Invariants of matrices under the action of the special orthogonal group

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Abstract

We prove the first fundamental theorem of invariant theory for the action of the special orthogonal group on m tuples of matrices by simultaneous conjugation. A basic relation between polynomial invariants is also studied.

1 Introduction

Throughout this paper we shall use the following notations and make the following assumptions: $\mathbb{K}$ denotes a field of characteristic zero, $Gl(n,\mathbb{K})$ denotes the group of invertible $n \times n$ matrices with coefficients in $\mathbb{K}$, $(\mathbb{K})_n^m$ denotes the space of m-tuples of $n \times n$ matrices.

$Gl(n,\mathbb{K})$ (and hence any subgroup $G \subseteq Gl(n,\mathbb{K})$) acts over $(\mathbb{K})_n^m$ by simultaneous conjugation

$$A \cdot (B_1, \dots, B_m) = (AB_1A^{-1}, \dots, AB_mA^{-1}) \qquad A \in Gl(n,\mathbb{K}),\ (B_1, \dots, B_m) \in (\mathbb{K})_n^m \quad (1)$$

Let $\mathbb{A}_n^m$ denote the ring of polynomial functions over $(\mathbb{K})_n^m$. For any $G \subseteq Gl(n,\mathbb{K})$ one can study the ring $(\mathbb{A}_n^m)^G$ of polynomials $f \in \mathbb{A}_n^m$ which are invariant under the action induced by (1), i.e. under the action

$$(A \cdot f)(B_1, \dots, B_m) = f(A^{-1}B_1A, \dots, A^{-1}B_mA) \qquad A \in Gl(n,\mathbb{K}),\ (B_1, \dots, B_m) \in (\mathbb{K})_n^m, f \in \mathbb{A}_n^m \quad (2)$$

Two classical problems about these actions are:

1. can we find a list of generators for $(\mathbb{A}_n^m)^G$?

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2. can we find a complete list of relations between the generators?

Solutions of these problems are called *the first and the second fundamental theorem* for G-invariants of m matrices respectively. In [3] the first and the second fundamental theorem are proved for the general linear group $Gl(n, \mathbb{K})$, for the orthogonal group $O(n, \mathbb{K})$ and for the symplectic group $Sp(n, \mathbb{K})$. In this paper we prove the first fundamental theorem for the special orthogonal group $SO(n, \mathbb{K})$ and study a basic identity between $SO(n, \mathbb{K})$ invariants.

Let us recall the first fundamental theorem for invariants of matrices for the action of $O(n, \mathbb{K})$ (see [3], Theorem 7.1, p.327).

Theorem 1.1 *Every orthogonal invariant of m matrices $(A_1, \ldots, A_m) \in \mathbb{K}_n^m$ is a polynomial in the invariants $\operatorname{tr}(U_{i_1}U_{i_2}\ldots U_{i_k})$ where $U_i = A_i$ or $U_i = A_i^t$ for each $i = 1, \ldots, m$.*

Remark 1.2 *Note that in the statement of Theorem 1.1, $U_{i_1}U_{i_2}\ldots U_{i_k}$ run over all possible noncommutative monomials in $A_1, \ldots, A_n$, $A_1^t, \ldots, A_n^t$.*

A classical way to express the content of Theorem 1.1 is that the trace is the only *typical* invariant for the action of $O(n, \mathbb{K})$ over m-tuples of matrices.

If we consider the action of the special orthogonal group, the structure of $(\mathbb{A}_n^m)^{SO(n,\mathbb{K})}$ turns out to depend on the parity of n:

Odd *n* Under the action of $SO(2n-1, \mathbb{K})$, no further invariants arise, i.e. $(\mathbb{A}_{2n-1}^m)^{SO(2n-1,\mathbb{K})} = (\mathbb{A}_{2n-1}^m)^{O(2n-1,\mathbb{K})}$ (see Theorem 2.11).

Even *n* One needs to introduce, beyond the trace, a new typical invariant Q, defined by

$$Q(A_1, \ldots, A_n) = \mathrm{Pf_L}\left(\frac{A_1 - A_1^t}{2}, \ldots, \frac{A_n - A_n^t}{2}\right)$$

where Pf_L is the complete polarization of the Pfaffian of an antisymmetric $2n \times 2n$ matrix (see Definition 2.1).

In order to describe more precisely $SO(2n, \mathbb{K})$ invariants of matrices it is convenient to introduce the notions of *even* and *odd* $SO(n, \mathbb{K})$ invariants.

Let W be any $O(n, \mathbb{K})$-module and let f be a polynomial function over W which is $SO(n, \mathbb{K})$-invariant:
If f is also $O(n, \mathbb{K})$-invariant we say that f is *even*;
If $f(g \cdot (w)) = \det(g)f(w)$ for all $g \in O(n, \mathbb{K})$ we say that f is *odd*.

Lemma 1.3 *Any $SO(n, \mathbb{K})$-invariant f can be written as the sum of an odd and an even invariant.*

In fact, let f' be obtained by f by applying any orthogonal transformation whose determinant is -1 . Then $\frac{f+f'}{2}$ is even, $\frac{f-f'}{2}$ is odd and $f = \frac{f+f'}{2} + \frac{f-f'}{2}$.

We are ready to state the analogue of Theorem 1.1 for $SO(2n, \mathbb{K})$-invariants which will be proved in Section 2. Note that, as we said above, $SO(2n-1, \mathbb{K})$ and $O(2n-1, \mathbb{K})$ invariants agree (Theorem 2.11).

Theorem 1.4 *Let F be a polynomial $SO(n, \mathbb{K})$-invariant of m matrices $(A_1, \ldots, A_m) \in \mathbb{K}_{2n}^m$.*

*If F is **even** it has the form already described in Theorem 1.1.*

*If F is **odd** it is a sum of terms of type*

$$Q(M_1, \ldots, M_n) f^*(A_1, \ldots, A_m)$$

where each M_i is a non commutative monomial in $A_1, \ldots, A_m, A_1^t, \ldots, A_m^t$ and f^* is an even invariant.

Since the product of two odd invariants is even we have in particular that for any pair of n-tuples of matrices $(X_1, \ldots, X_n), (Y_1, \ldots, Y_n) \in \mathbb{K}_{2n}^n$ the expression

$$Q(X_1, \ldots, X_n) Q(Y_1, \ldots, Y_n)$$

can be written as a polynomial in the basic even invariants $\mathrm{tr}(U_{i_1} U_{i_2} \ldots U_{i_k})$ described in Theorem 1.4. In Section 3 we shall give an explicit form for this polynomial.

2 Polynomial invariants of k-tuples of matrices for the action of $SO(n, \mathbb{K})$

The study of invariants of matrices in [3] is based on two main ideas:

1. the computation of general invariants can be reduced to the computation of multilinear invariants (*Aronhold method*);

2. the computation of multilinear invariants of m matrices under simultaneous conjugation can be reduced to the computation of multilinear invariants of $2m$ vectors under simultaneous left multiplication.

We shall show that these same ideas allow also the computation of $SO(n, \mathbb{K})$ matrix invariants.

Let us recall the basic facts about Aronhold method. The discussion is taken from the forthcoming book [4] where one can find all details (see also [1] and [2]).

First we need to introduce the operations of *full polarization* and *full restitution*. Let $U_1, \ldots, U_n$ be vector spaces. Recall that a polynomial $f : U_1 \times \cdots \times U_m \to \mathbb{K}$ is said to be *multihomogeneous of multidegree* $h_1, \ldots, h_m$, if

$$f(\lambda_1 u_1, \ldots, \lambda_m u_m) = \lambda_1^{h_1} \cdots \lambda_m^{h_m} f(u_1, \ldots, u_m) \qquad \forall \lambda_1, \ldots, \lambda_m \in \mathbb{K}, \quad \forall (u_1, \ldots, u_m) \in U_1 \times \cdots \times U_m \quad (3)$$

Given a multihomogeneous polynomial $f(u_1, \ldots, u_m)$ having multidegree $h_1, h_2, \ldots, h_m$, it is possible to "linearize" f in each variable, i.e, to get a polynomial $g(u_{11}, \ldots, u_{1h_1}; \ldots; u_{m1}, \ldots, u_{mh_m})$, the *full polarization* of f, which is linear in each vector variable $u_{ij} \in U_i$. We show how to proceed in the case $m = 1$; in the general case we need only to extend the procedure to each slot and it is only notationally more complicated.

Let V be a finite dimensional vector space over $\mathbb{K}$, let $\mathbb{K}[V]$ be the ring of polynomial functions over V and let $f \in \mathbb{K}[V]$ be a homogeneous polynomial of degree d. We can write

$$f(t_1 v_1 + \cdots + t_d v_d) = \sum_{s_1 + \cdots + s_d} t_1^{s_1} \cdots t_d^{s_d} f_{s_1 \ldots s_d}(v_1, \ldots, v_d) \quad (4)$$

where the polynomials $f_{s_1 \ldots s_d} \in \mathbb{K}[V^d]$ are multihomogeneous of degree $(s_1, \ldots, s_d)$.

Definition 2.1 (Full polarization) *The multilinear polynomial $f_{11\ldots1} \in \mathbb{K}[V^d]$ in the decomposition (4) is called the* full polarization *of f. It will be denoted by $\mathcal{P}f$.*

The full polarization $f \to \mathcal{P}f$ is a linear map. The inverse process is called full restitution.

Definition 2.2 (Full restitution) *For a multilinear $F \in \mathbb{K}[V^d]$ the homogeneous polynomial $\mathcal{R}F$ defined by $\mathcal{R}F(v) = \dfrac{F(v, v, \ldots, v))}{d!}$ is called the* full restitution *of F.*

Lemma 2.3 *The linear operators $\mathcal{P} : \mathbb{K}[V]_d \to \mathbb{K}[V^d]_{(1,1,\ldots,1)}$ and $\mathcal{R} : \mathbb{K}[V^d]_{(1,\ldots,1)} \to \mathbb{K}[V]_d$*

have the following properties (see [4] or [2], p. 34*):*

1. *$\mathcal{P}$ and $\mathcal{R}$ are equivariant for the group of all linear transformations;*

2. *$\mathcal{P}f$ is symmetric;*

3. *The maps $\mathcal{P}$ and $\mathcal{R}$ are inverse isomorphisms between the space of homogeneous forms of degree d and the space of symmetric multilinear functions in d variables.*

4. *If F is G-invariant then its full polarization is a multilinear G-invariant.*

5. *If F is a multilinear G invariant then its full restitution is G-invariant.*

As we said above we have defined full polarization and full restitution only for $m = 1$ but the procedure and the results remain the same for any m (see [4], p. 35).

In order to formalize Aronhold's method we consider an infinite sequence of n-dimensional vector variables $x_1, x_2, \dots$ where $x_i = (x_{1i}, x_{2i}, \dots, x_{ni})$. Let $\mathcal{A} = \mathbb{K}[x_{ij}]$ be the polynomial ring in the variables x_{ij}. $\mathcal{A}$ is a multigraded ring since we have the notion of being homogeneous with respect to any of the vector variables x_i.

Definition 2.4 *We say that a polynomial $F \in \mathcal{A}$ is quasi multilinear if it is homogeneous of degree* 0 *or* 1 *in each of the variables x_i.*

Given a subspace V of $\mathcal{A}$ we will denote by V_L the set of quasi multilinear elements of V. It is easy to prove (see [1] or [4]) that

Theorem 2.5 *If two subspaces V, W of $\mathcal{A}$ are stable under full polarization and full restitution and $V_L = W_L$, then $V = W$.*

Given this Theorem, the strategy to compute the space $W \subseteq \mathcal{A}$ of invariants under the action of a group $G \subseteq Gl(n, \mathbb{K})$ is the following. One produces some list of invariants forming a subspace $V \subseteq \mathcal{A}$ which is closed under polarization. In order to show that $V = W$ it is enough to look only at quasi multilinear invariants, i.e. to prove that $V_L = W_L$.

We shall use a slight generalization of Theorem 2.5.

Corollary 2.6 *If two subspaces V, W of $\mathcal{A}^{SO(n,\mathbb{K})}$ are stable under full polarization and restitution, $(V_L)_{odd} = (W_L)_{odd}$ and $(V_L)_{even} = (W_L)_{even}$, then $W_{odd} = V_{odd}$ and $W_{even} = V_{even}$.*

For each m-tuple of non negative integers $(h_1, \dots, h_m)$ the set $(\mathbb{A}_n^m)_{h_1,\dots,h_m}$ of multihomogeneous $F \in \mathbb{A}_n^m$ of multidegree $(h_1, \dots, h_m)$ is a vector space and we have the direct sum decomposition $\mathbb{A}_n^m = \oplus(\mathbb{A}_n^m)_{h_1,\dots,h_m}$, i.e. each $F \in \mathbb{A}_n^m$ can be written in a unique way as the sum of its multihomogeneous components. Since the action (2) preserves multidegree, F is G-invariant if and only if each of its multihomogeneous components is G-invariant.

Let $F \in (\mathbb{A}_n^m)_{h_1,\dots,h_m}$ and let $H = \mathcal{P}(F)$ be its full polarization. H is a multilinear function of $h = h_1 + \cdots + h_m$ matrices. In particular H (hence F by Lemma 2.3, point iii)) is completely determined by its restriction to h-tuples of rank one matrices

Now we discuss how multilinear invariants of matrices under simultaneous conjugation and multilinear invariants of vectors under simultaneous left multiplication are related.

Definition 2.7 *For each multilinear $H : \mathbb{K}_n^h \to \mathbb{K}$ let us define $\tilde{H} : (\mathbb{K}^n)^{2h} \to \mathbb{K}$ by*

$$\tilde{H}(u_1, v_1, \dots, u_h, v_h) = H(u_1 \otimes v_1, \dots, u_h \otimes v_h)$$

The following results follow easily from the definitions.

Lemma 2.8

1. *$\tilde{H}$ is multilinear.*
2. *H is invariant with respect to the action of $SO(n, \mathbb{K})$ by simultaneous conjugation if and only if $\tilde{H}$ is invariant with respect to the action of $SO(n, \mathbb{K})$ by simultaneous left multiplication.*
3. *H is even (resp. odd) if and only if $\tilde{H}$ is even (resp. odd).*

Because of Lemma 2.8, the computation of multilinear invariants H of m matrices will be reduced to the computation of multilinear invariants $\tilde{H}$ of $2m$ vectors. These are described in the next Theorem which follows immediately from [5], p. 53 (see also [4]).

Theorem 2.9 *Any multilinear $SO(n, \mathbb{K})$ even invariant of $2m$ vectors is a linear combination of the invariants*

$$\langle v_{j_1}, v_{j_2} \rangle \cdot \dots \cdot \langle v_{j_{2m-1}}, v_{j_{2m}} \rangle \tag{5}$$

where $(j_1, \dots, j_{2m})$ is any permutation of $(1, \dots, 2m)$.

Any multilinear $SO(n, \mathbb{K})$ odd invariant of $2m$ vectors is a linear combination of the invariants

$$\det[v_{j_1}, \dots, v_{j_n}] \langle v_{j_{n+1}}, v_{j_{n+2}} \rangle \cdot \dots \cdot \langle v_{j_{2m-1}}, v_{j_{2m}} \rangle \tag{6}$$

where $(j_1, \dots, j_{2m})$ is again any permutation of $(1, \dots, 2m)$

Remark 2.10 *If n is odd in Theorem 2.9 then no multilinear invariants of type (6) can appear i.e. for n odd there are only multilinear even invariants of $2m$ vectors.*

Note that the product of the two multilinear odd invariants

$$\det[x_1, \dots, x_n] \det[y_1, \dots, y_n], \qquad (x_1, \dots, x_n, y_1, \dots, y_n) \in (\mathbb{K}^n)^{2n}$$

is an even invariant, hence it can be written as a linear combination of even invariants. Infact

$$\det[x_1, \dots, x_n] \det[y_1, \dots, y_n] = \det \begin{bmatrix} \langle x_1, y_1 \rangle & \langle x_1, y_2 \rangle & \dots & \langle x_1, y_n \rangle \\ \langle x_2, y_1 \rangle & \langle x_2, y_2 \rangle & \dots & \langle x_2, y_n \rangle \\ & & \dots & \\ \langle x_n, y_1 \rangle & \langle x_n, y_2 \rangle & \dots & \langle x_n, y_n \rangle \end{bmatrix} \tag{7}$$

We can prove now that any $SO(2n-1, \mathbb{K})$-invariant of matrices is also $O(2n-1, \mathbb{K})$-invariant.

Theorem 2.11 *If* $F \in (\mathbb{A}_n^m)^{SO(2n-1,\mathbb{K})}$ *then* $F \in (\mathbb{A}_n^m)^{O(2n-1,\mathbb{K})}$

Proof Let $F_{h_1,\dots,h_m}$ be the multihomogeneous component of F of multidegree $(h_1, \dots, h_m)$. Let

$$H(A_{1,1}, \dots, A_{1,h_1}; \dots; A_{m,1}, \dots, A_{m,h_m})$$

be the full polarization of $F_{h_1,\dots,h_m}$ and let $h = h_1 + \dots + h_m$.

Let

$$\tilde{H}(u_1, v_1, \dots, u_h, v_h) = H(u_1 \otimes v_1, \dots, u_h \otimes v_h).$$

By Lemma 2.8 $\tilde{H}$ is a multilinear $SO(2n-1, \mathbb{K})$ invariant, hence by Remark 2.10 is even. Hence also H is even by Lemma 2.8. This implies that each multihomogeneous component of F, hence F itself, is even. □

In the following we shall identify the tensor product $\mathbb{K}^n \otimes \mathbb{K}^n$ of the standard $SO(n, \mathbb{K})$-module $\mathbb{K}^n$ with itself with the space of $2n \times 2n$ matrices. Let $e_1, \dots e_{2n}$ be the standard orthonormal basis of $\mathbb{K}^n$. We identify the reducible tensor $(a_1 e_1 + \dots + a_n e_n) \otimes (b_1 e_1 + \dots + b_n e_n)$ with the matrix $\{c_{ij}\} = \{a_i b_j\}$. This identification can be extended to all tensors by linearity. Under this identification the decomposition $\mathbb{K}^n \otimes \mathbb{K}^n = S^2(\mathbb{K}^n) \oplus \wedge^2(\mathbb{K}^n)$ corresponds to the decomposition of the space of square matrices in symmetric and antisymmetric ones. The map

$$\begin{array}{rcl} \mathbb{K}^n \otimes \mathbb{K}^n & \to & \wedge^2(\mathbb{K}^n) \\ A & \mapsto & \frac{A - A^t}{2} \end{array}$$

sends each rank one matrix $u \otimes v$ onto the antisymmetric matrix

$$u \wedge v = \frac{u \otimes v - v \otimes u}{2}$$

As anticipated in the Introduction, in order to describe all $SO(2n,\mathbb{K})$ invariants of matrices we need to add a further typical invariant to the trace. This new invariant, which we denote by Q, must be multilinear and such that $Q(u_1\otimes v_1,\dots,u_n\otimes v_n)$ coincides with $\det[u_1,v_1,\dots,u_n,v_n]$ ($u_i,v_i\in\mathbb{K}^{2n}$). Before defining Q we recall the definition of the Pfaffian of an antisymmetric, even dimensional matrix.

Definition 2.12 *Let $A=\{a_{ij}\}$ be an antisymmetric $2n\times 2n$ matrix. The Pfaffian of A is defined by*

$$\omega_A^n = n!\mathrm{Pf}(A)e_1\wedge\cdots\wedge e_{2n} \tag{8}$$

where

$$\omega_A=\sum_{i<j}2a_{ij}e_i\wedge e_j \tag{9}$$

Remark 2.13 *If $A=\{a_{ij}\}$ is any matrix, we can write $A=\sum a_{ij}e_i\otimes e_j$. If $A=(a_{ij})$ is antisymmetric A can be obviously written as a linear combination of the matrices $e_i\wedge e_j$. Note however the factor 2 in $A=\sum 2a_{ij}e_i\wedge e_j$ which explain the factor 2 in (9).*

Lemma 2.14 *Let $u_1,v_1,\dots,u_n,v_n\in\mathbb{K}^{2n}$ and let $X=u_1\otimes v_1+\cdots+u_n\otimes v_n$. Then*

$$\det[u_1,v_1,\dots,u_n,v_n]=\mathrm{Pf}\left(\frac{X-X^t}{2}\right).$$

Proof.

Let $e_1\wedge\cdots\wedge e_{2n}$ be the volume form in V (an element of the 1 dimensional exterior power $\bigwedge^{\dim(V)}V$). Recall that

$$\det[u_1,v_1,\dots,u_n,v_n]=\frac{u_1\wedge v_1\wedge\cdots\wedge u_n\wedge v_n}{e_1\wedge\cdots\wedge e_{2n}} \tag{10}$$

Then by (8)

$$\mathrm{Pf}\left(\frac{X-X^t}{2}\right)=\mathrm{Pf}(u_1\wedge v_1+\cdots+u_n\wedge v_n)=\frac{(u_1\wedge v_1+\cdots+u_n\wedge v_n)^n}{n!(e_1\wedge\cdots\wedge e_{2n})} \tag{11}$$

hence the result. □

We are finally ready to define Q by taking the complete polarization of the Pfaffian, (see Definition 2.1).

Definition 2.15 *Let* $\mathrm{Pf}_L = \mathcal{P}(\mathrm{Pf})$ *be the complete polarization of the Pfaffian. Let* $(A_1, \ldots, A_n) \in (\mathbb{K}^n_{2n})$*. We define* $Q(A_1, \ldots, A_n)$ *by*

$$Q(A_1, \ldots, A_n) = \mathrm{Pf}_L \left(\frac{A_1 - A_1^t}{2}, \ldots, \frac{A_n - A_n^t}{2} \right) \tag{12}$$

Q is obviously multilinear, let us show that it coincides with the determinant when $A_1, \ldots, A_n$ have rank one, as a consequence of Lemma 2.14.

Theorem 2.16 *Let* $u_1, v_1, \ldots, u_n, v_n \in \mathbb{K}^{2n}$*. Then*

$$Q(u_1 \otimes v_1, \ldots, u_n \otimes v_n) = \det[u_1, v_1, \ldots, u_n, v_n]$$

Proof. By Lemma 2.14,

$$\mathrm{Pf} \left(\frac{\lambda_1 u_1 \otimes v_1 + \cdots + \lambda_n u_n \otimes v_n - (\lambda_1 u_1 \otimes v_1 + \cdots + \lambda_n u_n \otimes v_n)^t}{2} \right) \tag{13}$$

coincides with

$$\lambda_1 \cdot \cdots \cdot \lambda_n \det[u_1, v_1, \ldots, u_n, v_n] \tag{14}$$

On the other hand we can expand (13) as a polynomial in $\lambda_1, \ldots, \lambda_n$ as

$$\sum \lambda_1^{s_1} \cdot \cdots \cdot \lambda_n^{s_n} \mathrm{Pf}_{s_1, \ldots, s_n} \left(\frac{u_1 \otimes v_1 - v_1 \otimes u_1}{2}, \ldots, \frac{u_n \otimes v_n - v_n \otimes u_n}{2} \right). \tag{15}$$

The expressions (14) and (15) coincide as polynomials in $\lambda_1, \ldots, \lambda_n$, hence $\mathrm{Pf}_{s_1, \ldots, s_n} = 0$ unless $(s_1, \ldots, s_n) = (1, \ldots, 1)$ and therefore

$$\det[u_1, v_1, \ldots, u_n, v_n] = \mathrm{Pf}_L \left(\frac{u_1 \otimes v_1 - v_1 \otimes u_1}{2}, \ldots, \frac{u_n \otimes v_n - v_n \otimes u_n}{2} \right) \tag{16}$$

hence the claim □

The following results are easy.

Lemma 2.17

1. $u \otimes v \cdot u' \otimes v' = u \otimes \langle v, u' \rangle v'$
2. $\mathrm{tr}(u \otimes v) = \langle u, v \rangle$
3. $(v \otimes w)^t = w \otimes v$

Corollary 2.18

$$\langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \ldots \langle x_{2n-1}, x_{2n} \rangle = \mathrm{tr}(x_{2n} \otimes x_1 \cdot x_2 \otimes x_3 \cdots x_{2n-2} \otimes x_{2n-1}) \tag{17}$$

In order to prove Theorem 1.4 we prove its multilinear version first.

Theorem 2.19 *Every multilinear special orthogonal* **even** *invariant of m matrices $A_1, \dots, A_m \in \mathbb{K}_{2n}^m$ is a linear combination of elements of type*

$$\mathrm{tr}(U_{k_1} \cdots U_{k_{p_1}}) \cdots \mathrm{tr}(U_{s_1} \cdots U_{s_{p_h}})$$

where each U_i either coincides with A_i or with A_i^t and

$$(k_1, \dots, k_{p_1}, \dots, s_1, \dots, s_{p_h})$$

is a permutation of $(1, \dots, m)$.

Every multilinear special orthogonal **odd** *invariant of m matrices $A_1, \dots, A_m \in \mathbb{K}_{2n}^m$ is a linear combination of elements of type*

$$Q(U_{r_1} \cdots U_{r_{p_1}}, \dots, U_{t_1} \cdots U_{t_{p_n}}) f^*(U_{j_1}, \dots, U_{j_k})$$

where each U_i either coincides with A_i or with A_i^t,

$$(r_1, \dots, r_{p_1}, \dots, t_1, \dots, t_{p_n}, j_1, \dots, j_k)$$

is a permutation of $(1, \dots, m)$ and $f^*(U_{j_1}, \dots, U_{j_k})$ is an even multilinear invariant of $k \le m - n$ matrices.

Proof The proof consists of an algorithm for expressing the odd and the even part of every multilinear special orthogonal invariant of m matrices in the required form. Let $H(A_1, \dots, A_m)$ be a multilinear special orthogonal invariant of m matrices. Since each element of type $\mathrm{tr}(U_{j_1} \cdots U_{j_k})$ and $Q(U_{j_1} \cdots U_{j_{p_1}}, \dots, U_{s_1} \cdots U_{s_{p_n}})$ appearing in the statement of the Theorem is multilinear, it is sufficient to prove the claim when $A_1, \dots, A_m$ have rank one, i.e. when

$$A_j = u_j \otimes v_j, \qquad j = 1, \dots, m$$

Let us define $\tilde{H}$ by

$$\tilde{H}(u_1, v_1, \dots, u_m, v_m) = H(u_1 \otimes v_1, \dots, u_m \otimes v_m).$$

In the rest of the proof we use the following conventions; w stands for u or v. Moreover, we put $\overline{u}_j = v_j$ and $\overline{v}_j = u_j$. Let us say that u_j, u_k are of the *same type*, as well as v_j, v_k while u_j, v_k are of *different type* and analogously that A_j, A_k are of the same type as well as A_j^t, A_k^t while A_j, A_k^t are of different type. Let us say moreover that w and $\overline{w}$ is a *coupled pair*.

Let $\tilde{H} = \tilde{H}_{even} + \tilde{H}_{odd}$ be the decomposition of $\tilde{H}$ in its even and odd parts. By Theorem 2.9, $\tilde{H}_{even}$ is a linear combination of expressions $\tilde{H}_\alpha$ like

$$\langle w_{j_1}, w_{j_2} \rangle \dots \langle w_{j_{2m-1}}, w_{j_{2m}} \rangle \tag{18}$$

and $\tilde{H}_{odd}$ is a linear combination of expressions $\tilde{H}_\beta$ like

$$\det[w_{j_1}, \dots, w_{j_n}]\langle w_{j_{n+1}}, w_{j_{n+2}}\rangle \dots \langle w_{j_{2m-1}}, w_{j_{2m}}\rangle \tag{19}$$

In each of the expressions (18) or (19) the list $(w_1, \dots, w_{2m})$ is obtained by allowing any permutation of the list $(u_1, v_1, \dots, u_m, v_m)$.

Monomials of type (18). These are dealt with as in [3]. For completeness we repeat the proof here. We begin with an example to clarify the procedure.

Example

Let H be such that

$$\tilde{H}(u_1, v_1, \dots u_6, v_6) = \langle u_3, u_5\rangle\langle u_2, v_4\rangle\langle u_4, v_3\rangle\langle v_2, v_5\rangle\langle v_1, v_6\rangle\langle u_6, u_1\rangle \tag{20}$$

We rearrange (20) and use Corollary 2.18 to get.

$$\begin{aligned}\tilde{H}(u_1, v_1, \dots u_6, v_6) = (\langle u_2, v_4\rangle\langle u_4, v_3\rangle\langle u_3, u_5\rangle\langle v_5, v_2\rangle)(\langle v_1, v_6\rangle\langle u_6, u_1\rangle) = \\ \mathrm{tr}(v_2 \otimes u_2 \cdot v_4 \otimes u_4 \cdot v_3 \otimes u_3 \cdot u_5 \otimes v_5)\mathrm{tr}(u_1 \otimes v_1 \cdot v_6 \otimes u_6) = \\ \mathrm{tr}(A_2^t \cdot A_4^t \cdot A_3^t \cdot A_5)\mathrm{tr}(A_1 \cdot A_6^t)\end{aligned} \tag{21}$$

We have verified that

$$H(A_1, \dots, A_6) = \mathrm{tr}(A_2^t \cdot A_4^t \cdot A_3^t \cdot A_5)\mathrm{tr}(A_1 \cdot A_6^t)$$

for all rank 1 matrices, hence for all matrices by multilinearity.

In general, we can write $\tilde{H}_{even} = \sum c_\alpha \tilde{H}_\alpha$ where each $\tilde{H}_\alpha$ is like in (18). We can rearrange any expression (18) as a product of cycles of scalar products of the form

$$\begin{aligned}\underbrace{(\langle w_{i_1}, \overline{w}_{i_2}\rangle\langle w_{i_2}, \overline{w}_{i_3}\rangle \cdots \langle w_{i_k}, \overline{w}_{i_1}\rangle)}_{first\ cycle} \cdot \underbrace{(\langle w_{j_1}, \overline{w}_{j_2}\rangle \cdots \langle w_{j_s}, \overline{w}_{j_1}\rangle)}_{second\ cycle} \cdots \\ \cdots \underbrace{(\langle w_{t_1}, \overline{w}_{t_2}\rangle\langle w_{t_2}, \overline{w}_{t_3}\rangle \cdots \langle w_{t_h}, \overline{w}_{t_1}\rangle)}_{last\ cycle}\end{aligned} \tag{22}$$

where in each cycle

$$(\langle w_{l_1}, \overline{w}_{l_2}\rangle\langle w_{l_2}, \overline{w}_{l_3}\rangle \cdots \langle w_{l_{p-1}}, \overline{w}_{l_p}\rangle\langle w_{l_p}, \overline{w}_{l_1}\rangle)$$

the first and the last element $w_{l_1}, \overline{w_{l_1}}$ are a coupled pair and all adjacent elements like

$$\cdots\langle\ ,\underbrace{w\rangle\langle\overline{w}}_{adjacent},\ \rangle\cdots$$

are coupled pairs.

By using Lemma 2.17 and Corollary 2.18 it is easy to verify that (22) can be written as

$$\tilde{H}_\alpha(u_1, v_1, \ldots, u_m, v_m) = \mathrm{tr}(U_{i_1} \cdot \cdots \cdot U_{i_k})\mathrm{tr}(U_{j_1} \cdot \cdots \cdot U_{j_s}) \cdots \mathrm{tr}(U_{t_1} \cdot \cdots \cdot U_{t_h})$$

where in each cycle we set $U_j = A_j$ or $U_j = A_j^t$, according to the following inductive rules

1. $U_{l_1} = A_{l_1}$ if $w_{l_1} = v_{l_1}$; $U_{l_1} = A_{l_1}^t$ if $w_{l_1} = u_{l_1}$.

2. Set $U_{l_{t+1}}$ of the same type as U_{l_t} if and only if $w_{l_t} = w_{l_{t+1}}$ have the same type.

We have therefore put each $\tilde{H}_\alpha$, hence H_{even}, in the required form.

Monomials of type (19). As above we shall begin with an example to clarify the procedure

Example

Let H be such that

$\tilde{H}(u_1, v_1, \ldots, u_7, v_7) = \det[u_1, v_3, v_7, v_1]\langle u_2 \cdot u_3\rangle\langle u_4, v_6\rangle\langle u_5, u_7\rangle\langle v_2, v_4\rangle\langle v_5, u_6\rangle$

then

$\tilde{H} = \det[u_1, v_1, v_3, \langle u_3, u_2\rangle\langle v_2, v_4\rangle\langle u_4, v_6\rangle\langle u_6, v_5\rangle\langle u_5, u_7\rangle v_7] =$

$Q(u_1 \otimes v_1, v_3 \otimes \langle u_3, u_2\rangle\langle v_2, v_4\rangle\langle u_4, v_6\rangle\langle u_6, v_5\rangle\langle u_5, u_7\rangle v_7)$

By Lemma 2.17, point i) this expression is equal to

$Q(u_1 \otimes v_1, v_3 \otimes u_3 \cdot u_2 \otimes v_2 \cdot v_4 \otimes u_4 \cdot v_6 \otimes u_6 \cdot v_5 \otimes u_5 \cdot u_7 \otimes v_7) =$

$Q(A_1, A_3^t \cdot A_2 \cdot A_4^t \cdot A_6^t \cdot A_5^t \cdot A_7)$

We have verified that

$$H(A_1, \ldots, A_7) = Q(A_1, A_3^t \cdot A_2 \cdot A_4^t \cdot A_6^t \cdot A_5^t \cdot A_7)$$

for all rank 1 matrices, hence for all matrices by multilinearity.

In general, we can write $\tilde{H}_{odd} = \sum c_\alpha \tilde{H}_\beta$ where each H_β is like (19), i.e.

$$\tilde{H}_\beta(u_1, v_1, \ldots, u_m, v_m) = \det[w_{i_1}, \ldots w_{i_n}]\langle w_{j_1}, w_{j_2}\rangle \cdot \cdots \cdot \langle w_{j_{2m-n-1}}, w_{j_{2m-n}}\rangle \tag{23}$$

First, let us rearrange the vectors inside the determinant, by possibly changing sign, to take all coupled pairs before any vector which is not coupled inside the determinant i.e. let us rewrite the factor $\det[w_{i_1}, \ldots w_{i_n}]$ in the form

$$\det[u_{s_1}, v_{s_1}, \ldots, u_{s_h}, v_{s_h}, w_{k_1}, \ldots, w_{k_{n-2s_h}}] \tag{24}$$

where each w which appears in (24) has its mate $\overline{w}$ outside the determinant.

Next, for each unmatched w inside the determinant, $\overline{w}$ appears in one and only one of the factors outside the determinant, say $\langle\overline{w}, w_{j_1}\rangle$. Then, either $\overline{w}_{j_1}$ is inside the determinant, and we stop the building of the chain of products, or it appears in one of the factor outside the determinant, say $\langle\overline{w}_{j_1}, w_{j_2}\rangle$. Iterating this procedure we extract a chain of products

$$\langle\overline{w}, w_{j_1}\rangle\langle\overline{w}_{j_1}, w_{j_2}\rangle \cdot\cdots\cdot \langle\overline{w}_{j_{p_1}}, w'\rangle \tag{25}$$

such that $\overline{w}'$ is inside the determinant. By possibly changing sign, starting from $w = w_{k_1}$ we can rearrange (23) by taking the chain (25) inside the determinant, in the following way.

$$\det[u_{s_1}, v_{s_1}, \dots, u_{s_h}, v_{s_h}, w, \langle\overline{w}, w_{j_1}\rangle\langle\overline{w}_{j_1}, w_{j_2}\rangle\cdot\cdots\cdot\langle\overline{w}_{j_{p_1}}, w'\rangle\overline{w}', \dots]\cdot\langle\ ,\ \rangle\cdots\langle\ ,\ \rangle \tag{26}$$

If there is another element inside the determinant which has its mate outside, we repeat the same procedure until we put $\tilde{H}_\beta$ in the following form

$$\begin{gathered}\det[u_{s_1}, v_{s_1}, \dots, u_{s_h}, v_{s_h}, w_{r_1}, \langle\overline{w}_{r_1}, w_{j_1}\rangle\langle\overline{w}_{j_1}, w_{j_2}\rangle \cdot\cdots\cdot \langle\overline{w}_{j_{p_1}}, w_{r_2}\rangle\overline{w}_{r_2}, \dots,\\ w_{r_{n-2s_h-1}}, \langle\overline{w}_{r_{n-2s_h-1}}, w_{t_1}\rangle\langle\overline{w}_{t_1}, w_{t_2}\rangle\cdots\langle\overline{w}_{t_{p_{m-s_h}}}, w_{r_{n-2s_h}}\rangle\overline{w}_{r_{n-2s_h}}]\cdot\langle w_{h_1}, w_{h_2}\rangle\cdots\langle\ ,\ \rangle\end{gathered} \tag{27}$$

By Theorem 2.16, we can write the determinant in (27) in the following way

$$\begin{gathered}Q[u_{s_1}\otimes v_{s_1}, \dots, u_{s_h}\otimes v_{s_h}, w_{r_1}\otimes\langle\overline{w}_{r_1}, w_{j_1}\rangle\langle\overline{w}_{j_1}, w_{j_2}\rangle\cdot\cdots\cdot\langle\overline{w}_{j_{p_1}}, w_{r_2}\rangle\overline{w}_{r_2}, \dots,\\ w_{r_{n-2s_h-1}}\otimes\langle\overline{w}_{r_{n-2s_h-1}}, w_{t_1}\rangle\langle\overline{w}_{t_1}, w_{t_2}\rangle\cdot\cdots\cdot\langle\overline{w}_{t_{p_{m-s_h}}}, w_{r_{n-2s_h}}\rangle\overline{w}_{r_{n-2s_h}}]\end{gathered} \tag{28}$$

By Lemma 2.17 part i), (28) is therefore equal to

$$\begin{gathered}Q[u_{s_1}\otimes v_{s_1}, \dots, u_{s_h}\otimes v_{s_h}, w_{r_1}\otimes\overline{w}_{r_1}\cdot w_{j_1}\otimes\overline{w}_{j_1}\cdot\cdots\cdot w_{j_{p_1}}\otimes\overline{w}_{j_{p_1}}\cdot w_{r_2}\otimes\overline{w}_{r_2}, \dots,\\ w_{r_{n-2s_h-1}}\otimes\overline{w}_{r_{n-2s_h-1}}\cdot w_{t_1}\otimes\overline{w}_{t_1}\cdot\cdots\cdot w_{t_{p_{m-s_h}}}\otimes\overline{w}_{t_{p_{m-s_h}}}\cdot w_{r_{n-2s_h}}\otimes\overline{w}_{r_{n-2s_h}})\end{gathered} \tag{29}$$

The products outside the determinant in (27) can be dealt with as for monomials of type (18), then we have put $\tilde{H}_\beta$, hence H_{odd}, in the required form. □

Proof of Theorem 1.4 Theorem 1.4 has just been proved for multilinear invariants. The general case follows from Corollary 2.6. In fact, let $W \subseteq \mathcal{A}$ be the subspace of $SO(2n, \mathbb{K})$ invariants and let V be the set of polynomials in $\mathrm{tr}(U_{j_1}\cdots U_{j_k})$ and $Q(U_{k_1}\cdots U_{k_{p_1}}, \dots, U_{s_1}\cdots U_{s_{p_n}})$ (notations as in Theorem 1.4). W is closed under full polarization and restitution by Lemma 2.3. In order to prove that V is closed under full polarization and restitution it is enough to use the linearity of the trace, the multilinearity of Q, the linearity of $\mathcal{P}$ and $\mathcal{R}$ and the Definitions 2.1 and 2.2.

By Theorem 2.19 we have $(V_L)_{even} = (W_L)_{even}$ and $(V_L)_{odd} = (W_L)_{odd}$, hence by Corollary 2.6, $W_{even} = V_{even}$, $W_{odd} = V_{odd}$ hence we get the proof Theorem 1.4 □

3 A basic relation

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two n-tuples of $2n \times 2n$ matrices. By Theorem 1.4 the even invariant

$$Q(X_1, \ldots, X_n) \cdot Q(Y_1, \ldots, Y_n) \tag{30}$$

can be written as a polynomial in $\operatorname{tr}(U_{i_1} U_{i_2} \ldots U_{i_j})$ (notations as in Theorem 1.1). In this Section we want to be more explicit about the form of this relation. Note that it is enough to find the form of this relation for rank one matrices $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ since (30) is multilinear in the variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$.

We begin with an example to illustrate the procedure. Let $n = 2$ and let $X_1 = x_1 \otimes x_2$, $X_2 = x_3 \otimes x_4$, $Y_1 = y_1 \otimes y_2$, $Y_2 = y_3 \otimes y_4$. Then by Theorem 2.16

$$Q(X_1, X_2) \cdot Q(Y_1, Y_2) = \det[x_1, x_2, x_3, x_4] \cdot \det[y_1, y_2, y_3, y_4].$$

By (7) this equals

$$\det \begin{bmatrix} \langle x_1, y_1 \rangle & \langle x_1, y_2 \rangle & \langle x_1, y_3 \rangle & \langle x_1, y_4 \rangle \\ \langle x_2, y_1 \rangle & \langle x_2, y_2 \rangle & \langle x_2, y_3 \rangle & \langle x_2, y_4 \rangle \\ \langle x_3, y_1 \rangle & \langle x_3, y_2 \rangle & \langle x_3, y_3 \rangle & \langle x_3, y_4 \rangle \\ \langle x_4, y_1 \rangle & \langle x_4, y_2 \rangle & \langle x_4, y_3 \rangle & \langle x_4, y_4 \rangle \end{bmatrix}$$

By expanding this determinant we get

$$\sum_{\rho \in S_4} \operatorname{sign}(\rho) \langle x_1, y_{\rho(1)} \rangle \langle x_2, y_{\rho(2)} \rangle \langle x_3, y_{\rho(3)} \rangle \langle x_4, y_{\rho(4)} \rangle$$

We need to write each product $\langle x_1, y_{\rho(1)} \rangle \langle x_2, y_{\rho(2)} \rangle \langle x_3, y_{\rho(3)} \rangle \langle x_4, y_{\rho(4)} \rangle$ as a monomial in the basic invariants of Theorem 1.1. This is achieved by using Corollary 2.18. For example, the factor corresponding to $\rho = (3, 2, 1, 4)$, i.e.

$$\langle x_1, y_3 \rangle \langle x_2, y_2 \rangle \langle x_3, y_1 \rangle \langle x_4, y_4 \rangle \tag{31}$$

can be rearranged as

$$\langle x_1, y_3\rangle\langle y_4, x_4\rangle\langle x_3, y_1\rangle\langle y_2, x_2\rangle \tag{32}$$

hence by Corollary 2.18 it can be rewritten in the form

$$\mathrm{tr}(x_2 \otimes x_1 \cdot y_3 \otimes y_4 \cdot x_4 \otimes x_3 \cdot y_1 \otimes y_2) = \mathrm{tr}(X_1^t \cdot Y_2 \cdot X_2^t \cdot Y_1) \tag{33}$$

By applying the same procedure to all terms we get the desired expression. We now describe the way to transform (30) in general.

Let $X_i = x_{2i-1} \otimes x_{2i}$ and $Y_i = y_{2i-1} \otimes y_{2i}$. Then

$$Q(X_1, \ldots, X_n) \cdot Q(Y_1, \ldots, Y_n) = \det[(\langle x_i, y_j\rangle)] = \sum_{\rho \in S_{2n}} \mathrm{sign}(\rho) \prod_{i=1,\ldots,2n} \langle x_i, y_{\rho(i)}\rangle$$

It is convenient to introduce some simple definitions and constructions related to permutations.

We say that a pair of integers (i, j) is *straight adjacent* if i is odd and $j = i+1$; is *reverse adjacent* if i is even and $j = i-1$. A pair which is either reverse or straight adjacent is said to be *adjacent*.

An *adjacency cycle* is an arrangement of an even number if integers

$$\begin{pmatrix} i_1 & \ldots & i_{2k} \\ j_1 & \ldots & j_{2k} \end{pmatrix} \tag{34}$$

such that:

1. the integers in each row are distinct;
2. (i_1, i_{2k}), $(i_2, i_3), (i_4, i_5), \ldots, (i_{2k-2}, i_{2k-1})$ are adjacent pairs;
3. $(j_1, j_2), (j_3, j_4), \ldots, (j_{k-1}, j_k)$ are adjacent pairs.

For example

$$\begin{pmatrix} 1 & 6 & 5 & 9 & 10 & 2 \\ 4 & 3 & 5 & 6 & 7 & 8 \end{pmatrix} \tag{35}$$

is an adjacency cycle.

If

$$\rho = \begin{pmatrix} i_1 & \ldots & i_{2n} \\ j_1 & \ldots & j_{2n} \end{pmatrix}$$

is a permutation of en even number of integers we can rearrange it by displaying its adjacency cycles as in the following example (the two adjacency cycles are separated by a vertical line).

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 3 & 7 & 1 & 4 & 9 & 5 & 8 & 6 & 2 \end{pmatrix} =$$
$$\left(\begin{array}{cccccc|cccc} 3 & 8 & 7 & 9 & 10 & 4 & 1 & 6 & 5 & 2 \\ 7 & 8 & 5 & 6 & 2 & 1 & 10 & 9 & 4 & 3 \end{array}\right) \tag{36}$$

In general we can describe a simple iterative algorithm to rewrite any permutation of an even number of integers in a form in which its adjacency cycles are displayed side by side as in (36). The result of the application of this algorithm will be called the *adjacency form* of the permutation.

Assume we have already found h adjacency cycles, i.e. assume we have written ρ in the form

$$\rho = \left(\begin{array}{ccc|ccc|c} i_1 & \dots & i_{k_1} & i_{k_1+1} & \dots & i_{k_1+k_2} & \dots \\ j_1 & \dots & j_{k_1} & j_{k_1+1} & \dots & j_{k_1+k_2} & \dots \end{array}\right.$$
$$\left.\begin{array}{c|ccc|ccc} \dots & i_{k_1+k_2+\dots+k_{h-1}+1} & \dots & i_{k_1+k_2+\dots+k_s} & i_{l_1} & \dots & i_{l_p} \\ \dots & j_{k_1+k_2+\dots+k_{h-1}+1} & \dots & j_{k_1+k_2+\dots+k_s} & j_{l_1} & \dots & j_{l_p} \end{array}\right) \tag{37}$$

where the first s blocks are in adjacency form. We rearrange the columns in the remaining part

$$\begin{pmatrix} i_{l_1} & \dots & i_{l_p} \\ j_{l_1} & \dots & j_{l_p} \end{pmatrix} \tag{38}$$

according to the following inductive rule. The first column will be the one containing the lowest index in the first row. The i-th column will be chosen as follows: if i is even, choose the column containing in the second row the index adjacent to that on the second row of the $(i-1)$-th column; if i is odd, choose the column containing in the first row the index adjacent to that on the first row of the $(i-1)$-th column. We keep on going with this rule until we get a column whose first row contains the index adjacent to the one in the first row of the first column (necessarily after an even number of steps). If there remain indexes outside the new adjacency cycle we repeat the procedure with the remaining indexes until we find all adjacency cycles. Let k_i be the number of columns in the i-th block. We finally rearrange the order of the blocks in such a way that $k_1 \geq k_2 \geq \dots \geq k_h$. If two blocks have the same length we consider first the one having the lowest index in the first row of the first column.

Lemma 3.1 *Let*

$$\rho = \begin{pmatrix} i_1 \dots & i_{2n} \\ j_1 \dots & j_{2n} \end{pmatrix} \tag{39}$$

be a permutation of an even number of integers in adjacency form. Let a be the number of reverse adjacent pairs in the first row and let b be the number of reverse adjacent pairs in the second row. Then

$$sign(\rho) = (-1)^{a+b} \tag{40}$$

Proof Let

$$\sigma = \begin{pmatrix} 1 \dots & 2n \\ i_1 \dots & i_{2n} \end{pmatrix} \tag{41}$$

and let

$$\tau = \begin{pmatrix} 1 \dots & 2n \\ j_1 \dots & j_{2n} \end{pmatrix} \tag{42}$$

Then $\tau^{-1} \circ \rho \circ \sigma$ is equal to the identity, hence $\text{sign}(\rho) = \text{sign}(\tau) \cdot \text{sign}(\sigma)$. The indexes $\tau(1), \dots \tau(2n)$ can be rearranged in increasing order by first transposing the b reverse adjacent pairs which gives the contribution $(-1)^b$ to the sign of τ and then permuting adjacent pairs, which do not change the sign. The same for σ □

We discuss now how to write each element

$$\langle x_1, y_{\rho(1)} \rangle \dots \langle x_{2n}, y_{\rho(2n)} \rangle \tag{43}$$

as a product of traces of products of the matrices X_i and Y_j.

First, write the permutation ρ in adjacency form. Let

$$\begin{pmatrix} i_1 & \dots & i_{2k} \\ \rho(i_1) & \dots & \rho(_{2k}) \end{pmatrix} \tag{44}$$

be any of its adjacency cycles. Let

$$\langle x_{i_1}, y_{\rho(i_1)} \rangle \langle x_{i_2}, y_{\rho(i_2)} \rangle \dots \langle x_{i_k}, y_{\rho(i_k)} \rangle \tag{45}$$

be the corresponding sub product of (43). We can write it as

$$\langle x_{i_1}, y_{\rho(i_1)} \rangle \langle y_{\rho(i_2)}, x_{i_2} \rangle \dots \langle y_{\rho(i_k)}, x_{i_k} \rangle \tag{46}$$

by reversing each product $\langle x_{i_{2s}}, y_{\rho(i_{2s})} \rangle$ at even position.

In this way the indexes of the elements in position $2i$, $2i+1$ are adjacent. Note that the indexes of the first and the last element are adjacent. By Corollary 2.18, the expression (46) equals

$$\text{tr}(x_{i_k} \otimes x_{i_1} \cdot y_{\rho(i_1)} \otimes y_{\rho(i_2)} \cdot x_{i_2} \otimes x_{i_3} \cdots y_{\rho(i_{k-1})} \otimes y_{\rho(i_k)}) \tag{47}$$

We introduce the last piece of notation. Let $\gamma = (\gamma_1, \gamma_2)$ be an adjacent pair of integers. We define

$$\mathcal{I}(\gamma) = \begin{cases} \gamma_2/2 & \text{if the pair } \gamma \text{ is straight adjacent} \\ \gamma_1/2 & \text{if the pair } \gamma \text{ is reverse adjacent} \end{cases}$$

and

$$\mathcal{S}(\gamma) = \begin{cases} 1 \text{ if the pair } \gamma \text{ is straight adjacent} \\ t \text{ (for transposition) if the pair } \gamma \text{ is reverse adjacent} \end{cases}$$

Hence (47) can be written

$$\operatorname{tr}\left(X^{\mathcal{S}(i_k,i_1)}_{\mathcal{I}(i_k,i_1)} \cdot Y^{\mathcal{S}(\rho(i_1),\rho(i_2))}_{\mathcal{I}(\rho(i_1),\rho(i_2))} \cdots Y^{\mathcal{S}(\rho(i_{k-1}),\rho(i_k))}_{\mathcal{I}(\rho(i_{k-1}),\rho(i_k))} \right) \tag{48}$$

We denote the expression (47) corresponding to an adjacency cycle c of ρ by tr_c. We have therefore

$$Q(X_1, \ldots, X_n)Q(Y_1, \ldots, Y_n) = \sum_{\rho \in S_{2n}} \operatorname{sign}(\rho) \prod_{c_i \in \mathcal{C}(\rho)} \operatorname{tr}_{c_i} \tag{49}$$

It is possible to have a more explicit form for (49) but we need to introduce some further notation.

Let $P(n)$ denote the set of all partitions of the integer n i.e. the set of all integer sequences

$$\mathbf{k} = (k_1, \ldots, k_h)$$

such that $k_1 \geq k_2 \geq \cdots \geq k_h \geq 1$ and $\sum_{i=1}^h k_i = n$ and let Q_n be the set of functions from $\{1, \ldots, n\}$ to the symbols $\{1, t\}$ (t stands for "transposition"). For each partition $\mathbf{k} \in P_n$, let $S_n(\mathbf{k})$ be the set of permutations $\sigma \in S_n$ such that

$$\begin{aligned} \sigma(1) &= \min\{\sigma(1), \ldots, \sigma(k_1)\} \\ \sigma(k_1+1) &= \min\{\sigma(k_1+1), \ldots, \sigma(k_1+k_2)\} \\ &\cdots \\ \sigma(k_1+\cdots+k_{h-1}+1) &= \min\{\sigma(k_1+\cdots+k_{h-1}+1), \ldots, \sigma(n)\}, \end{aligned}$$

and such that if $k_i = k_{i+1}$ then $\sigma(k_1 + \cdots + k_{i-1} + 1) < \sigma(k_1 + \cdots + k_i) + 1$.

Let

$$Q_n(\mathbf{k}) = \{f \in Q_n \ s.t.\ f(1) = f(k_1+1) = f(k_1+k_2+1) = \cdots = f(k_1+\cdots+k_{h-1}+1) = t\} \tag{50}$$

We are now ready to state our result.

Theorem 3.2 *The invariant* $Q(X_1,\ldots,X_n)Q(Y_1,\ldots,Y_n)$ *is equal to*

$$\sum_{\mathbf{k}\in P(n)} \sum_{\sigma\in S_n(\mathbf{k}),\tau\in S_n,\alpha\in Q_n(\mathbf{k}),\beta\in Q_n} \epsilon(\alpha,\beta)\mathrm{tr}(X_{\sigma(1)}^{\alpha(1)}Y_{\tau(1)}^{\beta(1)}\ldots X_{\sigma(k_1)}^{\alpha(k_1)}Y_{\tau(k_1)}^{\beta(k_1)})\cdot$$
$$\mathrm{tr}(X_{\sigma(k_1+1)}^{\alpha(k_1+1)}Y_{\tau(k_1+1)}^{\beta(k_1+1)}\ldots X_{\sigma(k_1+k_2)}^{\alpha(k_1+k_2)}Y_{\tau(k_1+k_2)}^{\beta(k_1+k_2)})\cdot$$
$$\ldots$$
$$\cdot\,\mathrm{tr}(X_{\sigma(k_1+\cdots+k_{h-1}+1)}^{\alpha(k_1+\cdots+k_{h-1}+1)}Y_{\tau(k_1+\cdots+k_{h-1}+1)}^{\beta(k_1+\cdots+k_{h-1}+1)}\ldots X_{\sigma(n)}^{\alpha(n)}Y_{\tau(n)}^{\beta(n)}) \tag{51}$$

where $\epsilon(\alpha,\beta)$ *is defined as follows. Let* h *be the number of integers in the partition* $\mathbf{k}$*, let* a *be the number of indexes* j *such that* $\alpha(j)=t$ *and let* b *be the number of indexes* j *such that* $\beta(j)=t$*; then* $\epsilon(\alpha,\beta)=(-1)^{a+b-h}$.

Proof Let $\rho\in S_{2n}$ be a permutation and let $\mathcal{C}(\rho)=\{c_1,\ldots,c_h\}$ be the set of adjacency cycles of ρ. Let $\mathbf{k}=(k_1,\ldots,k_h)$ be the partition of n associated to $\mathcal{C}(\rho)$. Let

$$c=\begin{pmatrix} i_1 & \ldots & i_{2p} \\ j_1 & \ldots & j_{2p}\end{pmatrix} \tag{52}$$

be an adjacency cycle $\rho\in S_{2n}$. Out of c we extract the following two ordered lists of adjacency pairs: from the first row

$$(i_{2p},i_1),(i_2,i_3),\ldots,(i_{2p-2},i_{2p-1})$$

and from the second row

$$(j_1,j_2),(j_3,j_4),\ldots,(j_{2p-1},j_{2p}).$$

Making this extraction cycle by cycle of the adjacency form of a permutation, we associate to a permutation two ordered lists of adjacency pairs: from the first row

$$\mu_1,\ldots,\mu_n$$

and from the second row

$$\nu_1,\ldots,\nu_n$$

Let us define two permutations $\sigma,\tau\in S_n$ by

$$\sigma(i)=\mathcal{I}(\mu_i) \qquad \tau(i)=\mathcal{I}(\nu_i)$$

and two functions $\alpha,\beta:\{1,\ldots,n\}\to\{1,t\}$ by

$$\alpha(i)=\mathcal{C}(\mu_i) \qquad \tau(i)=\mathcal{C}(\nu_i).$$

Note that if $\mathbf{k}$ is the partition associated to ρ, then $\sigma \in S_n(\mathbf{k})$ and $\alpha \in Q_n(\mathbf{k})$.

We have shown in this way how each factor

$$\prod_{c_i \in \mathcal{C}(\rho)} \mathrm{tr}_{c_i}$$

in (49) corresponds to one and only one product of traces in (51). For showing that (49) is equivalent to (51) it remains only to check that also the signs match and this follows from Lemma 3.1. $\square$

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