $w \in W$ and s a simple reflection, $rk(sw-e) = rk((s-e)w + w - e)$ so

$$rk(w-e) - 1 \leq rk(sw-e) \leq rk(w-e) + 1.$$

Since $\ell(sw) + rk(sw-e)$, $\ell(w) + rk(w-e)$ are both even it follows that $rk(sw-e) = rk(w-e) \pm 1$.

Changing w with sw if necessary we are reduced to discuss the following case.

Proposition. *Let α be the simple root associated to s. If $rk(sw-e) = rk(w-e) - 1$ then:*

$$P^w = \{x \in P^{sw} | (\alpha, x) = 0\}, \quad \alpha(P^{sw}) + \ell\mathbb{Z} = \mathbb{Z}.$$

Proof. Write $s_iw = s_{\beta_1}s_{\beta_2}\dots s_{\beta_k}$ for independent roots β_i. From the previous lemma applied to $w = s_is_iw$ we have that α_i is independent from the β_j, P^w is the orthogonal to the roots $\alpha_i, \beta_1, \beta_2, \dots, \beta_k$ while P^{sw} is the orthogonal to the roots $\beta_1, \beta_2, \dots, \beta_k$. The elements $\alpha_i, \beta_1, \beta_2, \dots, \beta_k$ are part of an A basis so the functions they induce on P_A by scalar product are part of a system of linear coordinates and this clearly proves the proposition. $\square$

Take w_1, w_2, s_i with $\ell(s_iw_1) = \ell(w_1) + 1$. We have two cases (notice that $P^{{w_1}^{-1}w_2} = w_2^{-1}P^{w_1{w_2}^{-1}}$).

Case 1) $rk(s_iw_1 - w_2) = rk(w_1 - w_2) + 1$. In this case $P^{{w_1}^{-1}w_2} \cap w_2^{-1}P^{s_i} = P^{(s_iw_1)^{-1}w_2}$. This implies that we have a surjective homomorphism:

$$T_{{w_1}^{-1}w_2} \times T_{s_i} \to T_{(s_iw_1)^{-1}w_2}.$$

Using the cartesian square 4.10.1 and the map 5.2.1 we obtain a morphism:

$$j : S_{s_i,e} \times S_{w_1,w_2} \to S_{s_iw_1,w_2}. \tag{5.3.1}$$

Take $(p, q) \in S_{s_i,e} \times S_{w_1,w_2}$ and V_p, V_q the corresponding irreducible representations of $F_\varepsilon[G]$. By Lemma 5.2 $V_p \otimes V_q$ has central character $p \cdot q := j(p, q)$. This point corresponds to an irreducible representation of the same dimension as $V_p \otimes V_q$ hence $V_p \otimes V_q = V_{p \cdot q}$.

Case 2) $rk(s_iw_1 - w_2) = rk(w_1 - w_2) - 1$. In this case $P^{(s_iw_1)^{-1}w_2} \cap w_2^{-1}P^{s_i} = P^{{w_1}^{-1}w_2}$. Consider the lattice $P^{{w_1}^{-1}w_2} + \ell P^{(s_iw_1)^{-1}w_2}$ which is the kernel of the surjective map $\alpha_i : P^{(s_i w_1)^{-1} w_2} \to \mathbb{Z}/(\ell)$. To this map corresponds a covering $\nu : S_{s_i w_1, w_2} \to \tilde{S}_{s_i w_1, w_2}$.

Reasoning as above we obtain a morphism:

$$(5.3.2) \qquad j : S_{s_i,e} \times S_{w_1,w_2} \to \tilde{S}_{s_i w_1, w_2}.$$

Take $(p, q) \in S_{s_i,e} \times S_{w_1,w_2}$ and V_p, V_q the corresponding irreducible representations of $F_\varepsilon[G]$. By Lemma 5.2 the coordinate ring of $\tilde{S}_{s_i w_1,w_2}$ acts on $V_p \otimes V_q$ by the central character $p \cdot q := j(p, q)$. Now this point corresponds to ℓ irreducible representations of dimension $dim(V_p \otimes V_q)/\ell$ in the fiber of the covering map ν.

Claim $V_p \otimes V_q = V_{p \cdot q}$ decomposes as the direct sum of all these representations.

In order prove it we choose a character $\lambda \in P^{(s_i w_1)^{-1} w_2}$ with $(\alpha, \lambda) = 1\ mod(\ell)$. The function $d_\lambda = d_{\lambda, s_i w_1, w_2}$ takes ℓ different values on the fiber $\nu^{-1}(u)$ of a point $u \in \tilde{S}_{s_i w_1, w_2}$. Suppose we prove that d_λ has at least ℓ distinct eigenvalues on $V_p \otimes V_q$ then our claim is proved, in fact, chosen such an eigenvalue of d_λ, its eigenspace is an $F_\varepsilon[G]$ subrepresentation of $V_p \otimes V_q$ which factors as $F_\varepsilon[G]_{s_i w_1, w_2}$ module with central character the point $r \in \nu^{-1}(p \cdot q)$ determined by the chosen eigenvalue for d_λ. From 1.8 we know that this eigenspace decomposes as sum of irreducible representations isomorphic to V_r. Since $\ell\, dim(V_r) = dim(V_p \otimes V_q)$ the claim follows.

It remains to perform the computation of the eigenvalues. It is clearly enough to prove that on V_p the operator $z_{s_i}^{w_1(\lambda)} {\zeta^\lambda}_e$ has ℓ distinct eivenvalues. Recall that on V_p this operator acts (up to a non zero constant) as $F_i {K_\lambda}^2$ hence the claim follows from the remarks in 5.1 ($F_i {K_\lambda}^2, K_{\omega_i}$ can be taken as the two generators x, y of a quantum torus).

Summarizing take w_1, w_2, s_i with $\ell(s_i w_1) = \ell(w_1) + 1$. We have proved the following theorem.

Theorem. *We have two cases:*

1) $rk(s_i w_1 - w_2) = rk(w_1 - w_2) + 1$. We obtain a morphism $j : S_{s_i,e} \times S_{w_1,w_2} \to S_{s_i w_1, w_2}$ If $(p, q) \in S_{s_i,e} \times S_{w_1,w_2}$ and V_p, V_q the corresponding irreducible representations of $F_\varepsilon[G]$. $V_p \otimes V_q$ is isomorphic to the irreducible representation $V_{j(p,q)}$.

2) $rk(s_i w_1 - w_2) = rk(w_1 - w_2) - 1$. We obtain morphisms $j : S_{s_i,e} \times S_{w_1,w_2} \to \tilde{S}_{s_i w_1, w_2}$; $\nu : S_{s_i w_1, w_2} \to \tilde{S}_{s_i w_1, w_2}$. $V_p \otimes V_q = V_{p \cdot q}$ decomposes as the direct sum of all the irreducible representations in the fiber $\nu^{-1}(j(p, q))$.

Similarly for pairs (w_1, w_2), $(w_1, w_2 s_i)$ with s_i a simple reflection such that $\ell(w_2 s_i) = \ell(w_2) + 1$.

□

Remarks. 1) Under the previous hypotheses one can easily check that the multiplication map (5.2.1) is not in general surjective. The complement of its image appears to be a remarkable set which should be analyzed.

2) Set for a pair (w_1, w_2), $X^0_{w_1,w_2}$ equal to the subset in X_{w_1,w_2} consisting of elements which lie in the image of the iterated multiplication map

$$\prod_{t=1}^{h} X_{s_{i_t}, e} \times \prod_{r=1}^{k} X_{e, s_{j_r}} \to X_{w_1, w_2},$$

for some reduced expressions $w_1 = s_{i_1} \cdots s_{i_h}$, $w_2 = s_{j_1} \cdots s_{j_k}$. Then an easy induction based on the previous theorem shows that, if we take two pairs (w_1, w_2) and (w_1', w_2') such that $\ell(w_1 w_1') = \ell(w_1) + \ell(w_1')$ and $\ell(w_2 w_2') = \ell(w_2) + \ell(w_2')$ and two elemnts $g \in X^0_{w_1,w_2}$ and $g' \in X^0_{w_1',w_2'}$, the tensor product of two irreducible $F_\varepsilon[G]$-modules, one lying above g and the other above g' is always semisimple.

References

[A-J-S] H. H. Andersen, J. C. Jantzen, W. Soergel, *Representations of Quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p*, preprint (1992).

[A] M. Artin, *On Azumaya algebras and finite–dimensional representations of rings*, J. Algebra **11**, 532–563.

[A-G1] M. Auslander, O. Goldman, *Maximal Orders*, Trans. A.M.S. **97** (1960), 1–24.

[A-G2] M. Auslander, O. Goldman, *The Brauer Group of a Commutative Ring*, Trans. A.M.S. **97** (1960), 367–409.

[Bo] N. Bourbaki, *Groupes et Algebre de Lie Ch 4,5,6*, Hermann, Paris, 1980.

[D-K-P] C. De Concini, V. G. Kac, C. Procesi, *Quantum coadjoint action*, Journal of AMS, **5** (1992), 151–190.

[D-K-P2] C. De Concini, V. G. Kac, C. Procesi, *Some remarkable degenerations of quantum groups*, Comm. Math. Phys., 1993.

[D-K-P3] C. De Concini, V. G. Kac, C. Procesi, *Some quantum analogues of solvable groups*, preprint, 1992.

[D-L] C. De Concini, V. Lyubashenko, *Quantum coordinate ring at roots of unity*, Adv. in Math. (to appear).

[D-P] C. De Concini, C. Procesi, *Quantum Groups*, Springer LNM 1565, 31–140.

[De] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. scient. Éc. Norm. Sup., 4^e série, t. 7, (1974) 53–88.

[D] V.G. Drinfeld, *Quantum groups*, Proc. ICM Berkeley 1 (1986), 789–820.

[Ji] M. Jimbo, *A q-difference analogue of $U(q)$ and the Yang-Baxter equation*, Lett. Math. Phys. 10 (1985),63-69..

[J] A. Joseph, *On the Prime and Primitive spectra of the Algebra of Functions on a Quantum Group*, preprint, 1992.

[H-L] T. Hodges, T. Levasseur, *Primitive Ideals of $\mathbb{C}_q[G]$*, preprint, 1993.

[L-R] V. Lakshmibai, N. Reshetikhin, *Quantum deformations of Flag and Schubert schemes*,.

[L-S] S. Z. Levendorskii, Ya. S. Soibelman, *Algebras of functions on compact quantum groups*, Schubert cells and quantum tori, Comm. Math. Physics 139 (1991), 141–170.

[L-S1] S. Z. Levendorskii and Ya. S. Soibelman, *Quantum Weyl group and multiplicative formula for the R-matrix of a simple Lie algebra*, Funct. Analysis and its Appl. **25** (1991), no.2, 143–145.

[L] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. in Math. **70** (1988), 237–249.

[L] G. Lusztig, *Quantum groups at roots of* 1, Geom. Ded. **35** (1990), 89–114.

[M] M. Matsumoto, *Generateurs et relations de groupes de Weyl generalises*, C.R. Acad. Sci. Paris **258** (1964), 3419–3422.

[R-T-F] Reshetikhin, Takhtadzyan, Faddeev, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J.1 (1990), 193–323..

[Ri] C. Ringel, *Hall algebras and quantum groups*, Inv. Math. 101 (1990), 583–592.

[Ro] M. Rosso, *Finite dimensional representations of the quantum analogue of the enveloping algebra of a semisimple Lie algebra*, Comm. Math. Phys. **117** (1988), 581–593.

[Sc] W. Schelter, *Azumaya algebras and Artin's theorem*, Jour. of Algebra **46** (1977), 303–304.

[S] Ya. S. Soibelman, *The algebra of functions on a compact quantum group, and its representations*, Leningrad Math. J. **2** (1991), 161–178.

[S2] Ya. S. Soibelman, *Quantum Weyl group and some of its applications*, Rend. Circolo Mat. Pal. s.II,**26** (1991), 232–235.

[Ta] T. Tanisaki, *Killing forms, Harish–Chandra isomorphisms, and universal R–matrices for quantum algebras*, Infinite Analysis Part A, Adv. Series in Math Phys. **16** (1992), 942–962.

[Ti] J. Tits, *Sur les constants de structure et le théorème d'existence des algebres de Lie semi–simple*, Publ. Math. IHES **31** (1966), 21–58.

Scuola Normale Superiore.
E-mail address: deconcin@ux1sns.sns.it

Università di Roma "La Sapienza".
E-mail address: Procesi@sci.uniroma1.it