Integral formulations of the geometric eikonal equation

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1 – Front propagation

1.1 – Level-set method.

We are interested in front propagations governed by the law

\[ V_{t,x} = c(t, x) \]  \hspace{1cm} (0.1)

where \( V_{t,x} \) denotes the normal velocity of the point \( x \) of the front at time \( t \).
1 – Front propagation

Références


1 – Front propagation

Assume that the front \( \Gamma(t) = \partial \Omega(t) \) is smooth, and that there exists a smooth \( u : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) such that

\[
\Omega(t) = \{ x \in \mathbb{R}^N ; u(t, x) > 0 \}, \quad \Gamma(t) = \{ x \in \mathbb{R}^N ; u(t, x) = 0 \}
\]

and \( Du(t, x) \neq 0 \) when \( x \in \Gamma(t) \), where \( Du \) is the gradient of \( u \) with respect to \( x \).

Then \( V_{t,x} = \frac{u_t(t, x)}{|Du(t, x)|} \) and \( u \) therefore satisfies the \textit{eikonal} equation :

\[
u_t(t, x) = c(t, x)|Du(t, x)|. \tag{0.2}\]
1 – Front propagation

To generalize the preceding evolution to non-smooth fronts, we realize the following program:

1. Find $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that $\Gamma(0) = \{x \in \mathbb{R}^N; u_0(x) = 0\}$

2. Solve in an appropriate sense the problem

   \begin{equation}
   \begin{cases}
   u_t(t, x) = c(t, x)|Du(t, x)| & \text{for } (t, x) \in (0, T) \times \mathbb{R}^N \\
   u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}^N
   \end{cases}
   \end{equation}

(0.3)

3. Set $\Gamma(t) = \{x \in \mathbb{R}^N; u(t, x) = 0\}$
1 – Front propagation

**Theorem 0.1** (M. Crandall, P.L. Lions).

*Under the following assumptions:*

\((H)\) \( c : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) is continuous, bounded, Lipschitz continuous with respect to the second variable,

the problem

\[
\begin{aligned}
    u_t(t, x) &= c(t, x)|Du(t, x)| \quad \text{for} \ (t, x) \in (0, T) \times \mathbb{R}^N \\
    u(0, x) &= u_0(x) \quad \text{for} \ x \in \mathbb{R}^N
\end{aligned}
\]

(0.4)

has a unique uniformly continuous viscosity solution on \([0, T] \times \mathbb{R}^N\) for all initial datum \(u_0\) that is uniformly continuous on \(\mathbb{R}^N\).
From now on, we restrict ourselves to sub-solutions.

A result from Barles, Soner and Souganidis ([3]) shows that any sub-solution $u$ of a geometric equation gives another sub-solution $1\{u \geq 0\}$.

We infer, for any closed set $K(0) \subset \mathbb{R}^N$, that there exists a family $(K(t))_{t \in [0,T]}$ such that

1. The graph of $K$, $\bigcup_{t \in [0,T]} \{t\} \times K(t)$ is closed in $\mathbb{R}^{N+1}$.

2. $(t, x) \mapsto 1_{K(t)}(x)$ is a sub-solution of the eikonal equation on $(0, T) \times \mathbb{R}^N$. 
1 – Front propagation

Proposition 0.2. If $K(0)$ is compact, the evolution is bounded: there exists $R > 0$ such that for all $t \in [0, T]$, $K(t) \subset B(0, R)$. 
If $\Gamma(t) = \partial \Omega(t)$ is a smooth hypersurface of $\mathbb{R}^N$ for $t \geq 0$, and $(\Gamma_t)_{t \geq 0}$ evolves smoothly in time, Hadamard’s formula states that:

For all $\phi \in C^1([0, +\infty) \times \mathbb{R}^N)$,

$$
\frac{d}{dt} \int_{\Omega(t)} \phi(t, x) \, dx = \int_{\Omega(t)} \frac{\partial \phi}{\partial t}(t, x) \, dx + \int_{\partial \Omega(t)} V_{t,x} \phi(t, x) \, d\mathcal{H}^{N-1}(x)
$$

(0.5)
In particular if $V_{t,x} \leq c(t,x)$ in the classical sense, we have for all $\phi \in C^1([0, +\infty) \times \mathbb{R}^N, \mathbb{R}_+)$:

$$\frac{d}{dt} \int_{\Omega(t)} \phi(t, x) \, dx \leq \int_{\Omega(t)} \frac{\partial \phi}{\partial t}(t, x) \, dx + \int_{\partial \Omega(t)} c(t, x) \phi(t, x) \, d\mathcal{H}^{N-1}(x) \quad (0.6)$$
When \((t, x) \mapsto 1_{K(t)}(x)\) is “only” a viscosity sub-solution of the eikonal equation, we don’t know anything about the regularity of \(K(t)\), and the term \(\int_{\partial K(t)} c(t, x) \phi(t, x) d\mathcal{H}^{N-1}(x)\) does not make any sense.
When \((t, x) \mapsto 1_{K(t)}(x)\) is “only” a viscosity sub-solution of the eikonal equation, we don’t know anything about the regularity of \(K(t)\), and the term \(\int_{\partial K(t)} c(t, x) \phi(t, x) \, d\mathcal{H}^{N-1}(x)\) does not make any sense.

However, under the conditions

\[ i. \ c(t, x) > 0 \ \forall (t, x) \in [0, T] \times \mathbb{R}^N, \]

\[ ii. \ K(0) \text{ has the interior ball property of radius } r > 0, \ i.e. \text{ is the union of closed balls of radius } r > 0, \]

it has been proved by O. Alvarez, P. Cardaliaguet and R. Monneau that \(K(t)\) also satisfies an interior ball condition, and that Hadamard’s formula still holds.
An essential point is the following theorem that gives a control on the perimeter of sets with the interior ball property:

**Theorem 0.3.** For all $r > 0$, there exists $M > 0$ such that for all closed set $E \subset \mathbb{R}^N$ having the interior ball property of radius $r$ and of diameter less than $1/r$, we have

$$\mathcal{H}^{N-1}(\partial E) \leq M.$$
From here on, we do not make any assumption either on the sign of $c$, nor on the regularity of the initial set.

We only consider a family $(K(t))_{t \in [0,T]}$ with closed graph such that $(t, x) \mapsto 1_{K(t)}(x)$ is a sub-solution of the eikonal equation on $(0, T) \times \mathbb{R}^N$.

Set for all $\varepsilon > 0$,

$$K^\varepsilon(t) = \{ x \in \mathbb{R}^N; d_{K(t)}(x) < \varepsilon \}$$

Then $\overline{K^\varepsilon(t)}$ has the interior ball property.

→ We want to generalize Hadamard’s formula to the evolution $t \to K^\varepsilon(t)$. 
To this end, we will have to modify the equation, which leads to introduce a perturbed velocity

\[ c^\varepsilon(t, x) = \max_{|y-x| \leq \varepsilon} c(t, y) \]
3 – The integral formulation

3.1 – Statement of the result

Theorem 0.4. Let $K : [0, T] \rightarrow \mathcal{P}(\mathbb{R}^N) \setminus \{\emptyset\}$ be such that

1. $K(0)$ is compact and $K(t) \rightarrow K(0)$ in the Hausdorff distance as $t \rightarrow 0$,

2. $\bigcup_{t \in [0, T]} \{t\} \times K(t)$ is closed in $\mathbb{R}^{N+1}$,

3. $u : (t, x) \mapsto 1_{K(t)}(x)$ is a viscosity sub-solution of the eikonal equation.
3 – The integral formulation

Then for all $t_1$ and $t_2$ satisfying $0 \leq t_1 < t_2 \leq T$, for almost all $\varepsilon > 0$, and for all $\phi \in C^1([0, T] \times \mathbb{R}^N, \mathbb{R}_+)$,

\[
\int_{t_1}^{t_2} \int_{K^\varepsilon(s)} \phi_t(s, x) \, dx \, ds + \int_{t_1}^{t_2} \int_{\partial K^\varepsilon(s)} c^\varepsilon(s, x) \phi(s, x) \, d\mathcal{H}^{N-1}(x) \, ds \\
\geq \left[ \int_{K^\varepsilon(s)} \phi(s, x) \, dx \, ds \right]_{t_1}^{t_2}
\]

(0.7)
3 – The integral formulation

3.2 – Steps of proof

Let \( w(t, x) = -d_{K(t)}(x) \). Let \( \theta : \mathbb{R} \rightarrow \mathbb{R} \) be smooth and non-decreasing such that \( \theta = 0 \) in \( (-\infty, -\varepsilon] \), \( \theta = 1 \) in \([0, \infty)\), and set \( w_\theta = \theta \circ w \).
3 – The integral formulation

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Let $w(t, x) = -d_{K(t)}(x)$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be smooth and non-decreasing such that $\theta = 0$ in $(-\infty, -\varepsilon]$, $\theta = 1$ in $[0, \infty)$, and set $w_\theta = \theta \circ w$.

1. $w_\theta$ is a sub-solution of $(w_\theta)_t = c^{\varepsilon}(t, x)|Dw_\theta|$ in $(0, T) \times \mathbb{R}^N$. 
3 – The integral formulation

3.2 – Steps of proof

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1. \( w_\theta \) is a sub-solution of \( (w_\theta)_t = c^\varepsilon(t, x)|Dw_\theta| \) in \( (0, T) \times \mathbb{R}^N \).

2. For all \( \phi \in C^1_c((0, T) \times \mathbb{R}^N, \mathbb{R}_+) \),

\[
\int_0^T \int_{\mathbb{R}^N} w_\theta(s, x) \phi_t(s, x) \, dx \, ds + \int_0^T \int_{\mathbb{R}^N} c^\varepsilon(s, x)|Dw_\theta(s, x)| \phi(s, x) \, dx \, ds \geq 0
\]

(this is obtained by a regularization in time and integration by parts).
As $\theta$ tends to $1_{(-\varepsilon, +\infty)}$:

$$\int_0^T \int_{\mathbb{R}^N} w_\theta(s, x) \phi_t(s, x) \, dx \, ds \to \int_0^T \int_{K^\varepsilon(s)} \phi_t(s, x) \, dx \, ds.$$
3 – The integral formulation

As $\theta$ tends to $1_{(-\frac{\varepsilon}{\varepsilon},+\infty)}$:

3. $\int_0^T \int_{\mathbb{R}^N} w_\theta(s, x) \phi_t(s, x) \, dx \, ds \to \int_0^T \int_{K^\varepsilon(s)} \phi_t(s, x) \, dx \, ds$.

4. The coarea formula shows that

$$
\int_{\mathbb{R}^N} c^\varepsilon(s, x) |Dw_\theta(s, x)| \phi(s, x) \, dx \\
= \int_0^1 \int_{\{w_\theta(s, \cdot) = \tau\}} c^\varepsilon(s, x) \phi(s, x) \, d\mathcal{H}^{N-1}(x) \, d\tau \\
\to \int_{\partial K^\varepsilon(s)} c^\varepsilon(s, x) \phi(s, x) \, d\mathcal{H}^{N-1}(x) \quad \text{for a.a. } \varepsilon > 0.
$$
Conclusion:
For almost all $\varepsilon > 0$, for all $\phi \in C^1_c((0, T) \times \mathbb{R}^N, \mathbb{R}_+)$,

$$\int_0^T \int_{K^\varepsilon(s)} \phi_t(s, x) \, dx \, ds + \int_0^T \int_{\partial K^\varepsilon(s)} c^\varepsilon(s, x) \phi(s, x) \, d\mathcal{H}^{N-1}(x) \, ds \geq 0.$$
The following theorem shows that the integral formulation characterizes the fact for \((t, x) \mapsto 1_{K(t)}(x)\) to be a viscosity sub-solution of the eikonal equation:
The following theorem shows that the integral formulation characterizes the fact for \((t, x) \mapsto 1_{K(t)}(x)\) to be a viscosity sub-solution of the eikonal equation:

**Theorem 0.5.** Let \(K : [0, T] \to \mathcal{P}(\mathbb{R}^N) \setminus \{\emptyset\}\) be such that

1. \(\bigcup_{t \in [0, T]} \{t\} \times K(t)\) is closed in \(\mathbb{R}^{N+1}\) and \(K\) is bounded.

2. For almost all small enough \(\varepsilon > 0\), for all \(\phi \in C^1_c((0, T) \times \mathbb{R}^N, \mathbb{R}_+)\),

\[
\int_0^T \int_{K_\varepsilon(s)} \phi_t(s, x)\, dx\, ds + \int_0^T \int_{\partial K_\varepsilon(s)} c_\varepsilon(s, x) \phi(s, x)\, d\mathcal{H}^{N-1}(x)\, ds \geq 0.
\]

Then \(u : (t, x) \mapsto 1_{K(t)}(x)\) is a viscosity sub-solution of \(u_t = c(t, x)|Du|\) in \((0, T) \times \mathbb{R}^N\).
5 – Regularity of the front

5.1 – BV functions and sets of finite perimeter

Let $\Omega$ be an open subset of $\mathbb{R}^N$.

**Definition 0.6.** An application $f \in L^1_{loc}(\Omega)$ is said to have locally bounded variations in $\Omega$ if for all open set $U \subset \subset \Omega$,

$$
\sup \left\{ \int_U f(x) \text{div} \phi(x) \, dx ; \phi \in C^1_c(U, \mathbb{R}^N) ; \|\phi\|_\infty \leq 1 \right\} < +\infty.
$$

We denote by $BV_{loc}(\Omega)$ the set of functions of locally bounded variations in $\Omega$.

Likewise, we say that $f \in L^1(\Omega)$ has bounded variations in $\Omega$ ($BV$) if the preceding definition holds for $U = \Omega$. We denote by $BV(\Omega)$ their set.
From the Riesz representation theorem, we deduce:

**Theorem 0.7.** Let \( f \in BV_{loc}(\Omega) \). Then there exists a Radon measure \( \mu \) on \( \Omega \) and a \( \mu \)-measurable application \( \sigma : \Omega \to \mathbb{R}^N \) such that:

1. \(|\sigma(x)| = 1 \) \( \mu \)-a.e.
2. \( \int_{\Omega} f(x) \text{div} \phi(x) \, dx = -\int_{\Omega} \langle \phi(x), \sigma(x) \rangle \, d\mu \quad \forall \phi \in C^1_c(\Omega, \mathbb{R}^N). \)

The measure \( \mu \) is called the variation measure of \( f \), and is denoted by \( \|Df\| \).
Definition 0.8. A $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$ is said to have (locally) finite perimeter in $\Omega$ if $1_E$ has (locally) bounded variations in $\Omega$.

The variation measure of $1_E$ is is this case denoted $\|\partial E\|$, and the function $-\sigma$ given by theorem 0.7 is denoted $\nu_E$.

We thus have for all $\phi \in C^1_c(\Omega, \mathbb{R}^N)$,

$$
\int_E \text{div} \phi(x) \, dx = \int_{\mathbb{R}^N} \langle \phi(x), \nu_E(x) \rangle \, d\|\partial E\|. 
$$

(0.8)
Definition 0.9. Let $E$ be a set of locally finite perimeter in $\Omega$. We say that $x \in \Omega$ belongs to the reduced boundary of $E$, denoted $\partial^* E$, if:

1. $\|\partial E\|(B(x,r)) > 0 \quad \forall r > 0$,
2. $\frac{1}{\|\partial E\|(B(x,r))} \int_{B(x,r)} \nu_E(y) \, d\|\partial E\| \to \nu_E(x) \quad r \to 0$,
3. $|\nu_E(x)| = 1$. 

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Then we have the following result:

**Theorem 0.10.** Let $E$ be a set of locally finite perimeter in $\Omega$. Then:
1. $\|\partial E\|(B) = \mathcal{H}^{N-1}(B \cap \partial^* E)$ for all Borel set $B \subset \Omega$.
2. Gauss-Green formula: For all $\phi \in C^1_c(\Omega, \mathbb{R}^N)$,
   \[
   \int_E \text{div } \phi(x) \, dx = \int_{\partial^* E} \langle \phi(x), \nu_E(x) \rangle \, d\mathcal{H}^{N-1}(x). \tag{0.9}
   \]
5.2 – Perimeter estimate

Theorem 0.11. Let \( K : [0, T] \to \mathcal{P}(\mathbb{R}^N) \) with closed graph in \( \mathbb{R}^{N+1} \) be such that

1. \( K(0) \) is compact,
2. \( K(t) \to K(0) \) in the Hausdorff distance as \( t \to 0 \),
3. \( u : (t, x) \mapsto 1_{K(t)}(x) \) is a viscosity sub-solution of the eikonal equation. Under the additional assumptions

   (A1) \( c \) is of class \( C^1 \), \( Dc \) is locally Lipschitz continuous with respect to the second variable,
   
   (A2) \( Dc(t, x) \neq 0 \) if \( c(t, x) = 0 \),
5 – Regularity of the front

We have:

1. For almost all $t \in [0, T]$, $c(t, \cdot) 1_{K(t)}$ has bounded variations in $\{c(t, \cdot) < 0\}$.

2. For almost all $t \in [0, T]$, $K(t)$ has locally finite perimeter in $\{c(t, \cdot) < 0\}$.

3. If we denote $(\cdot)_-$ the negative part of a quantity $(x)_- = \max(-x, 0)$, we have:

$$\int_0^T \int_{\partial^* K(s)} c_-(s, x) d\mathcal{H}^{N-1}(x) ds < +\infty.$$
Heuristic idea: apply the integral formulation with $\varepsilon = 0$ to $\phi = 1_{\{c < 0\}}$:

$$
\int_0^T \int_{K(s)} \phi_t(s, x) \, dx \, ds + \int_0^T \int_{\partial K(s)} c(s, x) \phi(s, x) \, d\mathcal{H}^{N-1}(x) \, ds
\geq \left[ \int_{K(s)} \phi(s, x) \, dx \, ds \right]_0^T
$$
5 – Regularity of the front

1. \[ \left[ \int_{K(s)} \phi(s, x) \, dx \, ds \right]_0^T = \left[ \int_{K(s)} 1_{\{c < 0\}}(s, x) \, dx \, ds \right]_0^T. \]
5 – Regularity of the front

1. \[ \left[ \int_{K(s)} \phi(s, x) \, dx \, ds \right]_0^T = \left[ \int_{K(s)} 1_{\{c<0\}}(s, x) \, dx \, ds \right]_0^T. \]

2. \[ \int_0^T \int_{K(s)} \phi_t(s, x) \, dx \, ds \leq \int_0^T \int_{K(s) \cap \{c(s, \cdot)=0\}} \frac{|c_t(s, x)|}{|Dc(s, x)|} \, d\mathcal{H}^{N-1}(x) \, ds. \]
5 – Regularity of the front

Thank you!