Homogenization of first order equations with u/ε-periodic Hamiltonians

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Outline

1. Introduction
   - Motivations from Physics
   - Setting of the problem
   - Main results

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   - Motivations from Physics
   - Setting of the problem
   - Main results
2. Proof of convergence / determining the cell problem
   - The classical case
   - Homogenize an ODE
   - The ODE case: ansatz?
   - The PDE case
3. Constructing correctors
   - Constructing approximate cell problems
   - Constructing approximate correctors
   - Approximate ergodicity implies exact ergodicity

Motivations from Physics

Defaults moving in crystals

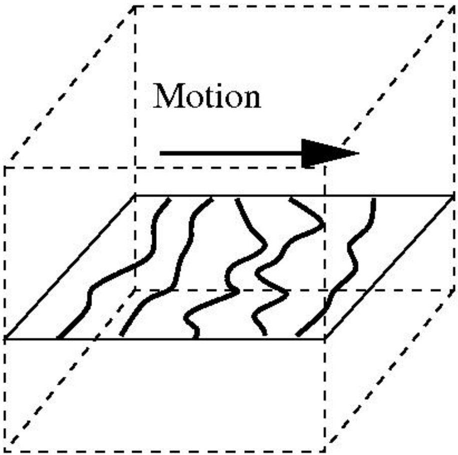

Figure: Dislocations in a slip plane

Physics of dislocations

New boom of physics of dislocations
New models for the dynamics of dislocations densities
(Groma, Balogh '99 / Groma, Czikor, Zaiser '03 / Sethna '04 *etc.*)

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The monotone case:

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} = h_1\left(\frac{x}{\varepsilon}, \nabla u^\varepsilon, [u^\varepsilon]\right) + h_2\left(\frac{u^\varepsilon}{\varepsilon}, \nabla u^\varepsilon\right) \\ u(0,x) = u_0(x) \end{cases}$$

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First difficulty: non-local HJ equation

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Main difficulty: the $\frac{u}{\varepsilon}$-dependance of the Hamiltonian

Setting of the problem

We reduce the problem to the main difficulty by considering the following ε-HJ equation:

$$(\mathrm{HJ})_\varepsilon \qquad \boxed{\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = H\left(\dfrac{u^\varepsilon}{\varepsilon}, \dfrac{x}{\varepsilon}, \nabla u^\varepsilon\right) \\ u(0,x) = u_0(x) \end{cases}}$$

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- $\dfrac{\partial u^\varepsilon}{\partial t} = c\left(\frac{x}{\varepsilon}\right) |\nabla u^\varepsilon| + h\left(\frac{u^\varepsilon}{\varepsilon}\right)$
- $\dfrac{\partial u^\varepsilon}{\partial t} = c\left(\frac{x}{\varepsilon}\right) (1 + |\nabla u|^2)^{\frac{1}{4}} + h\left(\frac{u^\varepsilon}{\varepsilon}\right)$

Setting of the problem (2)

Main assumptions

- **Regularity**

$$\left|\frac{\partial H}{\partial i}\right| \leq C \qquad (i = u, p) \qquad \left|\frac{\partial H}{\partial y}\right| \leq C(1 + |p|)$$

- **Periodicity**

$$H(u + l, y + k, p) = H(u, y, p) \qquad l \in \mathbb{Z}, \; k \in \mathbb{Z}^N$$

- **Coercivity**

$$H(x, u, p) \underset{|p| \to +\infty}{\longrightarrow} +\infty$$

Aims

- Determine the homogenized equation
- Construct correctors
- Prove convergence

Main results

The initial value "cell" problem (IVCP):

$$\begin{cases} \frac{\partial w}{\partial t} = H(p \cdot y + w, y, p + \nabla w) \\ w(0, y) = 0 \end{cases}$$

Ergodicity

There exists $\overline{H}(p)$, a unique $\lambda \in \mathbb{R}$ such that the continuous solution of (IVCP) satisfies: $\frac{w(\tau, y)}{\tau} \to \lambda$ as $\tau \to \infty$ unif wrt y.

Put $v = w - \lambda\tau$ and get the "cell" problem (CP):

$$\begin{cases} \lambda + \frac{\partial v}{\partial t} = H(\lambda\tau + p \cdot y + v, y, p + \nabla v) \\ v(0, y) = 0 \end{cases}$$

Main results (2)

Comments about correctors v

- They are bounded
- They are time-dependent
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The homogenized HJ equation:

$$\begin{cases} \frac{\partial u^0}{\partial t} = \overline{H}(\nabla u^0) \\ u^0(0,x) = u_0(x) \end{cases}$$

Convergence

The bounded continuous solution u^ε of the ε-HJ equation converges locally uniformly towards the bounded continuous solution u^0 of the homogenized HJ equation.

The results of Guy Barles

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Assumptions

- Regularity, periodicity, coercivity
- **Behaviour at infinity**
  $|H(x,u,p) - p \cdot \nabla_p H(x,u,p)| \leq C$

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Main idea

- Replace $\frac{u^\varepsilon}{\varepsilon}$ with $\frac{y}{\varepsilon}$ where y is a new variable
- Reduce the problem to homogenize a front equation

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Determining the cell problem

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- It is not clear (even if not surprising) that the cell problem we presented is the proper one.
- Does the classical proof of convergence (classical ansatz) applies to our case?

The classical case

The equation

$$\boxed{\frac{\partial u^\varepsilon}{\partial t} = H\left(\frac{x}{\varepsilon}, \nabla u^\varepsilon\right)}$$

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An ansatz

Look for v that is a good "corrector" between u^ε and u^0:

$$u^\varepsilon(t,x) = u^0(t,x) + \varepsilon v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) + \dots$$

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Comments

This extension is done around a fixed point (t_0, x_0) (r denotes the radius around this point)

The classical case (2)

Plugg into the ε-HJ equation and get:

$$\nabla_x u^\varepsilon = \nabla_x u^0 + \nabla_y v$$

$$\partial_t u^\varepsilon = \partial_t u^0 + \partial_\tau v$$

$$\partial_t u_0 + \partial_\tau v = H(y, \nabla_x u_0 + \nabla_y v)$$

The classical case (2)

Plugg into the ε-HJ equation and get:

$$\begin{aligned} \nabla_x u^\varepsilon &= \nabla_x u^0 + \nabla_y v \\ \partial_t u^\varepsilon &= \partial_t u^0 + \partial_\tau v \\ \partial_t u_0 + \partial_\tau v &= H(y, \nabla_x u_0 + \nabla_y v) \end{aligned}$$

Hope: if $p = \nabla_x u_0$ is fixed, there is a unique $\partial_t u_0 = \lambda = \overline{H}(p)$

Comments on the classical case.

- Look for bounded correctors v
- time-independent correctors v are ok

The ODE case

Threshold phenomenon for the homogenization of an ODE

- If the periodic function h vanishes, $u^\varepsilon \to 0$

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The ODE case

Threshold phenomenon for the homogenization of an ODE

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- If not, $0 < \alpha \leq h \leq A$, then $\alpha \leq \frac{u^\varepsilon}{t} \leq A$

Proper ansatz for $\boxed{\frac{du^\varepsilon}{dt}(t) = h\left(t, \frac{u^\varepsilon(t)}{\varepsilon}\right)}$ & $u^\varepsilon(0) = 1$?

Expected homogenized equation: $\frac{d}{dt}u^0 = \lambda(t)$ & $u^0(0) = 1$.

$$u^0(t) \simeq \boxed{u^0(t_0) - \lambda(t_0)t_0} + \lambda(t_0)t$$

and we try to **determine** $\lambda = \lambda(t_0)$.

Find the ansatz for the ODE case

Classical ansatz

$$
\begin{aligned}
u^\varepsilon(t) &= u^0(t) + \varepsilon v\left(\frac{t}{\varepsilon}\right) + \dots \\
&= \boxed{u^0(t_0) - \lambda t_0} + \lambda t + \varepsilon v\left(\frac{t}{\varepsilon}\right) + \dots
\end{aligned}
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$$\hookrightarrow \lambda + d_\tau v = h\left(\boxed{\frac{u^0(t_0) - \lambda t_0}{\varepsilon}} + \lambda \tau + v\right) \qquad \text{with } \tau = \frac{t}{\varepsilon}$$

The “error”: $\boxed{\quad \dots \quad} \simeq \frac{r}{\varepsilon}$.

Still oscillating!

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The "error": $\boxed{\dots} \simeq \frac{r}{\varepsilon}$. Still oscillating!

Find the ansatz for the ODE case (2)

Second ansatz

Add a fast variable: $\boxed{y = \frac{1}{\varepsilon}}$ and write:

$$u^\varepsilon(t) = u^0(t) + \varepsilon v\left(\frac{t}{\varepsilon}, \frac{u^0(t) - \lambda t}{\varepsilon}\right) + \dots$$

$$\hookrightarrow \lambda + \frac{d}{d\tau} v + \boxed{\quad \dots \quad} = h(\lambda \tau + y + v(\tau, y))$$

The error $\boxed{\quad \dots \quad}$ is small.

- To construct a bounded solution v, solve the PDE:

$$d_\tau w = h(y + w) \quad \& \quad w(0) = 0$$

and find λ such that $v := w - \lambda\tau$ is bounded.

- Note that the corrector have to depend on time

The PDE case:

Consider now the PDE case and write:

$$u^\varepsilon(t,x) = u^0(t,x \qquad) + \varepsilon v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \frac{u^0(t,x \qquad) - \lambda t}{\varepsilon}\right)$$

with $\lambda = \partial_t u_0(t_0, x_0)$ and plug it:

$$\lambda + \partial_\tau v + \boxed{\partial_{N+1} v \times (\partial_t u^0(t) - \lambda)}$$

$$= H(\lambda\tau \qquad + y_{N+1} + v, y, \boxed{\mathrm{p}} + \nabla_y v + \boxed{(\partial_{N+1} v \times \quad \nabla_x u^0 \qquad))})$$

with $p = \nabla u^0(t_0, x_0)$ and $\lambda = \partial_t u^0(t_0, x_0)$.

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To get $\boxed{\ \ldots\ } \ll 1$ **try to control** $\partial_{N+1} v$

The PDE case: twist a variable

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Outline

Constructing approximate cell problems

Recall: the initial value “cell” problem:

$$\begin{cases} \frac{\partial W}{\partial t} = H(p \cdot y + y_{N+1} + W, y, \nabla_y W) \\ W(0, Y) = 0 \end{cases}$$

and find λ such that $v = w - \lambda\tau$ is bounded

Aim: obtain regular sub- and supercorrectors.

How? By approximating the cell problem.

Which approximation?

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Which approximation?

Use coercivity of H to construct Lipschitz approx correctors

A gradient estimate

$$\boxed{\begin{cases} \frac{\partial U}{\partial t} = F(U, y, \nabla_Y U) \\ U(0, Y) = U_0(Y) \end{cases}}$$

Gradient estimate

If $F(W, y, p) =$ *constant* outside a starshaped compact set Ω, then the inclusion $\nabla U_0 \in \Omega$ is preserved:

$$\nabla U(t, \cdot) \cdot \in \Omega \quad \text{for any } t > 0.$$

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- It is now <u>constant</u> outside $\boxed{\Omega^+_{K,\delta}}$ starshaped compact.
  It implies the Lipschitz regularity of the solution.

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  It implies the Lipschitz regularity of the solution.

The approximate corrector is an "exact" supercorrector

Constructing approximate correctors

$$\begin{cases} \frac{\partial W}{\partial t} = H^{+}_{K,\delta}(P \cdot Y + W, y, \nabla_Y W) \\ W(0, Y) = 0 \end{cases}$$

Main steps in the construction of super-correctors.

Constructing approximate correctors

$$\begin{cases} \frac{\partial W}{\partial t} = H^+_{K,\delta}(P \cdot Y + W, y, \nabla_Y W) + \varepsilon \mathcal{I}[W] \\ W(0,Y) = 0 \end{cases}$$

Main steps in the construction of super-correctors.

- Perturb by a non-local 0 order operator to get a strong maximum principle

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Main steps in the construction of super-correctors.

- Perturb by a non-local 0 order operator to get a strong maximum principle ; get the gradient estimates

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Main steps in the construction of super-correctors.

- Perturb by a non-local 0 order operator to get a strong maximum principle ; get the gradient estimates
- Control Y-space oscillations of W

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Main steps in the construction of super-correctors.

- Perturb by a non-local 0 order operator to get a strong maximum principle ; get the gradient estimates
- Control Y-space oscillations of W
- Control τ-time oscillations of W

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Main steps in the construction of super-correctors.

- Perturb by a non-local 0 order operator to get a strong maximum principle ; get the gradient estimates
- Control Y-space oscillations of W
- Control τ-time oscillations of W
- Construct a global in time solution

Constructing approximate correctors

$$\begin{cases} \frac{\partial W}{\partial t} = H^+_{K,\delta}(P \cdot Y + W, y, \nabla_Y W) + \varepsilon \mathcal{I}[W] \\ W(0, Y) = 0 \end{cases}$$

Main steps in the construction of super-correctors.

- Perturb by a non-local 0 order operator to get a strong maximum principle ; get the gradient estimates
- Control Y-space oscillations of W
- Control τ-time oscillations of W
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- Pass to the limit as $\varepsilon \to 0$

Constructing approximate correctors

- Integer translations in y:

$$|v(\tau, y+k, y_{N+1}) - v(\tau, y, y_{N+1})| \leq 1$$

- v 1-periodic in y_{N+1}
- $u := y_{N+1} + v$ nondecreasing
- sliding method + strong maximum principle:

$$v(\tau, y, y_{N+1}) = v(0, y, \lambda\tau + y_{N+1})$$

$\hookrightarrow$ control v in $\lambda\tau$ and y_{N+1}

- $\lambda \leq F(|p|)$
- $\underline{v}(y) = \inf\{V(\tau, y, y_{N+1}) : \tau \geq 0\}$ is Lipschitz c:

$$|\nabla \underline{v}| \leq F(|p|) \quad \rightarrow \quad \text{control } v \text{ for small } y$$

Approximate ergodicity $\Rightarrow$ exact ergodicity

We construct correctors $V_K^{\pm}$ of:

$$\lambda_K^{\pm} + \partial_\tau V_K^{\pm} = H_K^{\pm}(\lambda_K^{\pm}\tau + P \cdot Y + V_K^{\pm}, y, P + \nabla_Y V_K^{\pm})$$

such that:

$$\boxed{|V_K^{\pm}| \leq C \quad \text{and} \quad |\lambda_K^{\pm}| \leq C}$$

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We next explain why as $K \to +\infty$:

$$\lambda_K^\pm \to \lambda \quad \text{for some } \lambda \in \mathbb{R}???$$

Approximate ergodicity $\Rightarrow$ exact ergodicity (2)

Define:
$$\lambda_u^\pm = \limsup \lambda_K^\pm \quad \& \quad \lambda_l^\pm = \liminf \lambda_K^\pm$$
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and this implies:

$W - \lambda\tau$ is bounded   <u>but not regular</u>

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- I., Monneau and Rouy. *First order equations with u/ε-periodic Hamiltonians. Part II: application to dislocation dynamics*, submitted

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These papers are available there:
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