THE TORIC VARIETY ASSOCIATED TO WEYL CHAMBERS

BY

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SUMMARY

We study the action of the Weyl group on the cohomology of the toric variety associated to the decomposition of Weyl chambers.

Introduction

Let V be a finite dimensional real Euclidean space, $\Phi \subset V$ a root system, W its Weyl group. In V we have the lattice $\Lambda = \{v \in V \mid (v, \alpha) \in Z, \quad \forall a \in \Phi\}$ which defines an integral structure and we can consider the rational polyhedral decomposition of V given by Weyl chambers, *i.e.* for each set Δ of simple roots in Φ we consider the cone $C_\Delta = \{v \in V \mid (v, \alpha) > 0 \quad \forall \alpha \in \Delta\}$ and its faces.

According to the combinatorial theory of toric varieties this decomposition defines a smooth projective torus embedding $\overline{T}$ of the torus T having as character group the lattice spanned by Φ.

In the cases of the root system of a semisingle Lie algebra or of the restricted roots of a symmetric variety, T is a maximal split torus of the adjoint group and the variety $\overline{T}$ plays a major role in the description of the "minimal wonderful compactification" (*cf.* [5]).

The purpose of this paper is to deduce a formula for the characters of the representations of W on the cohomology groups of $\overline{T}$. The interest for this motivation comes from the results of [2], [3] on the cohomology of complete symmetric varieties.

We give a general formula in paragraph 2 (THEOREM 2) and also a simpler computation for type A_n (following a suggestion of DE CONCINI) in paragraph 3.

We illustrate the effectiveness of this method computing explicitely for $n \leq 5$.

This work was essentially done during the conference on the symmetric group held in Durham in the summer 85 and it is the result of very useful discussions had with C. DE CONCINI, A. GARSIA, P. HANLON, R. STANLEY during this meeting.

1. Toric varieties

We recall briefly the method used by Danilov to compute the cohomology ring of toric varieties [1].

Let T be a torus, $\check{T}$ its character group, $V = \hom_Z(\check{T}, R)$. Let us also give a smooth torus embedding $\overline{T}$ and its associated rational polyhedral decomposition of V. Consider next the set of integral vectors $\{v_\alpha\}$ generating the 1 dimensional cones of the decomposition.

To each such vector v_α is associated a divisor D_α in $\overline{T}$ and we let $[v_\alpha] \in H^2(\overline{T}, Q)$ be the class dual to D_α.

Consider next the polynomial ring $A = Q[x_\alpha]$ in variables x_α corresponding to the vectors v_α.

In A we condiser two sets of polynomials :

I) The monomials $M_S = x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_k}$ for each set

$$S = \{v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_k}\}$$

of vectors which *do not* generate a cone in the given r.p.d. .

II) The linear forms

$$R_\chi = \sum_\alpha \chi(v_\alpha) x_\alpha$$

where χ varies in the character group $\check{T}$.

The cohomology of $\check{T}$ is given by the following THEOREM 1 (DANILOV). The map $Q[x_\alpha] \to H^*(\overline{T})$ given by $x_\alpha \to [v_\alpha]$ induces an isomorphism

$$Q[x_\alpha]/I \simeq H^*(\overline{T})$$

where I is the ideal generated by all the relations of type I) and II).

In fact the theorem is more precise.

Let $\overline{A}$ be the quotient of $Q[x_\alpha]$ modulo the relations of type I) only.

$\overline{A}$ is a Reisner-Stanley algebra associated to a simplicial decomposition of a sphere. So $\overline{A}$ is Cohen Macaulay and it has a canonical basis formed by the monomials with support in a cone of the given decomposition.

Furthermore if $\chi_1, \chi_2, \ldots, \chi_m$ are a basis of $\check{T}$ the elements $R_{\chi_1}, R_{\chi_2}, \ldots, R_{\chi_m}$ are a regular sequence in $\overline{A}$.

2. The Basic toric varieties

We now let $\overline{T}$ be the torus embedding associated to the decomposition into Weyl chambers. We apply all the results of the previous section keeping also the same notations. The computation follows the following steps :

a) Let B denote the subalgebra of $\overline{A}$ generated by the elements R_χ (or by $R_{\chi_1}, \ldots, R_{\chi_m}$). Since these elements form a regular sequence B is isomorphic to the symmetric algebra $S[U]$ on the vector space spanned by the $R'_\chi s$. Such a space is W isomorphic to $\check{T} \otimes_Z Q$.

b) $\overline{A}$ is a free B module. A basis of $\overline{A}$ over B is obtained liftig a Q basis of $\overline{A}/B^+\overline{A}$ (B^+ the augmentation ideal of B). Since W acts in a semisimple way we can construct a W representation M in $\overline{A}$ which maps isomorphically to $\overline{A}/B^+\overline{A}$. Therefore the multiplication $M \cdot B$ induces a W isomorphism $\overline{A} = M \otimes_Q B$.

c) As a graded vector space $H^*(\overline{T}, Q) \cong \overline{A}/B^+\overline{A} \cong M$. Let us now introduce a notation. If $N = \bigoplus N_i$ is a graded vector space define $P_N(t) = \sum \dim(N_i)t^i$ its Poincaré series. If each N_i is a representation of a group W we set

$$R_N(t) = \sum [N_i] t^i$$

where $[N_i]$ is the class of N_i in the character ring of W. The series $R_N(t)$ has his cœfficients in this ring.

The previous discussion gives immediately

LEMMA.

$$R_{\overline{A}}(t) = R_{H^\bullet(\overline{T},Q)}(t) \cdot R_B(t).$$

Proof. — From the formulas $\overline{A} \simeq M \otimes_Q B$

$$M \simeq H^*(\overline{T}, Q).$$

We make now some further remarks.

Since $B \simeq S[U]$ and U is isomorphic to the reflection representation of W we can apply Chevalley's theorem and we have :

$S[U]$ is a free module over $S[U]^W$.

The same analysis as before gives

$$S[U] \simeq N \otimes_Q S[U]^W,$$

for a representation N of W and hence :

$$R_B(t) = R_N(t) \cdot R_{S[U]} W(t).$$

In the case of a semisimple Lie algebra N is isomorphic, as graded representation, to the cohomology of the flag variety.

Next we wish to sompute $R_{\overline{A}}(t)$ in a different way as done by GARSIA, STANTON [6].

The set X of vectors $\{v_\alpha\}$ is the set of fundamental weights. Fix a Weyl chamber and its weights $w_1, w_2, \ldots, w_m$. Then X is the union of the disjoint orbits

$$Ww_1,\ Ww_2, \ldots,\ Ww_m.$$

Any set of vectors $v_{\alpha_1}, \ldots, v_{\alpha_k}$ which is contained in a closed chamber is in the W orbit of a subset of $w_1, \ldots, w_m$.

Thus $\overline{A}$ has, as basis, the union of the W orbits of the monomials in $w_1, w_2, \ldots, w_m$.

For every subset $\{w_{i_1}, \ldots, w_{i_\alpha}\} = J$ of $w_1, w_2, \ldots, w_m$ we collect all the monomials involving the elements of J exactly and let C_J be the span of all such monomials. If W_J is the subgroup of W fixing all elements in J (or J equivalently by general facts) we have that as representation

$$\overline{A} = \bigoplus_J \mathrm{Ind}_{W_J}^{W}(C_J)$$

where W_J acts trivially on C_J.

C_J is just the monomial in the elements $\{w_{i_1}, \ldots, w_{i_k}\}$ multiplied by the polynomial ring in these elements, thus setting as usual $|J| = k$, the number of elements of J, we have.

Setting also $M_J = \mathrm{Ind}_{W_J}^{W}(1)$

$$\begin{aligned} R_{\overline{A}}(t) &= \sum_J R_{\mathrm{Ind}_{W_J}^{w}}(t) \\ &= \sum_J [M_J] t^{|J|} \frac{1}{(1-t)^{|J|}} \\ &= \frac{1}{(1-t)^m} \sum_J [M_J] t^{|J|} (1-t)^{m-|J|}. \end{aligned}$$

Now we can view $1/(1-t)^m$ as the Poincaré series of $S[U]$ thought as a graded vector space.
So

$$\frac{1}{(1-t)^m} = P_N(t) \cdot P_{S[U]^W}(t).$$

Since

$$P_{S[U]^W}(t) = R_{S[U]^W}(t)$$

we have the identity

$$\Big[\sum_{J \subseteq \{1,2,\ldots,m\}} [M_J] t^{|J|} (1-t)^{m-|J|} \Big] P_N(t) \cdot P_{S[U]^W}(t) = R_{H^\bullet(\overline{T},Q)}(t) \cdot R_N(t) \cdot P_{S[U]^W}(t)$$

hence THEOREM 2.

$$\Big[\sum_{J\subseteq\{1,\dots,m\}} [M_J]t^{|J|}(1-t)^{m-|J|}\Big] P_N(t) = R_{H^*(\overline{T},Q)}(t)\cdot R_N(t).$$

Since $R_N(t)$ has constant term 1 it is invertible in the ring of power series and this given a formula for $R_{H^*(\overline{T},Q)}(t)$ in terms of $R_N(t)$, $P_N(t)$ and the $[M_J]'s$.

Alternatively one can view the identity of THEOREM 2 as defining $R_{H^*(\overline{T},Q)}(t)$ recursively. As in [6] one can see immediately, setting $t=1$ that the polynomial

$$\sum_J |M_J|T^{|J|}(1-t)^{m-|J|}$$

is associated to a graded form of the regular representation of W.

3. Type A_{n+1}

In this case we can follow a more direct approach which is also more effective for the computations. Let $\overline{T}_n$ denote the variety in consideration (type A_{n+1}). We give a formula for its cohomology recursively on n.

It is easily verified that $\overline{T}_n$ can be obtained from projective space P^n in a simple way.

So let $X = \mathsf{P}^n = \{(a_0,\dots,a_n)$ in homogeneous coordinates$\}$. The symmetric group S_{n+1} acts on P^n by permuting the coordinates.

For each subset I of $\{0,1,\dots,n\}$ we can consider the subspace π_I of P^n given by the vanishing of the coordinates in I.

We refer to such a subspace as a coordinate plane. Clearly if $\sigma\in S_{n+1}$ we have

$$\sigma(\pi_I) = \pi_{\sigma(I)}$$

and the stabilizer of π_I equals the stabilizer of I.

If $I=\{0,1,2,\dots,k\}$ this stabilizer is the subgroup $S_{k+1}\times S_{n-k}$ embedded in S_{n+1} in the obvious way.

Let

$$X_k = \bigcup_{|I|=n-k} \pi_I$$

X_k is the union of $\binom{n+1}{k+1}$ k-dimensional coordinate planes permuted transitively by S_{n+1}. We want to define a sequence of $(n-1)$ blow ups of P^n which we denote

$$\mathsf{P}^n \leftarrow Y^0 \leftarrow Y^1 \leftarrow \cdots \leftarrow Y^{n-2}.$$

By definition Y^0 is the blow up of P^n along X_0 while Y^{i+1} is the blow up of Y^i along the proper transform $\widetilde{X}_{i+1}$ of X_{i+1} in Y^i.

It is easy to verify that (*cf.* [8], [9]) :

1) $\widetilde{X}_{i+1}$ is the disjoint union of the proper transforms $\tilde{\pi}_I$ of the $i+1$ dimensional coordinate planes.

2) Each $\tilde{\pi}_I(|I| = i+1)$ is smooth and isomorphic to the toric variety $\overline{T}_{i+1}$.

3) The action of $S_{i+2} \times S_{n-i-1}$ on $\tilde{\pi}_{\{0,1,\ldots,i+1\}}$ is only via the first factor and for S_{i+2} coincides, in the isomorphism of part 2, with the standard action.

4) Y^{n-2} is isomorphic to $\overline{T}_n$.

We can visualize the sequence of operations in the combinatoric of torus embedding :

We start with a polyhedral decomposition of $\mathbf{R}^n$ in $n+1$ cones which combinatorially induces on the unit sphere the structure of a standard simplex and the we stepwise construct the baricentric subdivision

e.g. $n = 2$

The computation of the cohomology of $\overline{T}_n \simeq Y^{n-2}$ can then be done inductively by the following remarks :

1) In general if we blow up a smooth subvariety A of codimension k in a smooth complete variety B the cohomology of the blow up $\widetilde{B}$ is additively isomorphic to

$$H^*(B) \oplus H^*(A) \otimes H^+(p^{k-1})$$

where P^{k-1} is projective space and H^+ denotes the strictly positive cohomology.

2) If a group G acts on B preserving A the previous isomorphism is compatible with the natural group actions of G on $H^*(B)$, $H^*(B)$ and the trivial action on $H^+(\mathsf{P}^{k-1})$. The fact that G acts trivially a P^{k-1} follows from the fact that G acts linearly on the natural boundle N of A in B and so fixes the Chern class of the tautological bundle on the projectification of N.

In our case we have therefore :

$$\begin{aligned} H^*(Y^i) = H^*(P^n) \bigoplus H^*(\widetilde{X}_0) \otimes H^+(\mathsf{P}^{n-1}) \\ \bigoplus H^*(\widetilde{X}_1) \otimes H^+(\mathsf{P}^{n-2}) \bigoplus \cdots \\ \bigoplus H^*(\widetilde{X}_i) \otimes H^+(\mathsf{P}^{n-i-1}). \end{aligned}$$

By the previous analysis we get furthermore that as S_{n+1} representation

$$H^*(\widetilde{X}_i) \simeq \mathrm{Ind}_{S_{i+1}}^{S_{n+1}} \times S_{n-i} H^*(\overline{T}_i).$$

This gives the required inductive formula for $\overline{T}_n = Y^{n-2}$

$$H^*(\overline{T}_n) = H^*(\mathsf{P}^n) \bigoplus \sum_{i=0}^{n-2} \left(\mathrm{Ind}_S^{S_{n+1}} S_{i+1} \times S_{n-i} H^*(\overline{T}_i)\right) \otimes H^+(\mathsf{P}^{n-i-1}).$$

The most convenient way to treat this formula is obtained using the coding of representations of the symmetric groups S_{n+1} by Schur symmetric functions (in $\geq n+1$ variables). In this coding the representation associated to a Young diagram λ with $n+1$ boxes is given by the corresponding Schur function S_λ, the convertion is that if $\lambda = n+1$ the representation is the trivial representation and S_{n+1} is the sum of all monomials of degree $n+1$.

The convenience of this coding is given by the well known fact [7].

If we are given two representations V_λ, W_μ of S_k and S_{n+1-k} corresponding to Young diagrams and μ then the symmetric function corresponding to

$$\mathrm{Ind}_{S_k}^{S_{n+1}} \times S_{n+1-k} V_\lambda \otimes W_\mu$$

is $S_\lambda \cdot S_\mu$.

This is a particularly effective rule in many cases where the multiplication of Schur functions is easy to understand.

In our case we will need multiplications of type $S_\lambda \cdot S_h$, h a number (S_h corresponds to the trivial representation).

This is given by the simple rule : If $|\lambda| = k$

$$S_\lambda \cdot S_h = \sum S_{\lambda_i}$$

where λ_i runs over all diagrams with $h+k$ boxes which are contained in the diagram

(*i.e.* if λ has m rows and are obtained by removing m boxes from the rim (λ, h) is obtained from λ by adding a first column of length $m+h$).

We now write the graded character series with coefficients symmetric functions associated to our varieties. Since the cohomology is all even we give degree 2 to the variable t. (Writing t^i instead of t^{2i}).

Let A_n denote the series corresponding to $H^*(\overline{T}_n)$. The basic formula is

$$A_n = S_{n+1}\sum_{i=0}^{n} t^i + \sum_{i=0}^{n-2} S_{n-i}A_i\Big(\sum_{k=1}^{n-i-1} t^k\Big).$$

We can explicit easily for $n \leq 5$ we use starter notations indicating simply by $[k_1k_2\ldots] = S_{k_1k_2\ldots}$ or the corresponding representation of the symmetric group :

$$\begin{aligned}
A_0 &= [1]\\
A_1 &= [2](1+t)\\
A_2 &= [3](1+2t+t^2)+[21]t\\
A_3 &= [4]\big(1+3t+3t^2+t^3\big)+[31](2t+2t^2)+[22](t+t^2)\\
A_4 &= [5](1+t)^4+[41]3t(1+2t+t^2)+[32]t(2+5t+2t^2)\\
&\quad +[311]t^2+[221]t^2\\
A_5 &= [6](1+t)^5+[51]4t(1+t)^3+[42]3t\big(1+4t+4t^2+t^3\big)\\
&\quad +[33]t\big(1+5t+5t^2+t^3\big)\\
&\quad +[321]4t^2(1+t)+[411]t^2(1+t)+[222]t^2(1+t).
\end{aligned}$$

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