Numerical discretisation for a stochastic control problem with unbounded controls

a super replication problem arising in Finance

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This is a joint work with

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Outline

Our aim is to present a combination of recent and past technics in order to approximate a nonlinear PDE problem arising in Finance.

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A super replication problem arising in Finance

Let $T > 0$ be a fixed finite time horizon.
Let a given progressively measurable control process $(\rho, \xi) := \{(\rho(t), \xi(t)); 0 \leq t \leq T\}$ with values in $[-1, 1] \times \mathbb{R}_+$ and such that $\int_0^T \xi(t)^2 dt < \infty$.
We consider the controled 2-dimensional (positive) process $(X, Y) = (X^{\rho,\xi}_{t,x,y}, Y^{\rho,\xi}_{t,s,x})$ solution for $t \in [0, T]$ of:

$$\begin{cases} dX(s) = & \sigma(s, Y(s))X(s)\, dW^1(s) \\ dY(s) = -\mu(s, Y(s))\, ds + & \xi(s)Y(s)\, dW^2(s), \end{cases}$$

$$\langle dW^1(s), dW^2(s)\rangle = \rho(s)$$

$$X(t) = x\ ,\ Y(t) = y$$

X: "underlying" asset, Y: "derivative" asset (i.e., volatility);
μ: dividend; σ: volatility (typicaly: $\sigma(t, Y) = \sqrt{Y}$).

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σ and μ satisfy the following assumptions:

(A1) *$\sigma \geq 0$ and $\sigma^2 : [0,T] \times \mathbb{R}_+$ is a locally Lipschitz function, Lipschitz in time, with linear growth with respect to the second argument, and s.t.*

$$\sigma(t,0) = 0, \quad \forall t \in [0,T]$$

(A2) *$\mu : (0,T) \times \mathbb{R}^+ \to \mathbb{R}^+$ is a positive Lipschitz function, with*

$$\mu(t,0) = 0, \quad t \in [0,T].$$

We consider also a (Payoff) function $g : \mathbb{R}_+ \to \mathbb{R}$, and assume

(A3) *g is a bounded, Lipschitz function.*

We consider the following stochastic unbounded control problem:

$$v(t,x,y) = \sup_{(\rho,\xi)} \mathbf{E}\left[g\left(X_{t,x,y}^{\rho,\xi}(T)\right)\right] \tag{1}$$

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A first HJB equation

Let $\alpha = (\rho, \xi) \in [-1, 1] \times \mathbb{R}_+$ and

$$H_\alpha(v) := \mu \frac{\partial v}{\partial x} - \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 v}{\partial x^2} - \bar{\sigma}(\rho\xi)\frac{\partial^2 v}{\partial x \partial y} - \frac{1}{2}\xi^2 \frac{\partial^2 v}{\partial^2 y}$$

with $\bar{\sigma} := x\sigma$.

Theorem

v given by (1) is a viscosity solution of

$$\min_\alpha \left\{ -\frac{\partial v}{\partial t} + H_\alpha(v) \right\} = 0 \qquad (2)$$

Some difficulties:

- discretisation of the controls
- scheme definition
- error estimates

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A first HJB equation
... revisited

Let $H(v) = \min_\alpha H_\alpha(v)$. In fact, we have only (2) if $H(v) > -\infty$, and only get in general $-\frac{\partial v}{\partial t} + H(v) \geq 0$. Then, the exact sense for the HJB equation (2) should be the following (see [Pham 05'], [Soner & Touzi 02']).
Let $G(v) := G(t, x, D_x v, D_x^2 v)$, continuous, be such that

$$H(v) > -\infty \Leftrightarrow G(v) \geq 0.$$

Eq. (2) must be replaced by

$$\min\left\{ -\frac{\partial v}{\partial t} + H(v),\ G(v) \right\} = 0. \qquad (2')$$

Here

$$H(v) > -\infty \quad \Leftrightarrow \quad \left(-\frac{\partial^2 v}{\partial y^2} \geq 0, \ \text{ and } -\frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow -\frac{\partial^2 v}{\partial x \partial y} = 0 \right)$$
$$\Leftrightarrow \quad \Lambda_- \begin{pmatrix} 0 & -\frac{\partial^2 v}{\partial x \partial y} \\ -\frac{\partial^2 v}{\partial x \partial y} & \frac{\partial^2 v}{\partial y^2} \end{pmatrix} \geq 0$$

where $\Lambda_-(M)$ is the smallest eigenvalue of M.

A second HJB equation (case $\mu = 0$)

On the other hand, for v regular, let $M(v) := \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ where $a_{11}(v) := -\frac{\partial v}{\partial t} - \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 v}{\partial x^2}$, $a_{12}(v) := \bar{\sigma} \frac{\partial^2 v}{\partial x \partial y}$, and $a_{22}(v) := -\frac{1}{2}\frac{\partial^2 v}{\partial y^2}$.

$$\Lambda_-(M) = 0 \Leftrightarrow \inf_{\alpha_1^2+\alpha_2^2=1} \left\{ (\alpha_1 \quad \alpha_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right\} = 0$$

$$\Leftrightarrow \min \left\{ \inf_{\alpha_1>0, \alpha_2=\pm\sqrt{1-\alpha_1^2}} \left(a_{11}(v) + 2\frac{\alpha_2}{\alpha_1} a_{12}(v) + (\frac{\alpha_2}{\alpha_1})^2 a_{22}(v) \right), a_{22}(v) \right\} = 0$$

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Boundary conditions

Proposition. v satisfies
(i) The terminal condition

$$v^*(T,x,y) = g(x), \quad (x,y) \in (\mathbb{R}_+)^2.$$

(ii) The boundary condition on $y = 0$:

$$v(t,x,0) = g(x), \quad t > 0,\ x \in \mathbb{R}_+.$$

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approximation scheme

We rewrite the problem for $v(T-t)$ intead of $v(t)$. The equation is similar, with a reversed sign for $\frac{\partial v}{\partial t}$ and an initial condition instead of a terminal one.

Let $dt = T/N$, N integer.

We look for an approximation scheme for $\Lambda_-(M) = 0$, that will compute successive approximation $V^0, V^1, \ldots, V^N$ of the value function at times $t_n = n\Delta t$, $n = 0, \ldots, N$.

By [Souganidis- Barles], under a comparison result, we have a general convergence result if the scheme is

- (i) Monotone (i.e. $V^n \geq U^n \Rightarrow V^{n+1} \geq U^{n+1}$).
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method 1: Semi Lagrangian method

method 2: Generalized finite differences

- Generalized finite differences in space [Bonnans, Ottenwalter, Zidani]
- Euler Implicit (or better) in time

This method is
- local: utilizes close neighboring mesh points (ORDER p)
- can treat non-diagonal dominant diffusion matrices (in 2d)
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method 1: Semi Lagrangian method

method 2: Generalized finite differences

- Generalized finite differences in space [Bonnans, Ottenwalter, Zidani]
- Euler Implicit (or better) in time

This method is
- local: utilizes close neighboring mesh points (ORDER p)
- can treat non-diagonal dominant diffusion matrices (in 2d)
- can be adapted to treat border conditions

Let $\alpha = (\alpha_1, \alpha_2)$ and

$$\mathcal{A}_\alpha(v) := \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix} \begin{pmatrix} a_{11}(v) & a_{12}(v) \\ a_{12}(v) & a_{22}(v) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$= \alpha_1^2 \frac{\partial v}{\partial t} + \frac{1}{2} \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix} \begin{pmatrix} -\bar{\sigma}^2 \frac{\partial^2 v}{\partial x^2} & -\bar{\sigma} \frac{\partial^2 v}{\partial x \partial y} \\ -\bar{\sigma} \frac{\partial^2 v}{\partial x \partial y} & -\frac{\partial^2 v}{\partial y^2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

The spatial approximation for one given control α at time t_n is of the form

$$\mathcal{A}_\alpha(v) \simeq \alpha_1^2 \frac{V^{n+1} - V^n}{\Delta t} + A(\alpha) V^{n+1} - a(\alpha)$$

$$\simeq B(\alpha) V^{n+1} - b^{V^n}(\alpha)$$

where V^n vector, $B(\alpha) := \frac{\alpha_1^2}{\Delta t} I + A(\alpha)$ and $b(\alpha) := a(\alpha) + \frac{\alpha_1^2}{\Delta t} V^n$.

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The scheme in abstract form

- The scheme writes (without control discretisation):

$$\min_u (B(u)V^{n+1} - b^{V^n}(u)) = 0, \quad n = 0, \ldots, N-1$$

where
$u \in S^2_+ := \{(\alpha_1, \alpha_2),\ \alpha_1 \geq 0,\ \alpha_2 \in [0,1],\ \alpha_1^2 + \alpha_2^2 = 1\}$ (the right half unit circle).

- After the control discretisation, the scheme reads

$$\min_{k=1,\ldots,N_u} (B(u_k)V^{n+1} - b^{V^n}(u_k)) = 0, \quad n = 0, \ldots, N-1,$$

where $(u_k)_{k=1,\ldots,N_u}$ is chosen uniformly on the half-circle, with N_u controls, and with $u_1 = (0,1)$.

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Convergence and error control

Proposition The scheme is convergent (when $N_u \to \infty$, $h_x, h_y \to 0$, $\Delta t \to 0$, order $p \to \infty$).

Remark 1. Assume (A3). Then it is possible to derive explicit lower bound estimate. (Refs: [Barles-Jackobsen], [Krylov], [Maroso, Zidani, Bonnans]).

Remark 2. However, for the Financial problem, the upper bound would be more interesting !

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An Howard algorithm

Find a solution $X \in \mathbb{R}^q$ of $\min_\alpha B(\alpha)X - b(\alpha) = 0$, with $\alpha \in K^q$ where $K = \{u_1, \ldots, u_{N_u}\}$ finite.
Let $\alpha^{(0)}$ given, and consider for iterations $k = 0, 1, 2, \ldots$:

- Find $X^{(k)}$ such that $B(\alpha^{(k)})X^{(k)} - b(\alpha^{(k)}) = 0$
- If $X^{(k)} \neq X^{(k-1)}$, take $\alpha^{(k+1)} := argmin_\alpha B(\alpha)X^{(k)} - b(\alpha)$, otherwise stop.

Theorem (Convergence of the Howard algorithm)

Suppose $\forall\alpha$, $B(\alpha)$ monotonous ($B(\alpha)X \geq 0 \Rightarrow X \geq 0$). Then there exists a unique solution X and the Howard Algorithm converges to X with a finite number of iterations.

Remark. Neuman-type boundary conditions for large x, y, still monotonicity properties.

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Numerical results

cpu time / error test

$M_1 \times M_2 = 100^2$ space discretisation points
$N_u = 10$ controls
$N = 20$ time steps
Neighborhing order = 4.
On a 1.6 MHz cpu desktop computer

- Fast initialisation of a sparse generalized differences matrix
- An howard iteration : 2-4s (using a sparse solver)
- One time step: from 2 to 10 Howard iterations.
- Complete computation: ≤ 5 minutes
- Error test on $\Lambda_-(M(v)(t,x,y)) = f(t,x,y)$:
  relative L^∞ error $\simeq 5 \times 10^{-3}$.
- That's all folks !

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