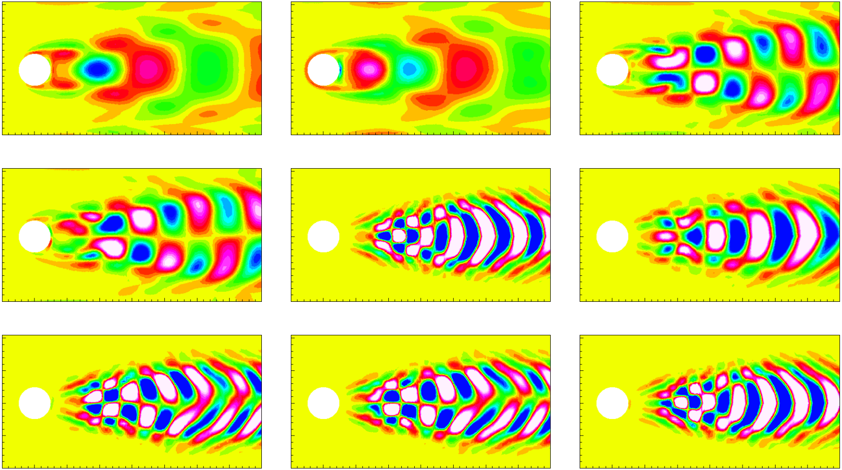
Reduced-Order Model Development and Control Design

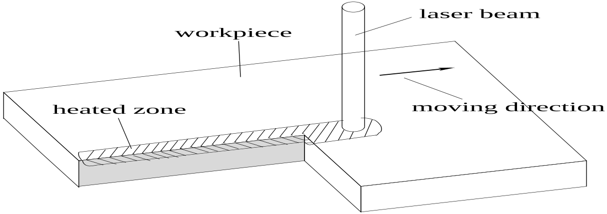
K. Kunisch

Department of Mathematics
University of Graz, Austria

Rome, January 06



Laser-Oberflächenhärtung eines Stahl-Werkstückes [WIAS-Berlin]:



Phasenübergänge:

Ferrit
Pearlit
Bainit
Martensit

} $\xrightarrow{\text{heizen}}$ Austenit $\xrightarrow{\text{kühlen}}$ {

Ferrit
Pearlit
Bainit
Martensit

Data: • $T > 0$, $\Omega \subset \mathbb{R}^d$ bounded, $d = 2$ or 3

• ϱ, c, k, L positive constants

• $\theta_\circ \in H^1(\Omega)$, $\alpha \in L^\infty(Q)$ non-negativ

Energy-Balance:

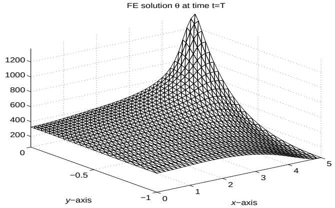
$$\begin{cases} \varrho c\theta_t - k\Delta\theta = \alpha u - \varrho L a_t & \text{in } Q = (0,T) \times \Omega \\ \frac{\partial\theta}{\partial n} = 0 & \text{on } \Sigma = (0,T) \times \partial\Omega \\ \theta(0,\cdot) = \theta_\circ & \text{in } \Omega \end{cases}$$

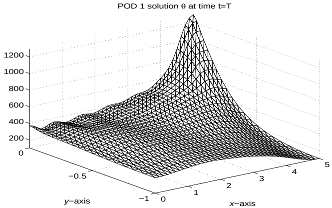
Austenit-phasetransition: $a_t = f(\theta, a)$ in Q, $\quad a(0,\cdot) = 0$ in Ω

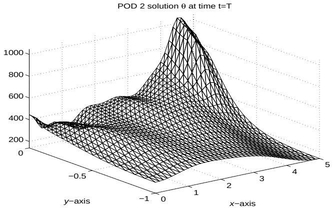
state-variables: θ and a

control-parameter: $u = u(t) \in L^2(0,T)$ (function of time)

c.f. S.Volkwein







note the scales

$$\Psi^i_{L^\infty} = \frac{\max\limits_{0\le j\le m} \|\theta^j_\ell - \theta^j_{FE}\|_{L^\infty(\Omega)}}{\max\limits_{0\le j\le m} \|\theta^j_{FE}\|_{L^\infty(\Omega)}}, \quad \Psi^i_{L^2} = \left(\frac{\sum\limits_{j=0}^{m} \|\theta^j_\ell - \theta^j_{FE}\|^2_{L^2(\Omega)}}{\sum\limits_{j=0}^{m} \|\theta^j_{FE}\|^2_{L^2(\Omega)}}\right)^{1/2} \begin{cases} i=1 & \text{POD with DQ} \\ i=2 & \text{POD without DQ} \end{cases}$$

	$X = L^2(\Omega)$				$X = H^1(\Omega)$			
ℓ	$\Psi^1_{L^\infty}$	$\Psi^2_{L^\infty}$	$\Psi^1_{L^2}$	$\Psi^2_{L^2}$	$\Psi^1_{L^\infty}$	$\Psi^2_{L^\infty}$	$\Psi^1_{L^2}$	$\Psi^2_{L^2}$
10	24.1%	40.6%	11.3%	12.1%	21.0%	40.1%	22.9%	11.8%
15	7.9%	38.4%	3.4%	6.8%	16.2%	37.8%	4.4%	6.7%
20	6.0%	35.3%	1.8%	4.3%	13.5%	34.3%	2.2%	4.3%
25	1.6%	26.9%	0.6%	2.9%	4.0%	24.6%	1.2%	2.9%

error-quotient: $\mathcal{E}(\ell) = \sum\limits_{i=1}^{\ell} \lambda_i \Big/ \sum\limits_{i=1}^{d} \lambda_i \cdot 100\% \ge 94\%$

	$\ell = 5$	$\ell = 10$	$\ell = 15$	$\ell = 20$	$\ell = 25$	$\ell = 30$
$\mathcal{E}(\ell)$, $X = L^2(\Omega)$	79.6	94.3	98.4	99.5	99.8	99.9
$\mathcal{E}(\ell)$, $X = H^1(\Omega)$	53.0	77.7	87.4	92.5	95.7	97.6

OS-POD

$$
(P^{\ell}_{OSP}) \begin{cases}
\min J^{\ell}(x,\psi,u) \text{ over } (x,\psi,u) \in L^2(\mathbb{R}^{\ell}) \times X^{\ell} \times L^2(0,T;\mathbb{R}^m), \\
\text{subject to} \\
E(\psi)\dot{x}(t) + A(\psi)x(t) + \mathfrak{N}(x(t),\psi) = B(\psi)u(t), \quad t \in (0,T], \\
E(\psi)x(0) = x_0(\psi), \\
\frac{d}{dt}y(t) + \mathcal{A}y(t) + \mathcal{N}(y(t)) = \mathcal{B}u(t), \quad t \in (0,T], \\
y(0) = y_0, \\
\mathcal{R}(y)\psi_i = \lambda_i\,\psi_i\,, \qquad i = 1,\dots,\ell, \\
\langle \psi_i,\, \psi_j \rangle = \delta_{ij} \qquad i,j = 1,\dots,\ell.
\end{cases}
$$

The following optimality system holds:

$$\begin{cases} -E(\psi)\,\dot{q}(t) + (\mathcal{A}(\psi) + \mathcal{N}_x^T(x,\psi))\,q(t) = -J_x^\ell(x,\psi,u), \\ q(T) = 0 \end{cases}$$

$$\begin{cases} -\dot{p}(t) + \mathcal{A}p(t) + \mathcal{N}'(y(t))^*\,p(t) \\ \qquad = \sum_{i=1}^{\ell} \langle y(t),\mu_i\rangle_X \mathcal{I}^{-1}\psi_i + \langle y(t),\psi_i\rangle_X \mathcal{I}^{-1}\mu_i \\ p(T) = 0 \end{cases}$$

$$\begin{cases} \eta_i = -\frac{1}{2}\,\langle \mathcal{G}_i(x,\psi,u,q),\psi_i\rangle_{X^*,X} \\ \mu_i = -(\mathcal{R} - \lambda_i\,I)^{-1}\,[2\,\eta_i\,\psi_i + \mathcal{I}\,\mathcal{G}_i(x,\psi,u,q)], \text{ for } i = 1,\ldots,\ell, \end{cases}$$

and

$$\mathbf{R}u(t) = B^T(\psi)\,q(t) + \mathcal{B}^*p(t).$$

TRUST REGION – POD

$$\text{Minimize}_{u \in U_{ad}} \; J(u)$$

Assume that current iterate u_k is given. Build quadratic model around u_k.

$$m_k(u_k + s) = J(u_k) + g_k^T s + \frac{1}{2} s^T H_k s$$

For the computation of the next step solve

$$\text{Minimize } m_k(u_k + s), \quad ||s||_2 \leq \delta_k$$

where δ_k is current trust region radius.

c.f. M.Fahl, E.Sachs

Full

$$\min \quad J(u) = \frac{1}{2}\int_0^T \|y(u;t,x) - y^d(t,x)\|_{L^2}^2 \, dt$$

where

- $y(t,x) = y(u;t,x)$ solution to the PDE for given control $u(x,t)$
- $y^d(t,x)$ desired state

Reduced

$$\min \quad f^{POD}(u) = \frac{1}{2}\int_0^T ||y^{POD}(u;t,x) - y^d(t,x)||_{L^2}^2 \, dt$$

where $y^{POD} = y^{POD}(u)$ POD solution for given control u(t).
? when update POD-basis ? dimension of POD basis

Consider

$$m_k(u_k + s) = J^{POD}(u_k + s)$$

as local model for J around u_k.

Unlike common quadratic trust region model functions we have

- $m_k(u_k + s)$ is not quadratic in s
- $m_k(u_k) \neq J(u_k)$
- $g_k := \nabla m_k(u_k) \neq \nabla J(u_k)$

Algorithm 1: Outline of TR-POD Alg.

Let $0 < \eta_1 < \eta_2 < 1$, $0 < \gamma_1 \leq \gamma_2 < 1 \leq \gamma_3$ and $u_1, \delta_1 > 0$, set $k = 1$.

1. Compute snapshot set corresponding to control u_k
2. Compute POD basis and build POD control model
3. Compute minimizer s_k of

$$\min_{||s||_2 \leq \delta_k} m_k(u_k + s)$$

4. Compute $J(u_k + s_k)$ and

$$\rho_k = \frac{J(u_k) - J(u_k + s_k)}{m_k(u_k) - m_k(u_k + s_k)}$$

5. Update trust region:
   - If $\rho_k \geq \eta_2$: set $u_{k+1} = u_k + s_k$ and increase trust region radius $\delta_{k+1} = \gamma_3 \delta_k$
   - If $\eta_1 < \rho_k < \eta_2$: set $u_{k+1} = u_k + s_k$ and decrease trust region radius $\delta_{k+1} = \gamma_2 \delta_k$
   - If $\rho_k \leq \eta_1$: set $u_{k+1} = u_k$ and decrease trust region radius $\delta_{k+1} = \gamma_1 \delta_k$
6. Set $k = k + 1$ and GOTO 1

Algorithm 2: Step determination algorithm (Toint '88)

Let $0 < \alpha < \beta < 1$, $0 < \nu_1 \leq 1$, $\nu_2 > 0$, $\nu_3 > 0$, $0 < \mu \leq 1$ be given.

Phase 1: Find λ_k^A such that

$$m_k(u_k - \lambda_k^A g_k) \leq m_k(u_k) - \alpha\lambda_k^A ||g_k||^2$$

$$||\lambda_k^A g_k||_2 \leq \delta_k$$

and

$$\lambda_k^A \geq \min\{\frac{\delta_k}{||g_k||}, \nu_2\} \text{ (there are alternatives)}$$

Phase 2: If $\delta_k \geq \nu_3$: Choose step s_k such that

$$m_k(u_k) - m_k(u_k + s_k) \geq \mu(m_k(u_k) - m_k(u_k - \lambda_k^A g_k))$$

$$||s_k||_2 \leq \delta_k$$

Convergence

Regularity assumption on f

Assumptions on model m_k

- each m_k is differentiable
- $\frac{||g_k - \nabla f(u_k)||}{||g_k||} \leq \varsigma$ for some $\varsigma > 0$ (Carter '91).

Sufficient decrease condition for Algorithm 2

$$J(u_k) - J(u_k + s_k) \geq c||g_k||^2 \min\{\delta_k, \|g_k\|^2\}$$

Theorem

Let J satisfy the standard assumptions. Assume that $\{u_k\}$ is a sequence of iterates generated by Algorithm 1 with step determination according to Algorithm 2 and

$$\frac{||g_k - \nabla J(u_k)||}{||g_k||} \leq \varsigma \quad \textit{for all } k$$

for some ς with $0 < \varsigma < 1 - \eta_2$.
Then

$$\lim_{k\to\infty} ||\nabla J(u_k)|| = 0.$$

Note: Consistency of gradients in POD framework true in practice, but no theory.

Example (Sachs et alt.)

k	$J(u_k)$	$m_k(u_k)$	δ_k	$\|\|s_k\|\|_2$	ρ_k	dim
0	0.229239	0.118274	2	2.00	2.36	5
1	0.149097	0.148143	4	4.00	1.27	5
2	0.080427	0.080532	8	6.84	1.47	5
3	0.006864	0.008835	16	2.87	0.93	6
4	0.000246	0.000795	32	0.29	3.61	8
5	0.000132					

BALANCED TRUNCATION and POD

$$\begin{cases} \dot{y} = Ay + Bu\,, \quad y(0) = y_0 \in, \mathbb{C}^n \\ z = Cy \end{cases}$$

$A \dots$ stable, i.e.: $Re\, \lambda(A) < 0$.

controllability operator

$\Psi_c : L^2(-\infty, 0] \to \mathbb{C}^n\,, \quad \Psi_c(u) = \int_{-\infty}^{0} e^{-At} Bu(t)dt$

controllability Gramian

$Z_c = \Psi_c \Psi_c^* = \int_{-\infty}^{0} e^{-At} B\, B^* e^{-A^* t} dt = \int_0^{\infty} e^{At} B\, B^* e^{A^* t} dt$

Z_c *pos. def. iff* (A, B) controllable

Liapunov equation $AZ_c + Z_c A^* + BB^* = 0$

ref: Dullerud-Paganini, A Course in Robust Control; C.W. Rowley, Int.J. Bifurcation and Chaos

observability operator

$$\Psi_o : \mathbb{C}^n \to L^2(0,\infty), \quad \Psi_o(y_o) = \begin{cases} C\, e^{At} y_o & \ldots\, t \geq 0 \\ 0 & \ldots\, t < 0. \end{cases}$$

observability Gramian:

$$Z_o = \Psi_o^* \Psi_o = \int_0^\infty e^{A^* s} C^* C\, e^{As} ds$$

$$Z_o \, pos.def.iff.(C, A) \text{ observable}, \;\; \|z\|^2_{L^2(o,\infty)} = y_o^* Z_o y_o.$$

Liapunov equation $A^* Z_o + Z_o A + C^* C = 0.$

$\{\eta_i\} \ldots EVE \; (Z_o^{1/2})$,

$\mathcal{E}_o = \{Z_o^{1/2} y_o : |y_o|_{\mathbb{C}^n} \leq 1\}$

recall

control-Liapunov equation $A Z_c + Z_c A^* + BB^* = 0.$

$$\mathcal{E}_c = \{\Psi_c u : \|u\|_{L^2} \leq 1\} = \{Z_c^{1/2} y_c : |y_c|_{\mathbb{C}^n} \leq 1\}$$

Change of basis

$$\tilde{y} = Ty\,, \quad \tilde{Z}_c = T\,Z_c T^*\,, \quad \tilde{Z}_o = (T^*)^{-1} Z_o\,T^{-1}$$

If (A, B, C) controllable and observable $\exists$ balanced realization, i.e. $\exists\ T$:

$$\tilde{Z}_c = \tilde{Z}_o = \Sigma = diag(\sigma_1, \ldots, \sigma_n). \quad \text{(Hankel singular values)}$$

$$Z_c\,Z_o\,T = T\,\Sigma^2, \text{ or}$$

$$Z_c = L\,L^* \text{ (Cholesky)}, \quad 0 < L^*\,Z_o\,L = U\,\Sigma^2\,U^*,\; U \ldots \text{orthogonal}$$

$$T = \Sigma^{1/2}\,U^*\,L^{-1}.$$

balanced truncation: project onto first r columns of T.
reduced state again balanced and stable

Error bounds

input-output operator: $(Gu)(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau$
transfer function: $\hat{G}(s) = C(sI - A)^{-1}B$

$$||\hat{G}||_\infty = \max_\omega \sigma_1(\hat{G}(i\omega)) = \max_u \frac{|Gu|_{L^2}}{|u|_{L^2}}$$

reduction to r dimensions: $||\hat{G}_r - G||_\infty > \sigma_{r+1}$

balanced truncation: $||\hat{G}_r^{bal} - G||_\infty < 2\sum_{j=r+1}^{n} \sigma_j$

but not optimal (Hankel-norm reductions)
1

[1] next we recall POD formulas

$\mathcal{Y}_n : \mathbb{R}^n \to X, \; \mathcal{Y}_n v = \sum_{j=0}^n \beta_j \, v_j \, y(t_j), \; \mathcal{Y}_n^* z = \left(\langle z, y(t_j)\rangle_X, .., \langle z, y(t_n)\rangle_X\right)$

$\mathcal{R}_n = \mathcal{Y}_n \mathcal{Y}_n^* = \sum_{j=0}^n \beta_j y(t_j) \otimes y(t_j) = \sum_{j=0}^n \beta_j y(t_j) \langle y(t_j), \cdot \rangle_X \in \mathcal{L}(X)$

$\mathcal{K}_n = \mathcal{Y}_n^* \mathcal{Y}_n = \langle y(t_i), y(t_j) \rangle_X \in \mathbb{R}^{n \times n}$

$$\mathcal{R}_n \ldots \{\sigma_j\}, \{\varphi_j\}; \quad \mathcal{K}_n \ldots \{\sigma_j\}, \{\psi_j\}; \qquad \varphi_k = \frac{1}{\sqrt{\sigma_k}} \sum_{j=0}^n (\psi_k)_j y_j$$

$$\mathcal{Y} : L^2(0, T; \mathbb{R}) \to X, \quad \mathcal{Y} v = \int_0^T v(t) y(t, \cdot) \, dt$$

$$\mathcal{R} = \mathcal{Y}\mathcal{Y}^* = \int_0^T y(t) \otimes y(t) \, dt \quad \in \mathcal{L}(X)$$

$$\mathcal{K} = \mathcal{Y}^* \mathcal{Y} = \int_0^T \bullet \langle y(t, \cdot), y(t, \cdot) \rangle_X \, d\tau \;\in\; \mathcal{L}(L^2, L^2) \text{ ... Hilbert-Schmidt}$$

X... finite dimensional, SVD: $\mathcal{Y}_n \psi_j = \sqrt{\sigma_j} \, \varphi_j, \quad \mathcal{Y}_n^* \varphi_j = \sqrt{\sigma_j} \, \psi_j$

$\mathcal{R}_n \varphi_j = \mathcal{Y}_n \mathcal{Y}_n^* \varphi_j = \sigma_j \varphi_j, \quad y_j = \sum_{i=0}^n \langle \varphi_i, y_j \rangle_X \varphi_i$

POD

$+$ is concept for nonlinear systems

$-$ high-energy modes are not necessarily most important to dynamics

Balanced truncation

$+$ takes into consideration input - output information

$-$ restricted to linear (stable) systems.

Connections POD–Balanced Truncation

controllability Gramian: $Z_c = \int_0^\infty e^{At} B B^* e^{A^* t} dt$

e.g.: $B = [b_1, \ldots, b_m]$

$y_i = e^{At} b_i, \quad Z_c = \int_0^\infty y_1(t) y_1(t)^t, \ldots, y_m(t) y_m(t)^t$

correlation matrix: $\mathcal{R} = \mathcal{Y}\mathcal{Y}^* = \int_0^T y(t)\, y(t)^t\, dt$

discretized $Z_c = Y_c Y_c^*, \quad Z_0 = Y_o^* Y_o \quad \in \mathbb{R}^{n \times n}$

$Y_c \in \mathbb{R}^{n \times m}$, rectangular ! SVD of $Y_o Y_c$

c.f. C.W. Rowley, Int. J. on Bifurcation and Chaos, to appear.

Connections POD–Balanced Truncation

controllability Gramian: $Z_c = \int_0^\infty e^{At} B B^* e^{A^* t} dt$

e.g.: $B = [b_1, \ldots, b_m]$

$y_i = e^{At} b_i, \quad Z_c = \int_0^\infty y_1(t) y_1(t)^t, \ldots, y_m(t) y_m(t)^t$

correlation matrix: $\mathcal{R} = \mathcal{Y}\mathcal{Y}^* = \int_0^T y(t) y(t)^t \, dt$

discretized $Z_c = Y_c Y_c^*, \quad Z_0 = Y_o^* Y_o \quad \in \mathbb{R}^{n \times n}$

$Y_c \in \mathbb{R}^{n \times m}$, rectangular ! SVD of $Y_o Y_c$

c.f. C.W. Rowley, Int. J. on Bifurcation and Chaos, to appear.

Connections POD–Balanced Truncation

controllability Gramian: $Z_c = \int_0^\infty e^{At} B B^* e^{A^* t} dt$

e.g.: $B = [b_1, \ldots, b_m]$

$y_i = e^{At} b_i, \quad Z_c = \int_0^\infty y_1(t) y_1(t)^t, \ldots, y_m(t) y_m(t)^t$

correlation matrix: $\mathcal{R} = \mathcal{Y}\mathcal{Y}^* = \int_0^T y(t)\, y(t)^t \, dt$

discretized $Z_c = Y_c Y_c^*, \quad Z_0 = Y_o^* Y_o \quad \in \mathbb{R}^{n \times n}$

$Y_c \in \mathbb{R}^{n \times m}$, rectangular ! SVD of $Y_o Y_c$

c.f. C.W. Rowley, Int. J. on Bifurcation and Chaos, to appear.

$(*) \qquad Y_o Y_c = U \Sigma V^*\,, \quad$ rectangular, $\Sigma \in \mathbb{R}^{n \times n}, \quad V \in \mathbb{R}^{m \times n}$

Theorem

If rank $\Sigma = n$, *then balanced truncation matrix is given by*

$$T = Y_c V \Sigma^{-1/2}\,, \quad T^{-1} = \Sigma^{-1/2} U Y_o,$$

and Σ *contains Hankel singular values.*

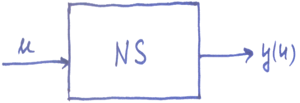
weighted inner product on $\mathbb{R}^n$: $(a,b)_{Z_o} = a^t\, Z_o\, b$

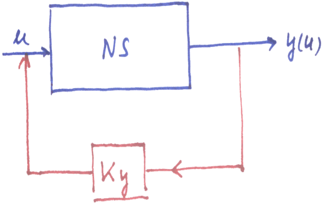
POD-modes: $\quad Y_c\, Y_c^*\, Z_o\, \tilde{\varphi} = Z_c\, Z_o\, \tilde{\varphi} = \tilde{\lambda}\, \tilde{\varphi}$

equivalently $\quad Y_c^*\, Z_o\, Y_c\, \tilde{\psi} = \tilde{\lambda}\, \tilde{\psi} \overset{(*)}{=} V\, \Sigma^2\, V^* \tilde{\psi}$

$\Rightarrow EVE(Y_c^*\, Z_o\, Y_c) = V\, \Sigma^{-2} \;\Rightarrow\; EVE(Z_c\, Z_o) = Y_c\, V\, \Sigma^{-1}$

CLOSED LOOP CONTROL BASED ON POD





$$(P)\begin{cases} \min_{u\in U_{ad}} \int_0^T e^{-\mu t} h(y) + \frac{\beta}{2}\int_0^T |u|^2 \\ \dot{y} = f(y) + Bu(t)\,, \quad y(0) = y_0 \end{cases}$$

f linear $u = -B^T \Pi y$; henceforth f nonlinear

$\upsilon = \upsilon(t, y)$ ... value functional $\qquad \nabla = \nabla_y$

$$\begin{cases} \upsilon_t + \mu\upsilon = \min_{u\in U_{ad}} \left((f(t,y) + B^T u, \nabla\upsilon) + h(y) + \frac{\beta}{2}|u|^2 \right) \\ \upsilon(T,\cdot) = 0 \end{cases}$$

$$u = P_{U_{ad}}\left(-\frac{1}{\beta} B^T \nabla \upsilon\right)$$

$$(HJB)\begin{cases} v_t + \mu v = \big(f(t,y), \nabla v(t,y)\big) + h(y) - \frac{\beta}{2}\,|P_U(-\frac{1}{\beta}B^T \nabla v(t,y))|^2 \\ v(T,\cdot) = 0 \end{cases}$$

close the loop $\qquad u(t) = P_{U_{ad}}(-\frac{1}{\beta}\,B^T \nabla\, v(t, y(t)))$

1) 2 point BVP-solutions

$\mu = 0$, no control constraints

$$(OS)\begin{cases} \dot{y} = f(y) + Bu\,, & y(0) = y_0 \\ \dot{\lambda} + f'(y)\lambda = -h'(y)\,, & \lambda(T) = 0 \\ u = \frac{1}{\beta}\,\lambda \end{cases}$$

$$u(t) = -\tfrac{1}{\beta} B^T \nabla_y v(t, y(t)) = \mathcal{F}_T(t, y(t)) = \tfrac{1}{\beta}\lambda(t; y_0)$$
$$= \mathcal{F}_{T-t}(0, y(t)) \sim \mathcal{F}_T(0, y(t)) = \tfrac{1}{\beta}\,\lambda\,(0; y(t))$$

finite dimensional realization

(i) choose grid $\sum \subset \mathbb{R}^n$ containing optimal trajectory

(ii) calculate $(y(\cdot, \bar{y}_0),\ \lambda(\cdot, \bar{y}_0))$ for all $\bar{y}_0 \in \sum$, thus $\mathcal{F}_T(0, \bar{y}_0)$ available

(iii) use interpolation for $\mathcal{F}_T(0, y)$, $y \in \mathbb{R}^n$ arbitrary.

Remark: (OS) must be solved often ! (gridpoints in one direction)$^{\#\text{ basis}}$ $\rightarrow$ POD use for open loop optimal control to estimate values in $\sum$.

Example

(Control-to-zero of Burgers equation).

	uncontrolled	FE	POD	closed loop
cost	0.3038	0.1694	0.1704	0.1733

2) POD-HJB

Problem Setting

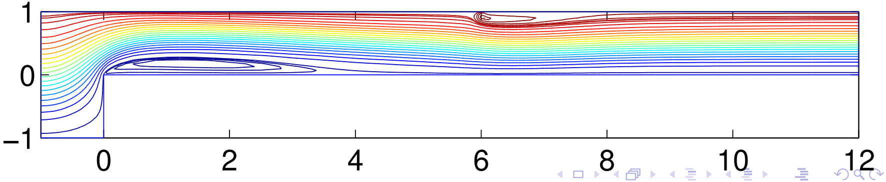
$$(NSt) \qquad \begin{cases} y_t - \frac{1}{Re}\Delta y + (y\cdot\nabla)y + \nabla p = B\mathbf{u} \\ div\, y = 0,\ y(0,\cdot) = y_0,\ \text{B.C.} \end{cases}$$

$$B\mathbf{u} = \sum_{i=1}^{m} \hat{\varphi}_i\, u_i(t)$$

$$\min J(u) = \frac{1}{2}\int_0^\infty \int_{\Omega_0} e^{-\mu t}|y - y_{\text{Stokes}}|^2 dxdt + \frac{\beta}{2}\int_0^\infty e^{-\mu t}|u(t)|^2 dt$$

subject to $u \in U_{ad}$ and (NSt).

Streamline, Re=1000



Dynamic Programming Principle

$$\begin{cases} \min J(y_0, u) = \int_0^\infty L(y(t), u(t)) e^{-\mu t} dt \\ \dot{y}(t) = f(y(t), u(t)), \quad y(0) = y_0,\ u \in \mathcal{U}_{ad}. \end{cases}$$

value function $v(y_0) = \min_{u \in \mathcal{U}_{ad}} J(y_0, u)$.

(DPP) $\quad v(y_0) = \min_{u \in \mathcal{U}_{ad}} \{\int_0^T L(y(t; y_0, u), u(t) e^{-\mu t} dt + v(y(T; y_0, u)) e^{-\mu T}\}$

(HJB) $\quad \mu v(y_0) = \min_{u \in \mathcal{U}_{ad}} \{\nabla v(y_0) \cdot f(y_0, u) + L(y_0, u)\}$
replace y_0 by $y^*(t)$.

$$u^*(t) = \mathcal{F}(y^*(t)) \qquad \text{where}$$

$\mathcal{F}(y^*(t)) \in \text{argmin}_{u \in \mathcal{U}_{ad}} \{\nabla v(y^*(t)) \cdot f(y^*(t), u) + L(y^*(t), u)\}$

close the loop

t-discretisation

$$t_j = jh$$

$$\begin{cases} y_{j+1} = y_j + h\,f(y_j, u_j) \\ y_0 \ \text{ given} \end{cases}$$

$$J_h(y_0, u_h) = \frac{h}{2}[L(y_0, u_0) + \sum_{j=1}^{\infty} e^{-\mu hj}(L(y_j, u_{j-1}) + L(y_j, u_j))].$$

$$\upsilon_h(y_0) = \inf_{u_h \in \mathcal{U}_{ad}^h} J_h(y_0, u_h)$$

$$(HJB_h) \quad \upsilon_h(y_0) = \inf_{u \in \mathcal{U}_{ad}^h} \{\frac{h}{2}[L(y_0, u) + e^{-\mu h} L(y_0 + hf(y_0, u), u)] + e^{-\mu h} \upsilon_h(y_0 + hf(y_0, u))\}$$

$$y_{j+1}^* = y_j^* + hf(y_j^*, S_h(y_j^*)), j \geq 0$$

numerically still infeasible

POD-Galerkin Ansatz:

$$y^l(t,x) = \sum_{i=1}^{l} \alpha_i(t)\varphi_i(x), \qquad \mathbf{u}(t,x) = \sum u_i(t)\,\hat{\varphi}_i\,(x),$$

where $\{\hat{\varphi}_i\}$ POD basis from $\{y^0\,(t_k) - y_{stokes}\}$

$y^0\,(t_k)$ ... no control $\;\rightarrow \hat{\varphi}_i$ injection on top filtered out.

$$\dot{\alpha} = -A\alpha - \alpha^T H\alpha + G(u)$$

$$A_{ij} = \frac{1}{Re}\langle\nabla\varphi_i,\ \nabla\varphi_j\rangle, \quad H_{ijk} = \langle\varphi_j \cdot \nabla\varphi_k,\ \varphi_i\rangle,$$

$$G_i = \langle\mathbf{u}, \varphi_i\rangle$$

Implementation Aspects

for NS: finite difference upwind scheme on staggered
nonuniform grid, fractional ϑ-scheme for time-integration.

$$(HJB_h)\ v_h(\alpha) = \inf_{u \in \mathcal{U}^h_{ad}} \{\frac{h}{2}\Big(L(\alpha,u) + e^{-\mu h} L(\alpha + hf(\alpha,u),u)\Big) + e^{-\mu h} v(\alpha + hf(\alpha,u))\}$$

hypercube Γ_h in $\mathbb{R}^n$ such that $\alpha^*(t) \in \Gamma_h$ for all t,
α_j grid points in Γ_h, $\alpha = \sum \lambda_j \alpha_j \quad , \quad \sum \lambda_j = 1$

$$v_h(\alpha) = \sum \lambda_j v_h(\alpha_j)$$

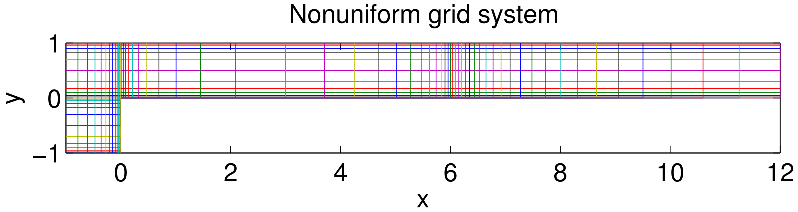
Solve (HJB_h) as fixed point equation,
(Bardi, Capuzzo-Dolcetta; Gonzales-Rofman)
nonuniform grid $\sim$ decay of POD eigenvalues.

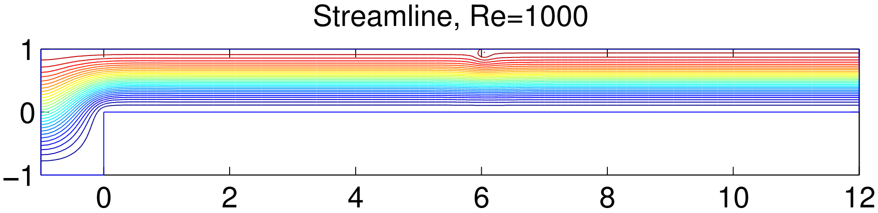
Constrained nonlinear programming problem at each grid point of Γ_h : in parallel.

Validation

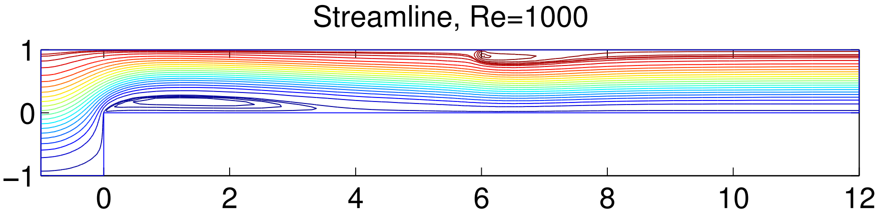
Reynolds number, *Re*	1000
Time horizon, T	10
No. of basis functions	6
No. of controls	2
Bounds on controls	$[-0.5, 0.5]$
Grid system	$12 \times 6 \times 6 \times 4 \times 3 \times 3$
Discount rate, μ	2
Observe region, Ω_o	$[-1, 12] \times [0, 1]$

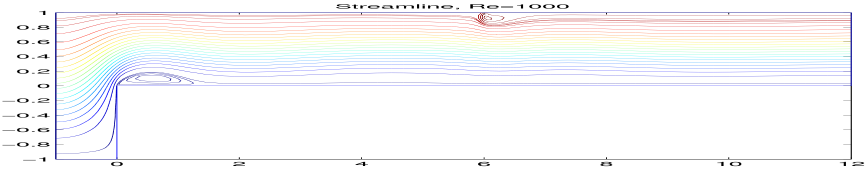
Table: Parameter Settings


Nonuniform grid system
y
1
0
–1
x
0
2
4
6
8
10
12


Streamline, Re=1000
1
0
–1
0
2
4
6
8
10
12

Stokes flow


Streamline, Re=1000
1
0
−1
0
2
4
6
8
10
12


Streamline, Re=1000
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
2
4
6
8
10
12

Boundary Control

no injection on top

construction of shape functions

$$\mathbf{u} = u(t)\,\chi(x)\,,\ x \in \Gamma_c$$

$$y_{ref} = y_{stokes}|_{u\equiv 2} - y_{stokes}|_{u\equiv 0}$$

$$\hat{\mathcal{V}} = span\{y(t_k)|_{u\equiv 0}\,,\ y(t_k)|_{u\equiv 2} - y_{ref}\}_{k=1}^{n}$$

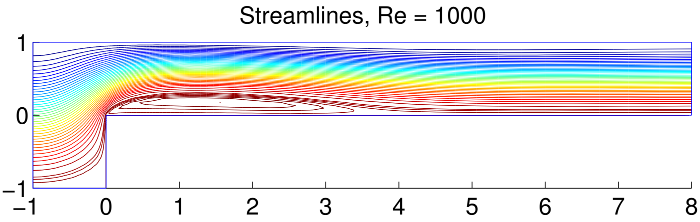
$$\mathcal{V} = \{\hat{\mathcal{V}} - y_m\}\,, \quad y_m = \frac{1}{2n}\sum_{k=1}^{2n}\hat{\mathcal{V}}$$

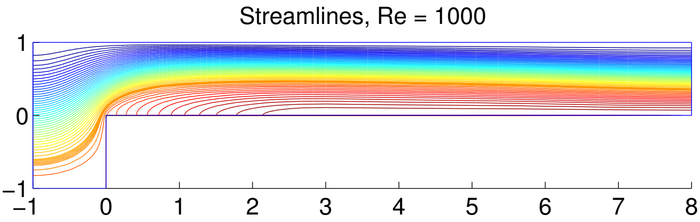
$$POD(\mathcal{V}) = \{\varphi_i\}$$

Ansatz for controlled solution

$y^n(t) = y_m + \sum_{i=1}^{2n} \alpha_i(t)\varphi_i + u(t)y_{ref}$.

Boundary Control





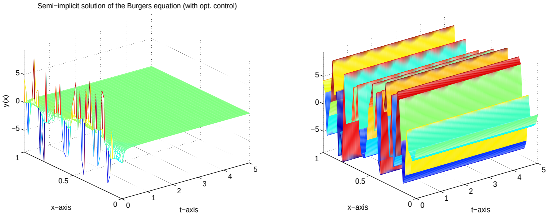


FIGURE 1. Optimal state with random noise (9.0) in the initial condition: feedback design (left) and open-loop design (right).

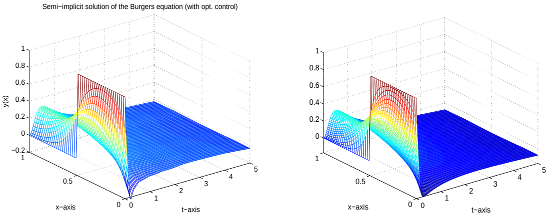


FIGURE 2. Optimal state with random noise (0.25) in the RHS: feedback design (left) and open-loop design (right).