Singular perturbations and Aubry-Mather theory

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July 1, 2006
Aim:

- To show the relation between singular perturbations problems arising in Large Deviations theory and the Aubry-Mather theory for Hamilton-Jacobi equations
- To reprove by simple PDE (viscosity) methods, some singular perturbation results which require hard probabilistic proofs

Plan of the talk:

- What is a Large Deviations result?
- The PDE approach to Large Deviations and when it fails
- The Aubry-Mather theory of Hamilton-Jacobi equations
- Improving the PDE approach to Large Deviations via Aubry-Mather theory
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Large Deviations

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\(E_\varepsilon\): a functional depending on the sample paths \(X_\varepsilon\) (f.e. \(E_\varepsilon(x) = \mathbb{E}[X_\varepsilon(\tau_\varepsilon)]\) or \(E_\varepsilon(x) = \mathbb{P}[X_\varepsilon(\tau_\varepsilon) \in \Gamma]\) with \(\Gamma \subset \partial D\)).
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The classical LD result is

\[-\epsilon \log(E_\epsilon(x)) \longrightarrow l(x) \quad \epsilon \to 0\]

where \(l > 0\) in \(D\) is the rate function, i.e.

\[E_\epsilon(x) = e^{-\frac{l(x)+O(1)}{\epsilon}} \quad \epsilon \to 0\]
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The classical LD result is

\[-\varepsilon \log(E_\varepsilon(x)) \longrightarrow I(x) \quad \varepsilon \to 0\]

where \(I > 0\) in \(D\) is the rate function, i.e.

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- Varadhan: Large deviations and applications, SIAM, 1984
The PDE approach to Large Deviations

Log-transforming $I_\epsilon(x) = -\epsilon \log(E_\epsilon(x))$ and interpreting $I_\epsilon$ as a solution of the singular perturbation problem

$$-\epsilon \Delta u + H(x, Du) = 0$$

$x \in D$

Boundary condition on $\partial D$

Pass to the limit for $\epsilon \to 0$ in the previous problem. If $I_\epsilon \to I$ for some subsequence, then $I$ solves the Hamilton-Jacobi equation $(HJ)$

$$H(x, Du) = 0$$

$x \in D$

Boundary condition on $\partial D$

Show uniqueness for $(HJ)$. Then $I_\epsilon \to I$ and we have the large deviations result

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for $\epsilon \to 0$.

Interpreting $(HJ)$ as a control problem, we also have a representation formula for $I$. 


The PDE approach to Large Deviations

- Perform the log-transform $I_\varepsilon(x) = -\varepsilon \log(E_\varepsilon(x))$ and interpret $I_\varepsilon$ as a solution of the singular perturbation problem

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- Show uniqueness for $(\text{HJ})$. Then $I_\varepsilon \to I$ and we have the large deviations result $-\varepsilon \log(E_\varepsilon(x)) \to I(x)$ for $\varepsilon \to 0$. Interpreting $(\text{HJ})$ as the control problem, we also have a representation formula for $I$. 


• W. Fleming ('81): logarithmic transformation and stochastic control methods
• Kamin, Eizenberg: classical solutions and strong convergence of $I_\varepsilon$, $D I_\varepsilon$ (estimates for $\|I_\varepsilon\|$, $\|D I_\varepsilon\|$, $\|D^2 I_\varepsilon\|$)
• Evans-Ishii ('85): continuous viscosity solutions and uniform convergence of $I_\varepsilon$ (estimates for $\|I_\varepsilon\|_\infty$, $\|D I_\varepsilon\|_\infty$)
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• Barles-Perthame ('90): discontinuous viscosity solutions and half-relaxed limits (estimates for $\| I_\varepsilon \|_\infty$)
There exists a strict subsolution to the Hamilton-Jacobi (i.e. $H(x, D\psi) < 0$ in $D$).

Levinson's condition: If $dX_{\varepsilon}(t) = b(X_{\varepsilon}(t))\, dt + \sqrt{\varepsilon}\, dW(t)$, then the trajectories of $\dot{x}(t) = b(x(t))$ must exit in a (uniformly bounded) finite time out of $D$.

$b$ cannot have equilibria inside $D \Rightarrow$ interesting problems in Large Deviation theory (e.g. Wentzell-Freidlin's theory) are excluded by the viscosity solution approach (Perthame (TAMS '90): the case of a single equilibrium point for $b$).
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A basic problem (Wentzell-Freidlin’s book, Ch. IV)

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\begin{align*}
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in bounded domain \(D\). Assume
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- $b(x) \cdot n_{ext}(x) < 0$ for $x \in \partial D$ ($D$ is invariant)
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in bounded domain \(D\). Assume

- \(b(x) \cdot n_{\text{ext}}(x) < 0\) for \(x \in \partial D\) (\(D\) is invariant)
- the set \(\Omega_b\) of the \(\omega\)-limits of \(\dot{x}(t) = b(x(t))\) is a \textit{class of equivalence} for the quasi-potential

\[
V(y, x) = \inf \{ \int_0^T \frac{1}{2} |\dot{\phi}(s) - b(\phi(s))|^2 ds : \\
\phi(0) = y, \phi(T) = x, T > 0 \}.
\]

(i.e. \(V(y, x) = V(x, y) = 0\) for \(x, y \in \Omega_b\))
Consider LD functional $\varepsilon(x) = \phi(\tau_\varepsilon(\tau_\varepsilon(x)))$ where $\tau_\varepsilon$ is the exit-time from $D$ and $\phi$ a given continuous function.

If there is a unique $y$ s.t. $V(\Omega^b, y) = \min_{x \in \partial D} V(\Omega^b, x)$, then $\varepsilon(x) \to \phi(y)$ for $\varepsilon \to 0$. This means that the stochastic trajectories $X_\varepsilon$ exit from $D$ close to $y$. 
Consider LD functional

\[ E_\varepsilon(x) = \mathbb{E}_x[\varphi(X_\varepsilon(\tau_\varepsilon))] \]

where \( \tau_\varepsilon \) is the exit-time from \( D \) and \( \varphi \) a given continuous function.
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The PDE approach (Kamin, Perthame)
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\[ E_\varepsilon(x) = \mathbb{E}_x[\varphi(X_\varepsilon(T_\varepsilon))] \] is a solution of

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\begin{cases}
  -\frac{\varepsilon}{2} \Delta u_\varepsilon + b(x) \cdot Du_\varepsilon = 0 & x \in D \\
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\end{cases}
\]

The invariant measure \( \nu^\varepsilon \) associated to the process satisfies the adjoint equation

\[
\begin{cases}
-\frac{\varepsilon}{2} \Delta \nu^\varepsilon + div(b(x)\nu^\varepsilon) = 0 & x \in D \\
\frac{\varepsilon}{2} \frac{\partial \nu^\varepsilon}{\partial n}(x) + b(x) \cdot n(x) \nu_\varepsilon = 0 & x \in \partial D
\end{cases}
\]
Set \( V_\varepsilon = -\varepsilon \log(v_\varepsilon) \), then \( V_\varepsilon \) is a solution of the singular perturbation problem

\[
\begin{cases}
-\varepsilon^2 \Delta V_\varepsilon + H(x, D_1 V_\varepsilon) = \varepsilon \text{div}(b) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad x \in D \\
\quad \partial V_\varepsilon \partial n(x) + 2b(x) \cdot n(x) = 0 \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad x \in \partial D
\end{cases}
\]

where \( H(x, p) = |p|^2 + b(x) \cdot p \) is the Hamiltonian associated to the Lagrangian \( L(x, q) = |q - b(x)|^2 \) in the quasi-potential.

Formally \( V_\varepsilon \to V \) where \( V \) is a solution of

\[
\begin{cases}
\quad H(x, D_1 V) = 0 \\
\quad \partial V \partial n(x) + 2b(x) \cdot n(x) = 0
\end{cases}
\]

Since Levinson's condition is violated (\( b \) has an attractor inside \( D \)), no uniqueness and the 3rd step in the PDE approach fails.
Set $V^\varepsilon = -\varepsilon \log(v^\varepsilon)$, then $V^\varepsilon$ is a solution of the singular perturbation problem

$$
\begin{cases}
-\frac{\varepsilon}{2} \Delta V^\varepsilon + H(x, DV^\varepsilon) = \varepsilon \text{div}(b) & \ x \in D \\
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where $H(x, p) = \frac{|p|^2}{2} + b(x) \cdot p$ is the Hamiltonian associated to the Lagrangian $L(x, q) = \frac{|q-b(x)|^2}{2}$ in the quasi-potential.
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where $H(x, p) = \frac{|p|^2}{2} + b(x) \cdot p$ is the Hamiltonian associated to the Lagrangian $L(x, q) = \frac{|q - b(x)|^2}{2}$ in the quasi-potential. Formally $V^\varepsilon \to V$ where $V$ is a solution of

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H(x, DV) &= 0 \quad x \in D \\
\frac{\partial V}{\partial n}(x) + 2b(x) \cdot n(x) &= 0 \quad x \in \partial D
\end{aligned}
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Since Levinson's condition is violated ($b$ has an attractor inside $D$), no uniqueness and the 3\textsuperscript{rd} step in the PDE approach fails.
Aim:

• To study the structure of the solutions of the 1st Neumann problem

\[ H(x, \nabla V) = 0 \quad x \in D \]
\[ \partial V \partial n(x) + 2b(x) \cdot n(x) = 0 \quad x \in \partial D \]

• To understand if the sequence of the solutions \( V_{\varepsilon} \) of the 2nd order problems selects a particular solution of the 1st order problem
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A short review of the Aubry-Mather theory for HJ equations
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Let $H(x, p)$ be a convex, coercive Hamiltonian and define

$$c = \inf\{\lambda : H(x, Du) \leq \lambda \text{ admits a subsolution in } D\}$$
A short review of the Aubry-Mather theory for HJ equations

Let $H(x, p)$ be a convex, coercive Hamiltonian and define

$$c = \inf\{\lambda : H(x, Du) \leq \lambda \text{ admits a subsolution in } D\}$$

For $\lambda \geq c$ set

$$Z_\lambda(x) = \left\{ p \in \mathbb{R}^N : H(x, p) \leq \lambda \right\}$$

$$\sigma_\lambda(x, q) = \sup \{ p \cdot q : p \in Z_\lambda(x) \}$$

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and define the distance

$$S_\lambda(x, y) = \inf \{ \int_0^1 \sigma_\lambda(\phi(s), \dot{\phi}(s))ds : \phi \in W^{1,\infty}([0, 1], D), \phi(0) = x, \phi(1) = y \}$$
Properties:

For any $x \in D$, $S_{\lambda}(x, \cdot)$ is a subsolution in $D$ and a supersolution in $D \setminus \{x\}$ to $H(y, Du) = \lambda$.

$u$ is a subsolution to $H(y, Du) = \lambda$ $\iff$ $u(x) - u(y) \leq S_{\lambda}(y, x)$ for any $x, y \in D$.

$H(y, Du) = \lambda + (BC)$ has a unique viscosity solution (or no viscosity solution) $\iff \lambda > c$ $\iff S_{\lambda}$ is locally equivalent to the Euclidean distance. For $\lambda = c$, a non-uniqueness phenomenon appears.
Properties:

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The Aubry set $\mathcal{A}$

The Aubry set is the set where $S_c$ fails to equivalent to the Euclidean distance.
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- **A metric definition:** $x \in \mathcal{A} \iff \exists \{\phi_n\}, \phi_n(0) = \phi_n(1) = x$, s.t.
  \[
  \inf_n \{\int_0^1 |\dot{\phi}_n(s)|\,ds\} \geq \delta > 0 \quad \text{(Euclidean length)}
  \]
  \[
  \inf_n \{\int_0^1 \sigma_c(\phi_n(s)), \dot{\phi}_n(s))\,ds\} = 0 \quad \text{(intrinsic length)}
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  $\inf_n \{ \int_0^1 |\dot{\phi}_n(s)| ds \} \geq \delta > 0$ (Euclidean length)
  $\inf_n \{ \int_0^1 \sigma_c(\phi_n(s)), \dot{\phi}_n(s) \} ds \} = 0$ (intrinsic length)

- **A PDE definition**: $x \in \mathcal{A} \iff S_c(x, \cdot)$ is a solution at $x$. 

The Aubry set $\mathcal{A}$

The Aubry set is the set where $S_c$ fails to equivalent to the Euclidean distance

- **A metric definition:** $x \in \mathcal{A} \iff \exists \{\phi_n\}, \phi_n(0) = \phi_n(1) = x$, s.t.
  \[
  \inf_n \left\{ \int_0^1 |\dot{\phi}_n(s)| \, ds \right\} \geq \delta > 0 \quad \text{(Euclidean length)}
  \]
  \[
  \inf_n \left\{ \int_0^1 \sigma_c(\phi_n(s)), \dot{\phi}_n(s) \right\} \, ds \right\} = 0 \quad \text{(intrinsic length)}
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- **A PDE definition:** $x \in \mathcal{A} \iff S_c(x, \cdot)$ is a solution at $x$.

**The main property:** There exists a subsolution to $H(x, Du) = c$, which is strict (i.e. $H(x, Du) < c$) out of $\mathcal{A}$. 

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**General Fact**: A unique solution to $H(x, Du) = c + (BC) \iff$ the value on $\mathcal{A}$ is prescribed, i.e. $\mathcal{A}$ is an uniqueness set for $H(x, Du) = c$.
Aubry-Mather theory for the Neumann problem

Recall that we want to study

\[
\begin{aligned}
&H(x, Dv) = 0 \quad x \in D \\
&\frac{\partial v}{\partial n}(x) + 2b(x) \cdot n(x) = 0 \quad x \in \partial D
\end{aligned}
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where \( H(x, p) = \frac{1}{2}|p|^2 + b(x) \cdot p \) is the LD Hamiltonian.
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\[ \begin{cases} \quad H(x, Dv) = 0 & x \in D \\ \quad \frac{\partial v}{\partial n}(x) + 2b(x) \cdot n(x) = 0 & x \in \partial D \end{cases} \]

where \( H(x, p) = \frac{1}{2}|p|^2 + b(x) \cdot p \) is the LD Hamiltonian. Set

\[ Z(x) = B(-b(x), |b(x)|) \]

\[ \sigma(x, q) = |b(x)||q| - b(x) \cdot q \]

\[ S(x, y) = \inf \left\{ \int_0^1 |b(\phi)||\dot{\phi}| - b(\phi) \cdot \dot{\phi} ds : \phi(0) = x, \phi(1) = y \right\} \]
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This explains the non-uniqueness of the solution to the Neumann problem.
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This explains the \textit{non-uniqueness} of the solution to the Neumann problem.

- $A$ is contained in the \textit{interior} of $D$. This fact is very important since we can interpret the \textit{Neumann boundary condition} in standard viscosity sense
The Comparison Theorem
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**Theorem:** $u$ subsolution and $v$ supersolution s.t. $u \leq v$ for $x \in A$
then

$$u \leq v \quad \text{for} \quad x \in \overline{D}$$

(i.e. $A$ is a uniqueness set for (HJ))
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(i.e. $\mathcal{A}$ is a uniqueness set for (HJ))

**Corollary:** If $g$ is such that $g(y) - g(x) \leq S(x, y)$ for any $x, y \in \mathcal{A}$ then

$$v(x) := \inf_{y \in \mathcal{A}} [g(y) + S(y, x)]$$

is the unique viscosity solution to (HJ) with value $g$ on $\mathcal{A}$. 
Aubry-Mather theory and Large Deviations
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The key point is to establish the relation between the LD objects $V$ (the quasi-potential), $\Omega_b$ (the $\omega$-limits set) and the PDE objects $S$ (the distance), $A$ (the Aubry set)
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- The quasi-potential

$$V(y, x) = \inf_{\phi(0)=y, \phi(T)=x, T>0} \left\{ \int_0^T \frac{1}{2} |\dot{\phi}(s) - b(\phi(s))|^2 \, ds \right\}.$$ 

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- $\Omega_b \subset \mathcal{A}$, $\mathcal{A}$ is forward invariant for $\dot{x} = b(x(t))$ and any subsolution is constant on the integral curve contained in $\mathcal{A}$. This implies that:
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- $\Omega_b \subset A$, $A$ is forward invariant for $\dot{x} = b(x(t))$ and any subsolution is constant on the integral curve contained in $A$.

This implies that:
- A subsolution of $H(x, Du)$ is constant on $A$
- $\Omega_b$ is a uniqueness set for the Neumann problem
The Large Deviations result

Let \( v \in \mathbb{E} \) be the solution of

\[
-\epsilon^2 \Delta v + \epsilon \text{div}(b) = 0 \quad \text{in} \quad D
\]

\[
\partial v \epsilon \partial_n(x) + 2b(x) \cdot n(x) = 0 \quad \text{on} \quad \partial D
\]

Set \( v \epsilon(x) = 0 \) for \( x \in A \) (a solution is defined up to a constant).

Then \( v \epsilon \to S(A, \cdot) \epsilon \to 0 \) where \( S(A, x) = \min \{ S(y, x) : y \in A \} \).

Remark: Recalling that \( S = V \), where \( V \) the quasi-potential, the previous theorem implies the Wentzell-Freidlin's large deviations result.
The Large Deviations result

**Theorem:** Let $v^\varepsilon$ be the solution of

$$\begin{cases}
-\frac{\varepsilon}{2} \Delta v^\varepsilon + H(x, Dv^\varepsilon) = \varepsilon \text{div}(b) & x \in D \\
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\end{cases}$$

Set $v^\varepsilon(\bar{x}) = 0$ for $\bar{x} \in \mathcal{A}$ (a solution is defined up to a constant).
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Set $v^\varepsilon(x) = 0$ for $x \in \mathcal{A}$ (a solution is defined up to a constant). Then

\[v^\varepsilon \to S(\mathcal{A}, \cdot) \quad \varepsilon \to 0\]

where $S(\mathcal{A}, x) = \min\{S(y, x) : y \in \mathcal{A}\}$.

**Remark:** Recalling that $S = V$, where $V$ the quasi-potential, the previous theorem implies the Wentzell-Freidlin’s large deviations result.
Proof:
By the Harnack’s inequality we have that $v_{\varepsilon}$ is uniformly Lipschitz continuous.

If $v_{\varepsilon} \to v$, then $v$ is a solution of the Neumann problem and $v(x) = 0$, hence $v(x) = 0$ for $x \in A$.

Recalling the representation formula $v(x) := \inf_{y \in A} \{ g(y) + S(y, x) \}$ we get $v(x) = S(A, x)$ for $x \in D$. 
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Recalling the representation formula

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we get

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Remarks:

- The previous result can be extended to the case \( \Omega = \bigcup_{i=1}^{N} K_i \), where \( K_i \) class of equivalence for the quasi-potential, \( K_1 \) attractive, \( K_2, \ldots, K_N \) repulsive.

- Then \( v \in S(K_1, \cdot) \) for \( \epsilon \to 0 \).

- With the same method it is possible to study other problems such as the Kamin and Eizenberg singular perturbation problem:

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-\epsilon \Delta v + H(x, Dv) - \epsilon c(x) = 0 \quad x \in Dv(x) = 0 \quad x \in \partial D
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where \( c \) is non-negative in \( D \).
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