WHERE TO FIND OPERATOR ALGEBRAS

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Abstract

Today, we hear often about quantum groups, subfactors, quantum Galois theory, Jones' index theory, knot theory, non-commutative geometry and so forth and are told that the theory behind these are operator algebras. But other than these abstract or fancy terminologies, what are they? Are operator algebras visible for us without a fancy back ground? I want to present some evidence that operator algebras are around in our mathematical every day life.

In the talk, I want to talk about a basic question in harmonic analysis. First we take the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$, the first Hilbert space which students meet in their mathematical life. The problem we want to tackle is that, fixing a pure periodic wave function, say V, on $\mathbb{R}$, i.e., $\exp(2\pi \mathrm{i}\theta x)$ and its translation U by fixed amount, say 1, we want to generate a lot of functions as a linear combinations of them. More precisely, set

$$(U\xi)(x) = \xi(x+1) \quad (V\xi)(x) = \exp(2\pi \mathrm{i}\theta x)\xi(x), \quad \xi \in L^2(\mathbb{R}).$$

Can we find a function $\xi_0 \in L^2(\mathbb{R})$ such that repeated applications of U and V to ξ_0 generate a dense set of $L^2(\mathbb{R})$? Of course, if $\theta \in \mathbb{Q}$, then it is rather easy to find a solution. So we want to study the case that θ is irrational. I want to explore with you into this question.