Lower semicontinuous value fonctions for target control problems with state constraints

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Outline

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Control problem with state constraints

$$v(T,x) := \text{Min } \varphi(y_x(T));$$

$$\begin{cases} \dot{y}_x(t) = f(y_x(t), u(t)), & t > 0, \\ y_x(0) = x, \end{cases} \quad (1a)$$

$$u(t) \in \mathcal{U} := L^\infty(0, +\infty; U) \text{ a.e. } t > 0, \quad (1b)$$

$$y_x(T) \in \mathcal{C}, \quad y_x(t) \in \mathcal{K} \text{ for } 0 \le t \le T. \quad (1c)$$

$\mathcal{C} \subset \mathcal{K}$ closed sets of $\mathbb{R}^d$ ($\mathcal{C} \neq \emptyset$), $U \subset \mathbb{R}^m$ a compact set, the functions $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ are Lipschitz, bounded, and $f(x, U)$ is convex for any $x \in \mathbb{R}^d$.

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Free state constraints ($\mathcal{K} = \mathcal{C} = \mathbb{R}^d$)

v is the unique bounded continuous viscosity solution of the HJB equation:

$$\begin{aligned} &\partial_t V - H(x, D_x V(x)) = 0 \qquad \text{in } x \in \mathbb{R}^d,\ t > 0 \\ &V(x, 0) = \varphi(x) \quad x \in \mathbb{R}^d, \end{aligned}$$

where $H(x, D_x V(x)) = \min_{a \in U} \Big(f(x, a) \cdot D_x V(x) \Big)$.

(Lions, Barles, Cappuzo-Dolcetta, ... etc)

RDV problem ($\mathcal{K} = \mathbb{R}^d, \ \mathcal{C} \subset \mathbb{R}^d$)

In this case, the value function v is the unique l.s.c. 'Bilateral' solution

$$\partial_t V - H(x, D_x V(x)) = 0 \qquad \text{in }]0, T[\times\mathbb{R}^d,$$
$$V(x,0) = \varphi(x)\chi_{\mathcal{C}}(x) \quad x \in \mathbb{R}^d,$$

(Barron-Jensen, Frankowska, ...)

RDV problem ($\mathcal{K} = \mathbb{R}^d, \ \mathcal{C} \subset \mathbb{R}^d$)

Bilateral solution: touching one side test function

Definition (Barron-Jensen, Frankowska)

A l.s.c. function v is a bilateral solution of the HJB equation, if

(i) for every $\phi \in C^1$ s. t. $u - \phi$ has a local minimum at $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+_*$,

$$\partial_t \phi(x,t) - H(x, D_x\phi(x,t)) = 0,$$

(ii) $v(x,0) = \varphi(x)\chi_{\mathcal{C}}(x) = \liminf_{\substack{y \to x \\ t \searrow 0+}} v(y,t)$,

$\forall x \in \mathbb{R}^d$

Known results

State constraints ($\mathcal{K} = \mathcal{C}$)

- Inward qualification constraint:

$$min_{a \in U} f(x, a) \cdot \eta_x < 0, \quad \forall x \in \partial\mathcal{K}$$

v is continuous on $\mathcal{K}$ (Ishii-Koike, Soner, ...)

- Outward qualification constraint:

$$\sup_{a \in U} f(x, a) \cdot \eta_x > 0, \quad \forall x \in \partial\mathcal{K}$$

The value function is the unique l.s.c. viscosity solution of the HJB equation on int($\mathcal{K}$) and supersolution on $\partial\mathcal{K}$. (Frankowska,Vinter, Plaskasz)

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Outline

Target problem with state constraints $(\mathcal{P})$

$$\vartheta(T,x) := \text{Min } \Psi(y_x(\tau));$$

$$\begin{cases} \dot{y}_x(t) = f(y_x(t), u(t)), & t > 0, \\ y_x(0) = x, \end{cases}$$

$$u(t) \in \mathcal{U} := L^\infty(0,+\infty; U) \text{ a.e. } t > 0,$$

$$\tau \in [0,T], \;\; y_x(\tau) \in \mathcal{C},$$

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$$\Psi(x) = \begin{cases} 0 & x \in \mathcal{C} \\ 1 & \text{otherwise} \end{cases} = \chi_{\mathcal{C}}(x).$$

- **Link with the optimal time problem**

$$\mathcal{T}(x) := \inf\{\tau \geq 0 \mid y_x(\tau) \in \mathcal{C}, y_x \subset \mathcal{K}\}.$$

Theorem

- $\vartheta(T, x) = 0 \Longleftrightarrow \mathcal{T}(x) \leq T$
- *For every* $x \in \mathbb{R}^d$,

$$\Big[\vartheta(T, x) = 1,\ \forall T \geq 0\Big] \Longleftrightarrow \mathcal{T}(x) = +\infty.$$

- $\mathcal{T}(x) = \inf\{T \geq 0;\ \vartheta(T, x) = 0\}.$

- **Link with front propagation:**

The target set $\mathcal{C}$: burned region at $t = 0$,

- **Link with the optimal time problem**

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- **Link with front propagation:**
  The target set $\mathcal{C}$: burned region at $t = 0$,
  $\vartheta(0, x) = \chi_{\mathcal{C}}(x)$.
  $\mathcal{C}_t := \{x \in \mathbb{R}^n,\ \vartheta(t, x) = 0\}$: burned region at time t.

"Evolution of regions" instead of the "level set approach".

$$\begin{cases} \dot{y}_x(t) = f(y_x(t), u(t)) \\ y_x(0) = x \end{cases} \rightsquigarrow \begin{cases} \dot{\tilde{y}}_x(t) = \lambda(t) f(\tilde{y}_x(t), \tilde{u}(t)) \\ \tilde{y}_x(0) = x \\ \lambda(t) \in \Lambda(\tilde{y}_x(t)) \end{cases}$$

$$\Lambda(z) = \begin{cases} \{0\} & \text{if } z \in \mathcal{K}^c \\ [0,1] & \text{if } z \in \mathcal{C} \cup \partial\mathcal{K} \\ \{1\} & \text{otherwise.} \end{cases}$$

Lemma 2.

Let $T \geq 0$, $x \in \mathcal{K}$. For every $\widetilde{y}_x : [0, T] \to \mathcal{K}$, there exist $0 \leq S \leq T$, $u \in \mathcal{U}$ such that:

$$\dot{y}_x(t) = f(y_x(t), u(t)) \;\; t \in (0, S), \quad y_x(0) = x,$$
$$y_x(S) = \widetilde{y}_x(T),$$

$$\{y_x(t); t \in (0, S)\} \equiv \{\widetilde{y}_x(t); t \in [0, T]\} \subset K.$$

Idea of the proof of Lemma 2:

$\widetilde{y}_x$ satisfies: $\exists \lambda : (0, T) \longrightarrow [0, 1]$ such that:

$$\begin{cases} \dot{\widetilde{y}}_x(t) = \lambda(t) f(\widetilde{y}_x(t), \widetilde{u}(t)), \\ \widetilde{y}_x(0) = x \end{cases} \qquad \lambda(t) \in \Lambda(\widetilde{y}_x(t)).$$

Let $\gamma(t) := \int_0^t \lambda(s) ds$. We construct y such that

$$y_x(\gamma(t)) = \widetilde{y}_x(\beta(t)); \quad S = \gamma(T);$$

where $\beta(t) = \inf\{0 \leq \tau \leq t,\ \gamma(\tau) = \gamma(t)\}$.

An other formulation of the target problem

$$\vartheta(T,x) = \inf\{\chi_C(\tilde{y}_x(t)),\ \dot{\tilde{y}}_x(t) \in F(\tilde{y}_x(t)),\ \tilde{y}_x(0) = x,\ \}$$

where $F(x) = \{\lambda f(x,a),\ a \in U,\ \lambda \in \Lambda(x)\}$ et

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Question

What is the HJB equation satisfied by ϑ?

Outline

Contingent epiderivative of a l.s.c function u:

$$D_{\uparrow}u(t,x)(s,p) := \liminf_{h\searrow 0, q\to p} \frac{1}{h}(u(t+hs, x+hq)-u(t,x))$$

When $F(x) := f(x,U)$ ($\mathcal{K} = \mathbb{R}^d$), F is cont.,

$$u = \vartheta \Leftrightarrow \begin{cases} u \text{ lsc, verifies (IC): } \liminf_{t\searrow 0,\ y\to x} u(y,t) = \chi_{\mathcal{C}}(x), \\ \forall t>0, x\in\mathbb{R}^n, \quad \sup_{p\in F(x)} -D_{\uparrow}u(t,x)(-1,p) \geq 0 \\ \forall t\geq 0,\ x\in\mathbb{R}^n, \quad \sup_{p\in F(x)} D_{\uparrow}u(t,x)(1,-p) \leq 0. \end{cases}$$

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Ref: Frankowska'93

$$Tr^-(x) := \left\{ \tilde{y}_x \,\middle|\, \exists \nu > 0 \ \dot{\tilde{y}}_x(\cdot) \subset F(\tilde{y}_x(\cdot)), \text{ on } [-\nu, 0], \text{ et } \tilde{y}_x(0) = x \right\},$$

$$\text{with } F(x) = \{\lambda f(x,a),\ a \in U,\ \lambda \in \Lambda(x)\}$$

$$D^F_\uparrow u(t,x)(\tilde{y}_x) := \liminf_{h \searrow 0} \frac{1}{h}(u(t+h, \tilde{y}_x(-h)) - u(t,x$$

Derivative with respect to trajectories arriving in x.

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Theorem (O. Bokanowski, HZ, N. Megdich)

$$u = \vartheta \Leftrightarrow \begin{cases} u \text{ is l.s.c, and satisfies the IC,} \\ \forall t > 0,\ x \in K, \quad \sup_{p \in F(x)} -D_{\uparrow} u(t,x)(-1,p) \geq 0, \\ \forall t \geq 0,\ x \in K, \quad \sup_{y \in Tr^{-}(x)} D_{\uparrow}^{F} u(t,x)(y) \leq 0, \end{cases}$$

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$$

Outline

Theorem (OB, HZ, NM)

The value function V is l.s.c on $\mathbb{R}^d \times \mathbb{R}^+_*$, and is a bilateral *solution of the HJB equation*

$$\min(\partial_t \vartheta(x,t) - \mathcal{H}(x, D_x\vartheta(x,t)), \vartheta - \chi_{\mathcal{K}}(x)) = 0,$$

$$V(x,0) = \Phi(x) \quad x \in \mathbb{R}^d,$$

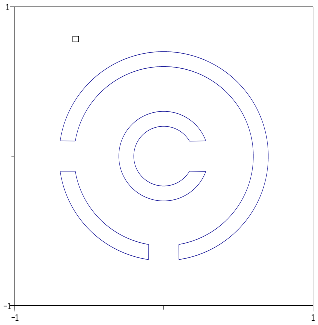
where $\mathcal{H}(x,p) = \inf\{\lambda f(x,a); \lambda \in \Lambda(x),\ a \in U\}$,

Algorithme: (without proof)

$$\min\left(\frac{V^{n+1}-V^{n}}{\Delta t}+[H(V^{n})],\ V^{n+1}-\chi_K\right)=0,$$

$$\Rightarrow\quad V^{n+1}=\max(V^{n}-\Delta t[H(V^{n})],\ \chi_K)$$

• 2d sparse method - The HJB-UltraBee (Bokanowski, Megdich, Zidani'05)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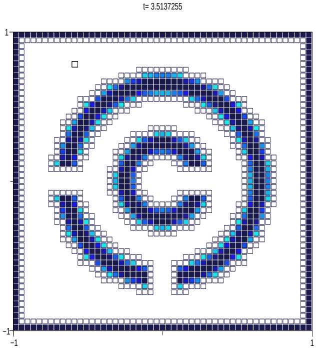
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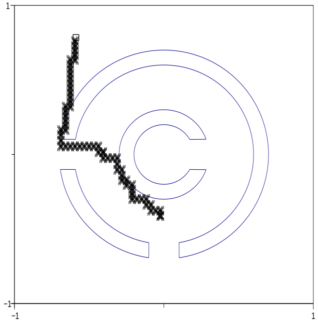
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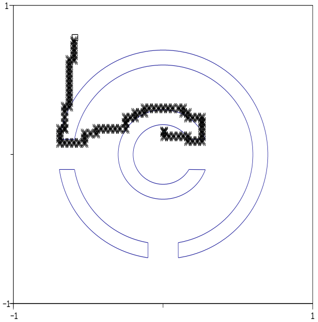
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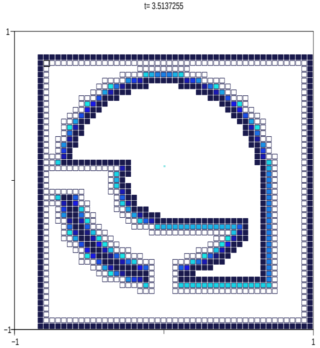
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