Proper Orthogonal Decomposition: Introductory Remarks [1]

1 Generalities

Let X be a separable, real Hilbert space, and $y \in L^2(0,T;X)$, typically the trajectory of a differential equation, and typically $X = L^2(\Omega)$, or $X = H_0^1(\Omega)$, or $X = H^1(\Omega)$. The POD-operator

$$\mathcal{R} : X \to X$$

associated to y is given by

$$\mathcal{R}\psi = \int_0^T \langle y(t), \psi \rangle_X \, y(t) dt. \tag{1.1}$$

Clearly $\mathcal{R}$ is a bounded selfadjoint, nonnegative, operator, which can be expressed as

$$\mathcal{R} = \mathcal{Y}\mathcal{Y}^*,$$

where

$$\mathcal{Y} : L^2(0,T;\mathbb{R}) \to X$$

is given by

$$\mathcal{Y}v = \int_0^T v(t) y(t) dt,$$

with adjoint $\mathcal{Y}^* : X \to L^2(0,T;\mathbb{R})$

$$\mathcal{Y}^* z(t) = \langle y(t), z \rangle_X .$$

We further define

$$\mathcal{K} : L^2(0,T) \to L^2(0,T)$$

by $\mathcal{K} = \mathcal{Y}^*\mathcal{Y}$, i.e.

$$(\mathcal{K}v)(t) = \int_0^T \langle y(t), y(s) \rangle_X \, v(s) ds.$$

[1] January 5, 2006

Note that $\mathcal{K}$ is a bounded, nonnegative selfadjoint operator. Moreover for the kernel of the integral operator $\mathcal{K}$ we have

$$\int_0^T \int_0^T |\langle y(t), y(s)\rangle_X|^2 ds\, dt = \int_0^T |y(t)|^2 dt \int_0^T |y(s)|^2 ds < \infty,$$

hence $\mathcal{K}$ is a Hilbert-Schmidt operator, in particular $\mathcal{K}$ is a compact operator with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq 0$.

Theorem 1.1. *Let $y \in L^2(0,T;X)$. Then the operator $\mathcal{K}$ is compact. Moreover, except for possibly 0, $\mathcal{K}$ and $\mathcal{R}$ possess the same eigenvalues which are positive with identical multiplicities and ψ is eigenvector of $\mathcal{R}$ if and only if $\mathcal{Y}^*\psi = \langle y(t,\cdot), \psi\rangle_X \, y(t,\cdot)$ is an eigenvector of $\mathcal{K}$.*

Proof. Since the kernel of $\mathcal{K}$ is square integrable on $(0,T)\times(0,T)$ the integral operator $\mathcal{K}$ is Hilbert-Schmidt and therefore compact. Its non-zero spectral values are necessarily eigenvalues with finite multiplicity and their only possible accumulation point is 0. If φ is an eigenvector of $\mathcal{K}$ with eigenvalue $\lambda \neq 0$ then $\mathcal{Y}^*\mathcal{Y}\varphi = \lambda\varphi$ and thus $\mathcal{Y}\varphi$ is an eigenvector of $\mathcal{R}$. Analogously, if ψ is an eigenvector of $\mathcal{R}$ with eigenvalue $\lambda \neq 0$ then $\mathcal{Y}^*\psi$ is an eigenvector of $\mathcal{K}$. Let $\lambda \neq 0$ be an eigenvalue of $\mathcal{K}$ and let $ker\{\mathcal{K} - \lambda I\} = span\{\varphi_i\}_{i=l}^r$, with $\{\varphi_i\}_{i=l}^r$ linearly independent. Then $\{\mathcal{Y}\varphi_i\}_{i=l}^r$ are linearly independent. If not, then there exist α_i with $\Pi_{i=l}^r \alpha_i \neq 0$ such that $\sum_{i=l}^r \alpha_i \, \mathcal{Y}\varphi_i = 0$. This implies

$$0 = \sum_{i=l}^r \alpha_i \, \mathcal{K}\, \varphi_i = \lambda \sum_{i=l}^r \alpha_i \, \varphi_i,$$

which is impossible, since $\lambda \neq 0$ and $\{\varphi_i\}_{i=l}^r$ are linearly independent. Hence $dim\, ker\{\mathcal{R} - \lambda I\} \geq dim\, ker\{\mathcal{K} - \alpha I\}$. The converse inequality follows analogously and hence

$$dim\, ker\{\mathcal{R} - \lambda I\} = dim\, ker\{\mathcal{K} - \lambda I\}.$$

□

Definition 1.1. *The nontrivial, decreasing eigenvalues of $\mathcal{R}$ are called the POD eigenvalues associated to y. The corresponding orthonormalized eigenvectors are called the POD eigenvectors.*

Remark 1.1. The POD eigenvectors depend on X. - If $y \in L^2(0,T;H) \cap L^2(0,T;V)$ then $X = Hand\ X = V$ lead to different POD bases.

Remark 1.2. (Method of snapshots). Let $y \in C(0,T;X) \subset L^2(0,T;X)$ and choose time instances $0 \leq t_1 \leq \ldots \leq t_n = T$. Set

$$y_i = y(t_i),$$

$$\mathcal{R} : X \to X$$

$$\mathcal{R}\psi = \sum_{i=1}^{n} \langle y_i, \psi \rangle y_i \tag{1.2}$$

If $X = \mathbb{R}^N$, and $y_i \in \mathbb{R}^N$, column vectors, then

$$\mathcal{R} = YY^*, \quad \text{where}$$

$$Y = (y_1, \ldots, y_n) \in \mathbb{R}^{N \times n}.$$

In fact,

$$(\mathcal{R}\psi)_k = \sum_i \sum_j \langle y_{ji}, \psi_j \rangle y_{ki} = \sum_j \sum_i y_{ki}\, y_{ji}\, \psi_j.$$

$$\Longrightarrow \mathcal{R}_{\text{finite}} = YY^* \in \mathbb{R}^{N \times N}.$$

$$\mathcal{Y} : \mathbb{R}^n \to X\,, \quad \mathcal{Y}(v) = \sum_{i=1}^{n} v_i\, y_i$$

$$\mathcal{Y}^* : X \to \mathbb{R}^N\,, \quad \mathcal{Y}^* z = col\langle z, y_i \rangle\,, \quad \mathcal{R} = \mathcal{Y}\mathcal{Y}^*.$$

$$\mathcal{K} = \mathcal{Y}^*\mathcal{Y} : \mathbb{R}^n \to \mathbb{R}^n$$

$$(\mathcal{K}v)_j = \sum_{i=1}^{n} v_i \langle y_i, y_j \rangle_X, \quad \text{correlation matrix.}$$

If $X = \mathbb{R}^N$ then $\mathcal{K}_{\text{finite}} = Y^*Y \in \mathbb{R}^{n \times n}$.

In numerical practice the POD eigenvalues/eigenvectors are calculated from $\mathcal{K}_{\text{finite}}$ respectively $\mathcal{R}_{\text{finite}}$, whichever has smaller dimension. From the proof of Theorem 1.1 we can transfer between them. In fact, if

$$\mathcal{K}v_k = \lambda_k v_k,$$

then

$$\mathcal{R}\mathcal{Y}v_k = \lambda_k \mathcal{Y}v_k \text{ , i.e.}$$

$$\tilde{\psi}_k = \mathcal{Y}v_k = \sum_{i=1}^{n} (v_k)_i \, y_i$$

satisfies

$$\mathcal{R}\tilde{\psi}_k = \lambda_k \, \tilde{\psi}_k,$$

i.e. $\tilde{\psi}_k$ is a POD-eigenvector with POD-eigenvalue λ_k.

If $|v_k|_{\mathbb{R}^n} = 1$ then

$$|\psi_k|_X^2 = |\mathcal{Y}v_k|^2 = (\mathcal{Y}^*\mathcal{Y}v_k, v_k)_{\mathbb{R}^n} = (\mathcal{K}v_k, v_k) = \lambda_k |v_k|^2 = \lambda_k,$$

and hence $\psi_k = \frac{1}{\sqrt{\lambda_k}}\mathcal{Y}v_k$ is a normalized eigenvector. Note that the method of snapshots coincides with the "continuous POD" calculations explained above, if y is piecewise constant.

For reasons of scaling $\int_0^T$ in (1) is sometimes replaced by $\frac{1}{T}\int_0^T$, and $\sum_{i=1}^n$ in (1.2) by $\frac{1}{n}\sum_{i=1}^n$. Some authors investigate "weighted POD", i.e. (1.1) is replaced by

$$\mathcal{R}_w(\psi) = \int_0^T \langle y(t), \psi \rangle_X \, y(t) \, w(t) \, dt, \tag{1.3}$$

for a nonnegative weight function w. If one admits distributions for w, then (1.2) can be considered as a special case of (1.3).

Remark 1.3. Consider the linear control system

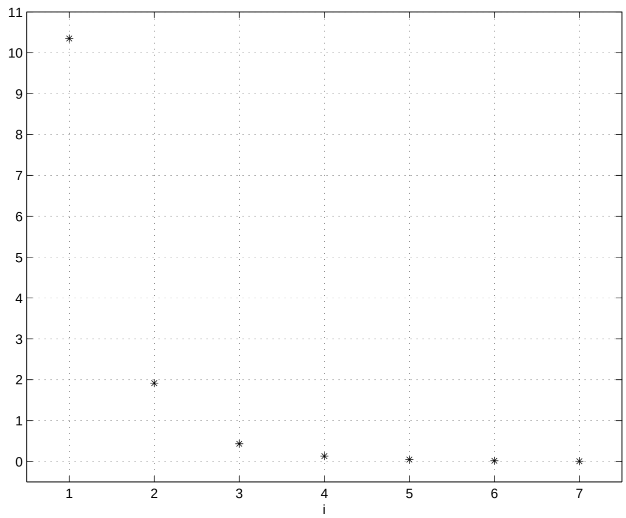
$$\dot{x} = Ax + Bu \, , \quad A \in \mathbb{R}^{n \times n} \, , \quad B \in \mathbb{R}^{n \times 1}, \tag{1.4}$$

For $u = \delta$, the Delta-distribution with weight at 0, $x_\delta = e^{tA}B$ is the input response. The associated POD operator is

$$\mathcal{R} = W_c = \int_0^T e^{tA} BB^* e^{tA^*} dt \in \mathbb{R}^{n \times n},$$

which in control theory is referred to as the controllability matrix. System (1.4) is controllable if $rank(W_c) = n$.

In the following plot the decay of the POD eigenvalues is plotted. We point at the rapid decay of these values which is typical for many POD-computations.



2 Properties of POD

Theorem 2.1. *The POD-eigenvectors associated to $y \in L^2(0,T;X)$ are characterized by*

$$\int_0^T \frac{\langle y(t),\, \psi\rangle_X^2}{\langle \psi, \psi\rangle_X} dt = \max_{\varphi \in X\setminus\{0\}} \int_0^T \frac{\langle y(t), \varphi\rangle_X^2}{\langle \varphi, \varphi\rangle_X} dt. \tag{2.1}$$

For $l \geq 1$ the vector ψ_{l+1} is the solution to

$$\int_0^T \frac{\langle y(t),\, \psi\rangle_X^2}{\langle \psi, \psi\rangle_X} dt = \max_{\varphi \in \ span(\psi_1,\ldots,\psi_l)^\perp} \int_0^T \frac{\langle y(t), \varphi\rangle_X}{\langle \varphi, \varphi\rangle_X} dt. \tag{2.2}$$

Proof. Note that (2.1) (equivalently (2.2)) is equivalent to

$$\frac{\langle R\psi, \psi\rangle_X}{\langle \psi, \psi\rangle_X} = \max_{\varphi \in X\setminus\{0\}} \frac{\langle R\varphi, \varphi\rangle_X}{\langle \varphi, \varphi\rangle_X}. \tag{2.3}$$

Clearly ψ_1 solves (2.3). Alternatively, if ψ is solution to (2.3), then for $\varphi \in X \setminus \{0\}$

$$f_\varphi(s) = \frac{\langle R(\psi + s\varphi), \psi + s\varphi\rangle_X}{\langle \psi + s\varphi,\, \psi + s\varphi\rangle_X}.$$

Note that $f_\varphi(s) \geq f_\varphi(0)$, for all s, and hence, since f_φ is differentiable,

$$f'_\varphi(0) = 0,$$

which can equivalently be expressed as

$$\langle \mathcal{R}\psi, \varphi\rangle\langle \psi, \psi\rangle = \langle \mathcal{R}\psi, \psi\rangle\langle \psi, \varphi\rangle, \quad \text{for all } \varphi \in X.$$

Thus $\lambda = \frac{\langle \mathcal{R}\psi, \psi\rangle}{\langle \psi, \psi\rangle}$ is an eigenvalue of $\mathcal{R}$ with associated eigenvector ψ. Moreover ψ solves (2.3), so λ is the largest eigenvalue. $\square$

Remark 2.1. Thus, among all normalized vectors, ψ_1 is in average most aligned to y.

Theorem 2.2. *Let $\{\psi_k\}$ be the POD-vectors associated to $\mathcal{R}$ in X. Then, for all $l \geq 0$ and all orthonormal families $\{\varphi_k\}$ we have*

$$\int_0^T |y(t) - \sum_{k=1}^{l} \langle y(t), \psi_k\rangle_X\, \psi_k|^2 dt \leq \int_0^T |y(t) - \sum_{k=1}^{l} \langle y(t), \varphi_k\rangle_X\, \varphi_k|^2 dt, \tag{2.4}$$

and

$$\int_0^T |y(t) - \sum_{k=1}^{l} \langle y(t), \psi_k\rangle_X \psi_k|^2 dt = \sum_{k=l+1}^{\infty} \lambda_k \xrightarrow{l \to \infty} 0 \tag{2.5}$$

Proof. Let $v_k = \frac{1}{\sqrt{\lambda_k}} \mathcal{Y}^* \psi_k$ denote the POD-vectors of $\mathcal{K}$. From Hilbert-Schmidt theory the kernel $\tilde{k}(t,s) = \langle y(t), y(s)\rangle_X$ can be expressed as

$$\tilde{k}(t,s) = \sum_{k=1}^{\infty} \lambda_k\, v_k(t)\, v_k(s).$$

Therefore

$$|y|^2_{L^2(0,T;X)} = \int_0^T \langle y(t), y(t)\rangle_X \, dt = \int_0^T \tilde{k}(t,t)dt$$
$$= \sum_{k=1}^{\infty} \lambda_k \int_0^T v_k^2(t)dt = \sum_{k=1}^{\infty} \lambda_k,$$

and thus $\sum_{k=1}^{\infty} \lambda_k < \infty$. Moreover

(2.6)
$$\int_0^T |y(t) - \sum_{k=1}^{l} \langle y(t), \psi_k\rangle_X \, \psi_k|^2 dt = \int_0^T |y(t)|^2 dt - \sum_{k=1}^{l} \int_0^T \langle y(t), \psi_k\rangle_X^2 dt$$
$$= \int_0^T |y(t)|^2 dt - \sum_{k=1}^{l} \langle \mathcal{R}\psi_k, \psi_k\rangle_X = \int_0^T |y(t)|^2 dt - \sum_{k=1}^{l} \lambda_k,$$

so that (2.5) follows, and

$$y(t) = \sum_{k=1}^{\infty} \langle y(t), \psi_k\rangle_X \, \psi_k \quad \text{for a.e. } t \in (0,T). \tag{2.7}$$

Let $Y = cl\, span\{\psi_k\}$, and $X = Y \bigoplus Y^{\perp}$.

Now let $\{\varphi_k\}$ be an arbitrary orthonormal family in X. Then

$$\int_0^T |y(t) - \sum_{k=1}^{l} \langle y(t), \varphi_k\rangle \varphi_k|_X^2 dt = |y(t)|^2_{L^2} - \sum_{k=1}^{l} \int_0^T \langle y(t), \varphi_k\rangle_X^2 \, dt. \tag{2.8}$$

Since $y(t) \in Y$ we henceforth assume that $\{\varphi_k\} \subset Y$. For all k there exists $\{\varphi_k^n\}_{n=1}^{\infty} \subset span\{\varphi_k\}$ such that

$$\varphi_k^n \to \varphi_k \quad \text{in} \quad X \quad \text{as} \quad n \to \infty,$$

and w.l.o.g, for all n

$$\{\varphi_k^n\}_{1 \le k \le l} \quad \text{is an orthonormal set.}$$

We can then pass to the limit

$$\lim_{n\to\infty} \int_0^T \langle y(t), \varphi_k^n\rangle_X^2 dt = \int_0^T \langle y(t), \varphi_k\rangle_X^2 \, dt$$

and (2.4) follows, provided (2.4) is verified for orthonormal sets $\{\varphi_k\}_{1\le k\le l} \subset span\{\psi_k\}_{k=1}^{\infty}$. Thus let $\{\varphi_k\}_{1\le k\le l}$ be an orthonormal set in $span\{\psi_k\}_{k=1}^{\infty}$. Then there exists $n_0 \ge l$, such that $\varphi_k \in span\{\psi_i\}_{i=1}^{n_0}$, for $1 \le k \le l$. We complete $\{\varphi_k\}_{k=1}^{l}$ to an ONB $\{\varphi_k\}_{k=1}^{n_0}$ of $span\{\psi_i\}_{i=1}^{n_0}$. Let $[\mathcal{R}_\psi]$ (resp. $[\mathcal{R}_\varphi]$) denote the matrix representation of $\mathcal{R}|span\{\psi_i\}_{i=1}^{n_0}$ with respect to the basis $\{\psi\}_{k=1}^{n_0}$ (resp. $\{\varphi\}_{k=1}^{n_0}$). Then

$$[\mathcal{R}\psi] = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_l & & & \\ & & & \lambda_{l+1} & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_{n_0} \end{pmatrix}.$$

Then

$$[\mathcal{R}\psi] = A^{-1}[\mathcal{R}_\varphi]A =: \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

where P is of dimension $l \times l$ and A is the orthogonal matrix characterizing the change of basis. Note that

$$\sum_{k=1}^{l} \int_0^T \langle y(t), \varphi_k\rangle_X^2 \, dt = \sum_{k=1}^{l} (\mathcal{R}\varphi_k, \varphi_k)_X = trace(P), \tag{2.9}$$

$$\sum_{k=1}^{l} \int_0^T \langle y(t), \psi_k\rangle_X \, dt = \sum_{k=1}^{l} (\mathcal{R}\psi_k, \psi_k)_X = \sum_{k=1}^{l} \lambda_k. \tag{2.10}$$

Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_1 is $l \times l$. Since $A^{-1} = A^T$ we find

$$P = A_1^T \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_l \end{pmatrix} + \begin{pmatrix} \lambda_{l+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{n_0} \end{pmatrix} A_3.$$

For each $1 \le i \le l$, noting that $\sum_{k=1}^{n_0} a_{ki}^2 = 1$,

$$p_{ii} = \sum_{k=1}^{n_0} \lambda_k \, a_{ki}^2 \le \sum_{k=1}^{l} \lambda_k \, a_{ki}^2 + \lambda_l (1 - \sum_{k=1}^{l} a_{ki}^2)$$
$$\le \sum_{k=1}^{l} (\lambda_k - \lambda_l) a_{ki}^2 + \lambda_l,$$

and thus

$$trace(P) \le \sum_{i=1}^{l} \sum_{k=1}^{l} (\lambda_k - \lambda_l) a_{ki}^2 + l\lambda_l$$
$$\le \sum_{k=1}^{l} (\lambda_k - \lambda_l) \sum_{i=1}^{l} a_{ki}^2 + l\lambda_l \le \sum_{k=1}^{l} (\lambda_k - \lambda_l) + l\lambda_l = \sum_{k=1}^{l} \lambda_k.$$

This inequality, together with (2.8)-(2.10), imply (2.4).

□

Inequality (2.4), in fact, characterizes the POD-vectors. We have

Theorem 2.3. *Let $\{\psi_k\}_{k\ge 1}$ be an orthonormal family in X so that* (2.4) *holds for all $l \ge 1$ and all orthonormal families $\{\varphi_k\}_{k\ge 1}$. Then $\{\psi_k\}_{k\ge 1}$ is a family of POD-vectors associated to y.*

Proof. Let $\{\hat{\psi}_k\}_{k\ge 1}$ be a family of POD-vectors. Then by (2.4) and (2.6) we find that

$$\langle \mathcal{R}\psi_1, \psi_1 \rangle_X = \langle \mathcal{R}\hat{\psi}_1, \hat{\psi}_1 \rangle_X.$$

By induction we deduce that

$$\langle \mathcal{R}\psi_k, \psi_k \rangle_X = \langle \mathcal{R}\hat{\psi}_k, \hat{\psi}_k \rangle_X,$$

for all $k = 1, \dots$. The claim now follows from the proof of Theorem 2.2, see (2.3).

□

3 Galerkin-POD Approximation

Let V and H be real separable Hilbert spaces with V densely embedded in H, and $V \subset H \subset V^*$ the usual Gelfand triple. Let $a : V \times V \to \mathbb{R}$ be a bilinear continuous elliptic form satisfying for positive constants β and κ

$$|a(\varphi, \psi)| \leq \beta\, |\varphi|_V\, |\psi|_V$$

$$\kappa\, |\varphi|_V^2 \leq a(\varphi, \varphi)$$

for all $\varphi, \psi \in V$. For $y_0 \in H$ and $f \in L^2(0, T; V^*)$ consider

$$(E) \qquad \begin{cases} \frac{d}{dt}(y(t), v)_H + a(y(t), v) = \langle f(t), v\rangle_{V^*,V}, \quad t \in (0, T] \\ (y(0), v)_H = (y_0, v), \quad \text{for all } v \in V. \end{cases}$$

It is known that (E) admits a solution $y \in W(0,T) =$
$= \{y \in L^2(0, T; V) : y_t \in L^2(0, T; V^k)\}$.

(3.1) Choose X as V or H.

Let $\{\psi_k\}$ denote the POD-family associated to y. Using (1.1), note that $\{\psi_k\} \subset V$ regardless of the choice of X. Let $V^l = span\{\psi_k\}_{k=1}^l \subset X$, with $P^l : X \to V^l$ the orthogonal projection We consider the Galerkin approximation to (E)

$$(E^l) \qquad \begin{cases} \frac{d}{dt}(y_l(t), v)_H + a(y, v) = \langle f(t), v\rangle_{V^*,V}, \quad t \in (0, T] \\ (y_l(0), v)_H = (y_0, v)_H\,, \quad \text{for all } v \in V^l. \end{cases}$$

Theorem 3.1. *$y_l \to y$ in $W(0,T)$ as $l \to \infty$.*

Note that the basis elements depend on the solution y.- We next turn to rate of convergence results and consider first the method of snap-shots. Let $t_i = \frac{iT}{n}$ define an equidistant grid in $[0, T]$ and take snapshot $\{y(t_i)\}$. They generate a POD space of dimension $d \leq n$. Consider an implicit Euler scheme for the time discretization of (E^l). Here for simplicity of presentation the grid for the time-discretization is taken identical with the snap-shot grid.

Theorem 3.2. *Let $X = V$ and denote by Y_i the solution to the implicit Euler POD-Galerkin approximation to (E). If $y \in W^{2,2}(0, T; V)$, then there exists C independent of l such that for $\Delta t = \frac{T}{n}$*

$$\frac{1}{n}\sum_{i=1}^{n} |Y_i - y(t_i)|_H^2 \leq C(|y_0 - P_{y_0}^l|_H^2 + (\Delta t)^2 + (\frac{1}{(\Delta t)^2} + 1) \sum_{i=l+1}^{d} \lambda_i).$$

Similar results can be obtained for other time-discretizations.- To avoid the factor $\frac{1}{(\Delta t)^2}$ one can take a different snapshot set, namely

$$\{y_i, \frac{1}{\Delta t}(y_i - y_{j-1})\}_{i=1}^{n}. \tag{3.2}$$

Note that, despite linear dependence of these snapshots, they lead to a different POD family than $\{y_i\}$! (Note also, that we have not addressed the choice of l in applications, so far).

Theorem 3.3. *If the snapshots are taken according to* (3.2)*, and otherwise the same assumptions as in Theorem 3.2 hold, then*

$$\frac{1}{n}\sum_{k=1}^{n} |Y_i - y(t_i)|_H^2 \leq C(|y_0 - P^l y_0|^2 + (\Delta t)^2 + \sum_{i=l+1}^{d} \lambda_i).$$

Theorem 3.4. *If $X = H$ and the snapshots are taken as in* (3.2)*, then we have for the implicit Euler POD-Galerkin approximation to (E):*

$$\frac{1}{n}\sum_{i=1}^{n} |Y_i - y(t_i)|_H^2 \leq C(\|S\|_2\, |y_0 - P^l y_0|_H^2 + (\Delta t)^2 + \sum_{i=l+1}^{d} \lambda_i),$$

where $S \in \mathbb{R}^{l \times l}$ denotes the stiffness matrix with elements $a(\psi_i, \psi_j)$. For v in the POD-subspace we have $|y|_V \leq \sqrt{\|S\|_2}\ |y|_H$, i.e. $\|S\|_2 \to \infty$ if the dimension of the POD-subspace $\to \infty$.

Remark 3.1. Results analogous to Theorems 3.2- 3.4 also hold for certain nonlinear equations, including e.g. the Burgers and Navier Stokes equations.

We now turn to rate of convergence for the semi-continuous Galerkin-POD approximation (E^l), combined with 'continuous-POD'. Let

$$\tilde{y}_l(t) = \sum_{k=l+1}^{\infty} (y(t), \psi_k)_V \, \psi_k.$$

From Theorem 3.1 we have that $\tilde{y}_l\,(t) \to 0$ as $l \to \infty$ for $t \in [0,T]$. Let α denote the embedding constant of $V \to H$.

Theorem 3.5. *Choose $X = V$ and $\varepsilon > 0$ such that $K - \frac{\alpha}{2\varepsilon} - \beta/2 > 0$,*

$$\rho_l := \frac{1}{k - \frac{\alpha}{2\varepsilon} - \frac{\beta}{2}} \, (\frac{\varepsilon}{2} \, |\frac{d\tilde{y}_l}{dt}|^2_{L^2(H)} + \frac{1}{2} \, |\tilde{y}_l\,(0)|^2_H).$$

Then for the solution y_l of (E^l) we have that

$$|y - y_l|^2_{L^2(V)} \leq \rho_l + \frac{(\kappa - \frac{\alpha}{2\varepsilon})T}{\kappa - \frac{\alpha}{2\varepsilon} - \frac{\beta}{2}} \sum_{i=l+1}^{\infty} \lambda_i \xrightarrow{l \to \infty} 0.$$

Remark 3.2. For a POD-scheme with respect to y and $\frac{d}{dt}y$ a similar result as in Theorem 3.5 with smaller constants can be obtained.

Remark 3.3. The choice of l is frequently based on setting some percentage $\delta \in [0,1]$, typically $\delta \in (.95, .99)$ and determining l as the smallest integer such that

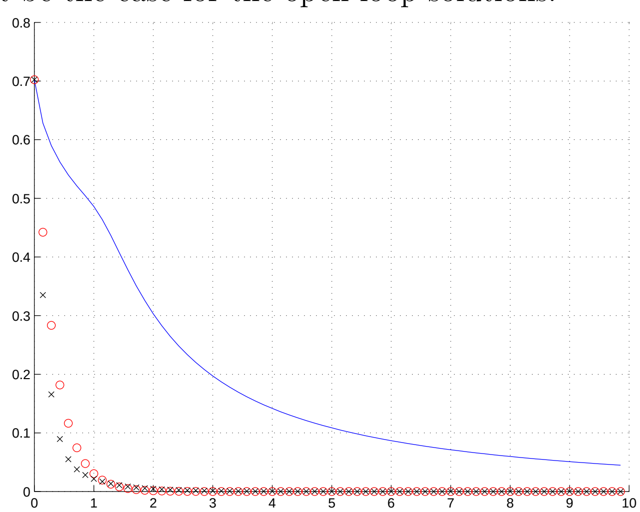
$$\sum_{k=1}^{l} \lambda_k \, / \sum_{k=1}^{\infty} \lambda_k \geq \delta.$$

The motivation for this choice is that $\sum_{k=1}^{\infty} \lambda_k$ represents the "total energy" of the system (E).

Remark 3.4. Let y be a solution to (E), and let $\{\psi_k\}$ be an associated POD-family. By $y^i,\ i = 1, 2$, we denote the POD-Galerkin approximation to (E^l) based on one basis element, ψ_1, and ψ_2, respectively. An example can be constructed, such that

$$|y - y^1|_{L^2(V)} > |y - y^2|_{L^2(V)}.$$

In the following figure the values of a cost-functional for an optimal control problem are plotted against time. The solid line gives the cost for the uncontrolled problem, circles and crosses show the values for the open and closed loop controls. The closed loop controls are obtained by means of solving the HJB-equation, with POP-model reduction of the underlying infintedimensional dynamical system, which is the Burgers equation in this case. We point out that it would be numerically infeasible to solve the HJB equation for a standard finite-element or finite difference discretisation of the Burgers equation. We further point to the fact that the values for the open and closed loop solution are quite close. These are results without noise added to the system. With noise the closed loop solutions are stable, whereas this may not be the case for the open loop solutions.



Most of these notes are based on parts of

T. Henri: Réduction de modèles par des méthodes de décomposition orthogonales propre, thesis, Rennes, 2004.

which in turn uses

K. Kunisch and S. Volkwein: Galerkin proper orthogonal decompostion for a generalized equation in fluid dynamics, SIAM J. Numer. Anal., 40(2002), 492-512.