Dislocations dynamics and mean curvature motion

Nicolas Forcadel

ENPC, CERMICS

Work in collaboration with F. Da Lio and R. Monneau

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Plan

1. Convergence of dislocations dynamics to mean curvature motion
2. Numerical scheme for dislocations dynamics
3. Numerical Scheme for the mean curvature motion

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Slepčev Formulation

- We consider the solutions u^ε of:

$$\begin{cases} u^\varepsilon_t(x,t) = \left(\left(c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t)>u^\varepsilon(x,t)\}}\right)(x) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|Du^\varepsilon| \\ u^\varepsilon(\cdot,0) = u_0(\cdot) \end{cases} \quad (1)$$

where

$$c_0^\varepsilon = \frac{1}{\varepsilon^{n+1}|\ln \varepsilon|} c_0\left(\frac{x}{\varepsilon}\right)$$

with the particular kernel

$$c_0 \in L^\infty(\mathbb{R}^n), \quad \begin{cases} c_0(x) = \frac{1}{|x|^{n+1}} g\left(\frac{x}{|x|}\right) & \text{if} \quad |x| > 1 \\ c_0(-x) = c_0(x) \end{cases}$$

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Definition of the solutions

Definition

- *We say that* u^ε *is sub-solution (resp. super-solution) if* $u^\varepsilon(\cdot, t=0) \le u_0$ *(resp.* $u^\varepsilon(\cdot, t=0) \ge u_0$*) and for all* (x_0, t_0) *and for every test function* $\Phi \in C^1(\mathbb{R}^n \times (0,T))$ *such that* $u^\varepsilon - \Phi$ *has a maximum (resp.minimum) at* (x_0, t_0)*, the following holds*

$$\Phi_t \le \left(\left(c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0) \ge u^\varepsilon(x_0,t_0)\}} \right)(x_0) - \frac{1}{2} \int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi|$$

$$\left(\text{resp. } \Phi_t \ge \left(\left(c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0) > u^\varepsilon(x_0,t_0)\}} \right)(x_0) - \frac{1}{2} \int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi| \right).$$

Existence and uniqueness

Theorem (Da Lio, F, Monneau)

Under certain assumptions of regularity, there exists a unique solution u^ε. *Moreover, for* $\varepsilon \in (0, \frac{1}{2})$, *we have:*

$$|Du^\varepsilon| \leq |Du_0|$$

$$|u^\varepsilon(x, t+h) - u^\varepsilon(x,t)| \leq C|Du_0|\sqrt{h} \quad \forall t,\ h \in \mathbb{R},\ \forall x \in \mathbb{R}^n$$

Convergence to anisotropic MCM

Theorem (Da Lio, NF, Monneau)

Under certain assumptions of regularity, when $\varepsilon \to 0$, u^ε *converges locally uniformly on compact sets to* u^0, *which is the unique solution of the limit problem:*

$$\begin{cases} u^0_t + F(D^2u^0, Du^0) = 0 \\ \\ u^0(\cdot, 0) = u_0 \end{cases}$$

with

$$F(M,p) = -\text{trace}\left(MA\left(\frac{p}{|p|}\right)\right)$$

$$A\left(\frac{p}{|p|}\right) = \int_{\theta \in \mathbf{S}^{n-1} \cap \{p^\perp\}} \frac{1}{2} g(\theta)\theta \otimes \theta d\theta$$

Similar Results

- Garroni, Muller (Gamma convergence, stationnary problem)
- Algorithm of Merriman, Bence, Osher
- [Evans], [Barles, Georgelin], [Ishii], [Ishii, Pires, Souganidis], [Bellitini, Chambolle, Novaga]

Error Estimate

Theorem (NF)

Under certain assumptions of regularity, we have the following error estimate between u^ε and u^0:

$$|u^\varepsilon - u^0| \leq C\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\,T\right)^{\frac{1}{2}}.$$

Plan

Monotone scheme

- Scheme:

$$v_I^0 = \tilde{u}_0(x_I),$$

$$v_I^{n+1} = v_I^n + \Delta t c^{\Delta}[v](x_I, t_{n+1}) E^{sign(c^{\Delta}[v](x_I,t_{n+1}))}(D^+ v_I^{n+1}, D^- v_I^{n+1}) \quad (2)$$

- Non-local velocity:

$$c^{\Delta}[v](x_I, t_{n+1}) = \sum_{J \in \mathbb{Z}^N} \bar{c}_0(x_{I-J}) 1_{\{v_J^{n+1} \geq v_I^{n+1}\}} \Delta x_1 \ldots \Delta x_N - \frac{1}{2} \sum_{J \in \mathbb{Z}^N} \bar{c}_0(x_J) \Delta x_1 \ldots \Delta x_N \quad (3)$$

- Kernel:

$$\bar{c}_0(x_I) = \frac{1}{|Q_I|} \int_{Q_I} c_0(x)\, dx \quad (4)$$

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Monotone scheme

where Q_I is the square cell

$$Q_I = [x_{i_1} - \Delta x_1/2, x_{i_1} + \Delta x_1/2] \times ... \times [x_{i_N} - \Delta x_N/2, x_{i_N} + \Delta x_N/2]$$

- $E^{\pm}$; approximation of the Euclidean norm ([Osher-Sethian]):

$$E^{+} = \left\{ \sum_i \max(D^{+}_{x_i} v^n_I, 0)^2 + \sum_i \min(D^{-}_{x_i} v^n_I, 0)^2 \right\}^2$$

$$E^{-} = \left\{ \sum_i \min(D^{+}_{x_i} v^n_I, 0)^2 + \sum_i \max(D^{-}_{x_i} v^n_I, 0)^2 \right\}^2$$

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Definition of the solutions

Definition (Numerical sub, super and solution of the scheme)

We say that v is a discrete sub-solution (resp super-solution) of the scheme (2) if for all $I \in \mathbb{Z}^N$, $n \in \mathbb{N}$, we have

$$v_I^{n+1} \leq v_I^n + \Delta t\ c^\Delta[v](x_I, t_{n+1}) E^{Sign(c^\Delta[v](x_I,t_{n+1}))}(D^+ v_I^{n+1}, D^- v_I^{n+1})$$

(resp.

$$v_I^{n+1} \geq v_I^n + \Delta t\ \tilde{c}^\Delta[v](x_I, t_{n+1}) E^{Sign(\tilde{c}^\Delta[v](x_I,t_{n+1}))}(D^+ v_I^{n+1}, D^- v_I^{n+1}))$$

where

$$\tilde{c}^\Delta[v](x_I, t_{n+1}) = \sum_{J \in \mathbb{Z}^N} \bar{c}_0(x_{I-J}) 1_{\{v_J^{n+1} > v_I^{n+1}\}} \Delta x_1 \ldots \Delta x_N - \frac{1}{2} \sum \bar{c}_0(x_J) \Delta x_1 \ldots \Delta x_N$$

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Discrete-continuous error estimate

Theorem (NF)

Under certain assumptions of regularity, we have the following error estimate between the continuous solution u of the dislocations dynamics equation and its numerical approximation v:

$$\sup_{Q_T^\Delta} |u - v| \leq K|c_0|^2_{L^1}|Dc_0|_{L^1}\sqrt{T}\,(\Delta x + \Delta t)^{1/2} + \sup_{Q^\Delta} |u_0 - \tilde{u}_0|$$

provided $|\Delta X| + \Delta t \leq \dfrac{1}{|c_0|^4_{L^1}|Dc_0|^2_{L^1}K^2}$.

Plan

Numerical scheme for mean curvature motion

- Using the previous result, we get:

$$|u^0 - v| \leq |u^0 - u^\varepsilon| + |u^\varepsilon - v|$$
$$\leq C \left(\frac{\ln|\ln \varepsilon|}{|\ln \varepsilon|} T \right)^{\frac{1}{2}} + \frac{K}{\varepsilon^4 |\ln \varepsilon|^3} \sqrt{T} \sqrt{|\Delta X| + \Delta t} + \sup_{Q^\Delta} |u_0 - \tilde{u}_0|.$$

- By optimizing in ε, we get the following Theorem:

Theorem (NF)

Under certain assumptions of regularity, we have the following error estimate between the continuous solution u^0 of the mean curvature motion and its numerical approximation v:

$$\sup_{Q_T^\Delta} |u^0 - v| \leq C \left(\frac{\ln|\ln \sqrt{\Delta X + \Delta t}|}{|\ln \sqrt{\Delta X + \Delta t}|} T \right)^{\frac{1}{2}} + \sup_{Q^\Delta} |u_0 - \tilde{u}_0|.$$

Numerical scheme for mean curvature motion

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\leq & C \left(\frac{\ln|\ln \varepsilon|}{|\ln \varepsilon|} T \right)^{\frac{1}{2}} + \frac{K}{\varepsilon^4 |\ln \varepsilon|^3} \sqrt{T} \sqrt{|\Delta X| + \Delta t} + \sup_{Q^\Delta} |u_0 - \tilde{u}_0|.
\end{aligned}
$$

- By optimizing in ε, we get the following Theorem:

Theorem (NF)

Under certain assumptions of regularity, we have the following error estimate between the continuous solution u^0 of the mean curvature motion and its numerical approximation v:

$$
\sup_{Q^\Delta_T} |u^0 - v| \leq C \left(\frac{\ln|\ln \sqrt{\Delta X + \Delta t}|}{|\ln \sqrt{\Delta X + \Delta t}|} T \right)^{\frac{1}{2}} + \sup_{Q^\Delta} |u_0 - \tilde{u}_0|.
$$

References

- [Crandall, Lions], [Oberman]
- Algorithm of Merriman, Bence, Osher
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