UNIVERSITA' STUDI NAPOLI FEDERICO II

Metastable states in a model of spin dependent point interactions

Claudio Cacciapuoti, Raffaele Carlone, Rodolfo Figari

Multiscale Analysis for Quantum Systems and Applications

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Istituto Nazionale di Alta Matematica

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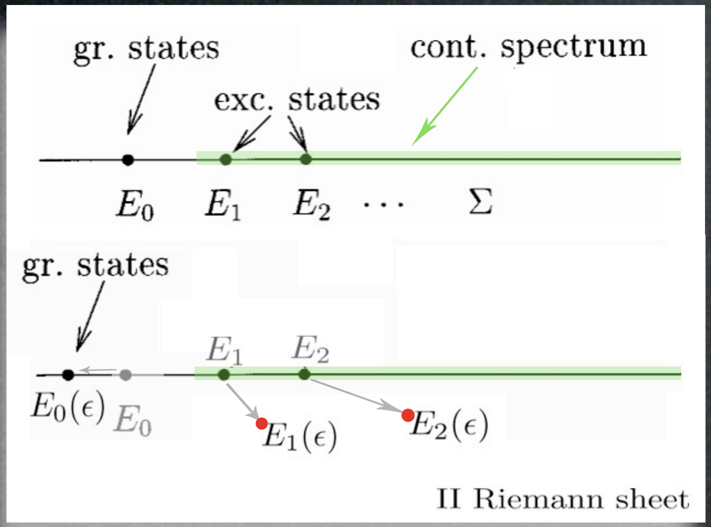

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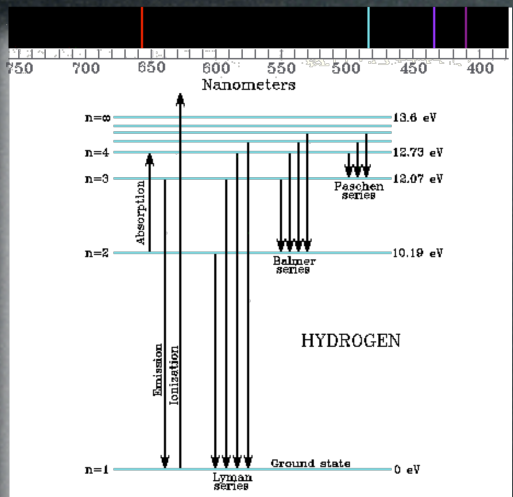

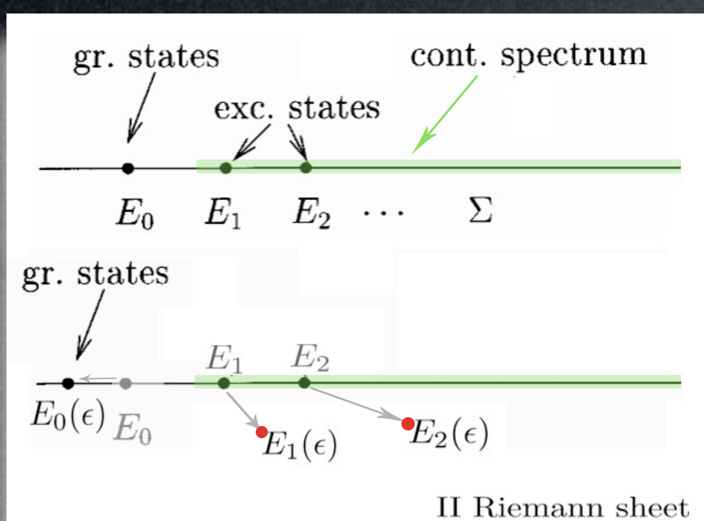

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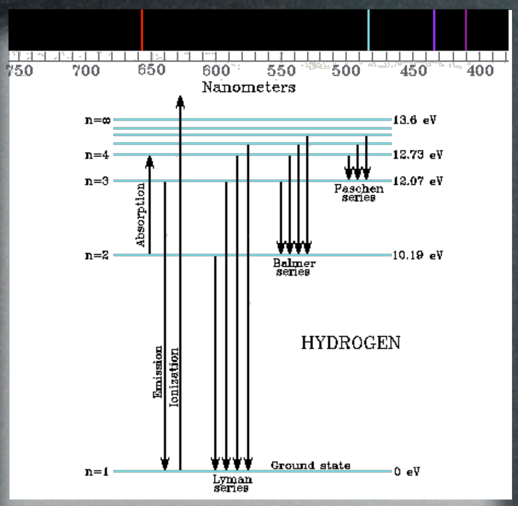
METASTABLE STATES IN A MODEL OF SPIN DEPENDENT POINT INTERACTIONS



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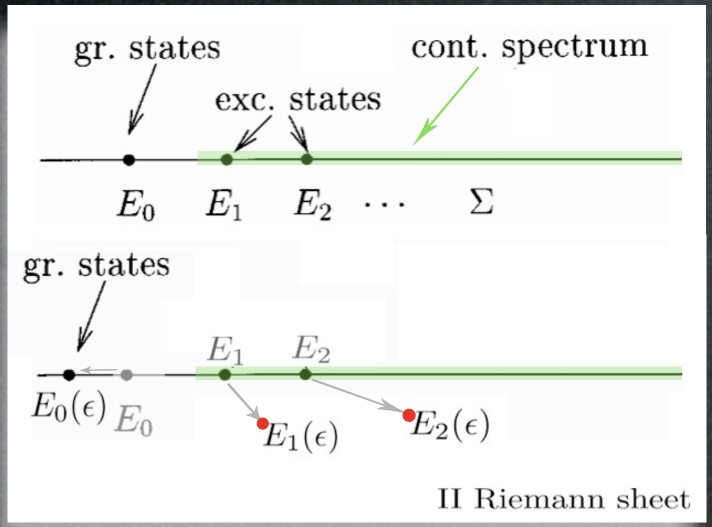

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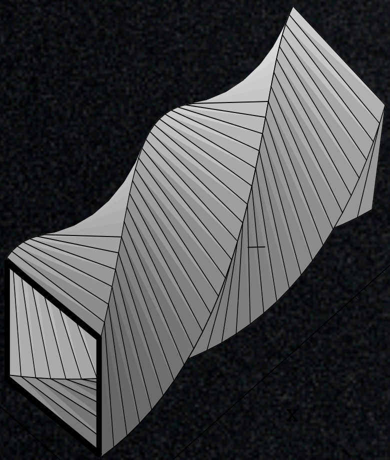
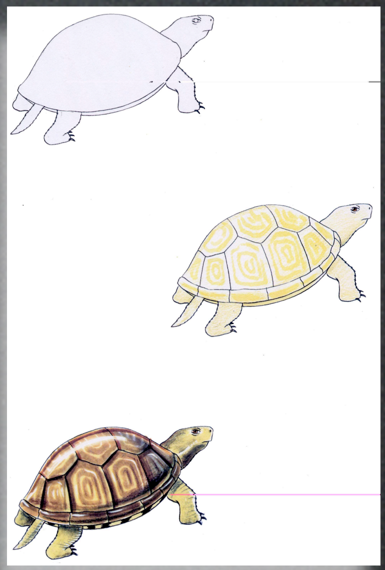


Resonances in twisted quantum waveguides
H. Kovarik, A. Sacchetti
J. Phys. A: Math. Theor. 40 8371-8384 (2007)



Quantum Zeno subspaces
P. Facchi, S. Pascazio
Phys. Rev. Lett. 89, 080401 (2002)

Metastable states in a model of spin dependent point interactions

The analysis of time decay of resonances will proceed as follows:

(1) define an "unperturbed" Hamiltonian $\hat{H}_0$ in a way such that the spectrum has one eigenvalue embedded in the continuous spectrum

(2) define Hamiltonian $\hat{H}$ as a self-adjoint perturbation, in some suitable sense, of $\hat{H}_0$

(3) show that the embedded eigenvalue turnes in a resonance

(4) estimate the decay times of such a metastable

Point Interaction: a very short introduction

$$(H_\alpha - k^2)^{-1} = (H - k^2)^{-1} + \frac{4\pi}{4\pi\alpha - i\,k}(G^{\bar{k}}(\cdot - y), \cdot\,)G^k(\cdot - y)$$

Being $D(H_\alpha) = \mathrm{Ran}[(H_\alpha - k^2)^{-1}]$ it is easily seen that

$$D(H_\alpha) = \Big\{\psi \in L^2(\mathbb{R}^3) : \psi = \psi^k + qG^k(\cdot - y);\ \psi^k \in H^2(\mathbb{R}^3),$$
$$q = \frac{4\pi\psi^k(y)}{4\pi\alpha - i\,k},\ k^2 \in \rho(H_\alpha), k > 0,\ -\infty < \alpha \le \infty\Big\}$$

Function $\psi^k(x)$ is called regular part and often constant q is referred to as charge.

$$G^k(x - y) = \frac{e^{ik|x-y|}}{4\pi|x - y|} \qquad k > 0$$

satisfies, in the sense of distributions, the equation $(-\Delta - z)G^z = \delta_y$, where δ_y is the three dimensional Dirac delta centered in y.

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this operator matches up with the point Hamiltonian defined formally in the sense that if $y \notin \mathrm{supp}[\psi^k]$ functions ψ and ψ^k coincide and H_α acts on ψ as $-\Delta$.

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$$H_\alpha\psi = -\Delta\psi \qquad \forall\psi \in C_0^\infty(\mathbb{R}^3\backslash y)$$

Point Interaction: A Very Short Introduction

The spectrum of H_α:

$$\sigma_{ac}(H_\alpha) = [0, \infty)$$

If $\alpha < 0$, H_α has one eigenvalue

$$\sigma_{pp}(H_\alpha) = \{-(4\pi\alpha)^2\} \qquad -\infty < \alpha < 0$$

the corresponding normalized eigenfunction is

$$\phi_0 = \sqrt{2|\alpha|}\frac{e^{4\pi\alpha|x-y|}}{|x-y|}$$

If $\alpha \geq 0$, then $\sigma_{pp} = \varnothing$.

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For every $k \in \mathbb{R}^3$ the generalized eigenfunction of H_α corresponding to the energy $E = |k|^2$ in the continuous spectrum is given in closed form by

$$\Phi^y_\pm(x,k) = e^{ikx} + \frac{e^{iky}}{4\pi\alpha \pm i|k|}\frac{e^{\mp i|k||x-y|}}{|x-y|}$$

Spin dependent point interaction

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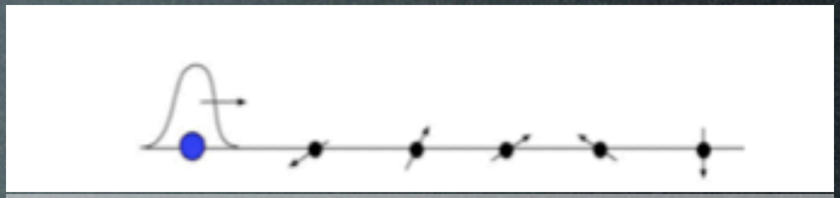
Spin dependent point potentials in one and three dimensions.
Claudio Cacciapuoti, Raffaele Carlone, Rodolfo Figari.
J. Phys. A: Math. Theor. 40 (2007) 249-261



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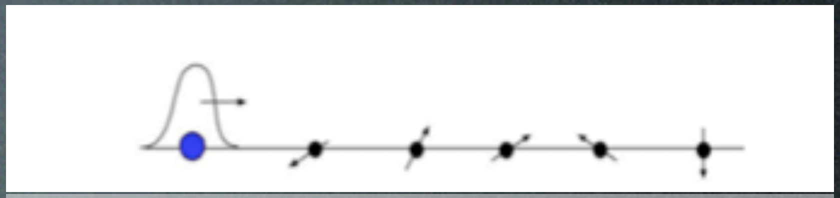

The Hilbert space for a system made up of one particle in dimension d and one spin $1/2$ can be written

$$\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathbb{C}^2 \qquad d = 1, 2, 3\,.$$

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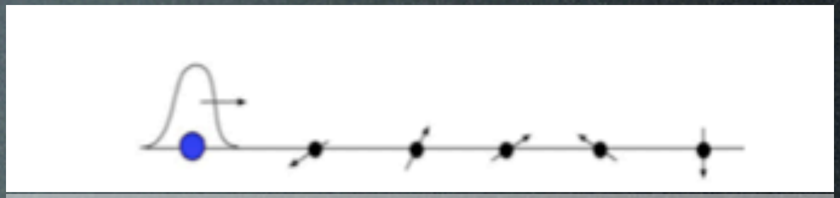

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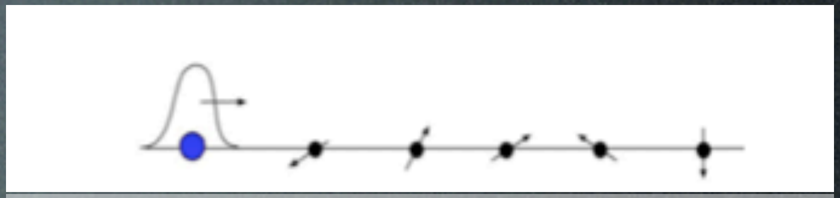

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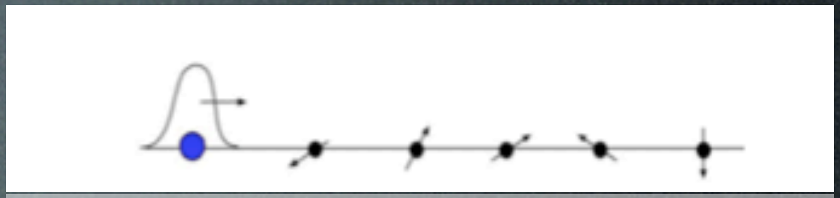

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$$D(S) := C_0^\infty(\mathbb{R}^d \backslash \underline{0}) \otimes \mathbb{C}^2 \qquad d = 1, 2, 3$$

$$S := -\Delta \otimes \mathbb{I}_{\mathbb{C}^2} + \mathbb{I}_{L^2} \otimes \beta \hat{\sigma}^{(1)} \qquad \beta \in \mathbb{R}^+\,,$$

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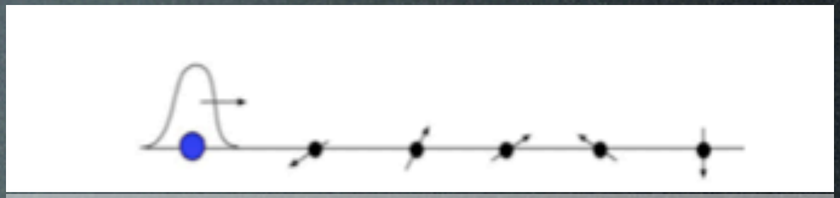

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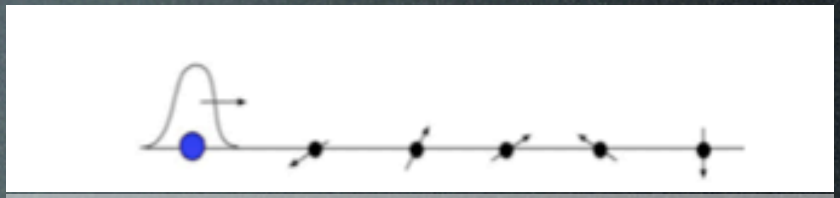

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$$H := -\Delta \otimes \mathbb{I}_{\mathbb{C}^2} + \mathbb{I}_{L^2} \otimes \beta \hat{\sigma}^{(1)} \qquad \beta \in \mathbb{R}^+\,,$$

$$H\Psi = \sum_\sigma \left(-\Delta + \beta\,\sigma \right) \psi_\sigma \otimes \chi_\sigma \qquad \Psi \in D(H)\,.$$

$$R(z)\Psi = \sum_\sigma \left(-\Delta - z + \beta\,\sigma \right)^{-1} \psi_\sigma \otimes \chi_\sigma \qquad \Psi \in \mathcal{H};\ z \in \rho(H)\,,$$

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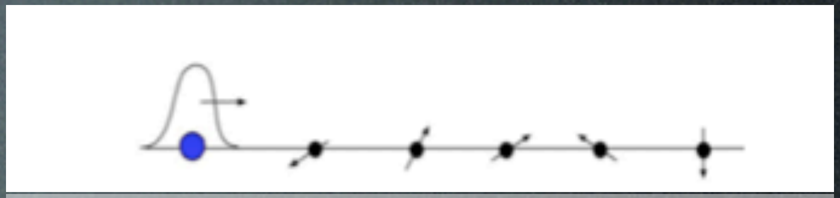

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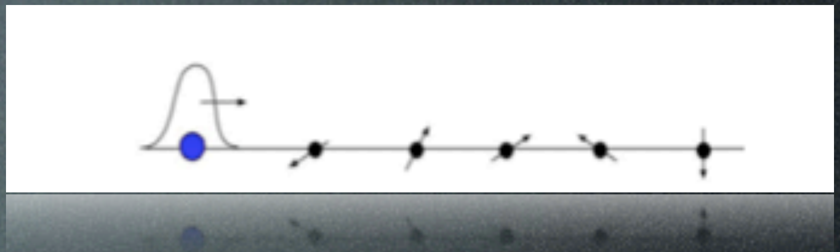

$$\hat{R}_0(z) = R(z) + \sum_{\sigma,\sigma'} \left((\Gamma_0(z))^{-1}\right)_{\sigma,\sigma'} \langle \Phi^{\bar{z}}_{\sigma'}, \cdot \rangle \Phi^z_\sigma \qquad z \in \mathbb{C}\backslash\mathbb{R}\,,$$

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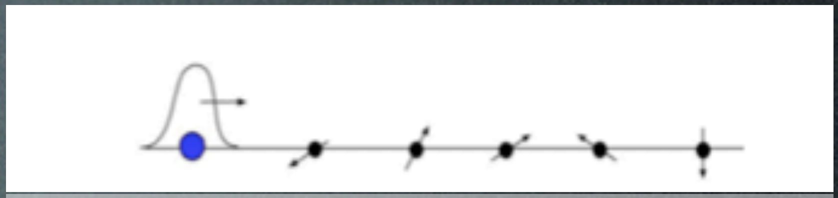

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$$q_\sigma = -\alpha f_\sigma \, , \; d = 1 \, ; \; \alpha q_\sigma = f_\sigma \, , \; d = 2 \, ; \; \alpha q_\sigma = f_\sigma \, , \; d = 3 \Big\}$$

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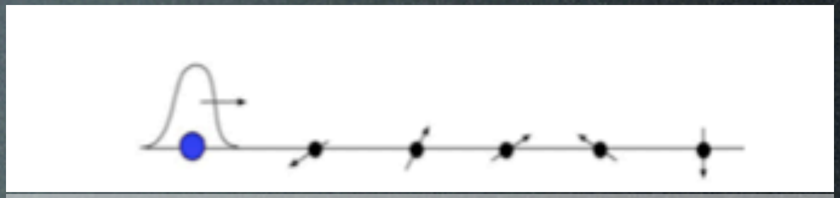

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$$d=1 \quad \Gamma_0(z) = \begin{pmatrix} -\frac{i}{2\sqrt{z-\beta}} - \frac{1}{\alpha} & 0 \\ 0 & -\frac{i}{2\sqrt{z+\beta}} - \frac{1}{\alpha} \end{pmatrix}$$

$$d=2 \quad \Gamma_0(z) = \begin{pmatrix} \frac{\ln(\sqrt{z-\beta}/2) + \gamma - i\pi/2}{2\pi} + \alpha & 0 \\ 0 & \frac{\ln(\sqrt{z+\beta}/2) + \gamma - i\pi/2}{2\pi} + \alpha \end{pmatrix}$$

$$d=3 \qquad \Gamma_0(z) = \begin{pmatrix} \frac{\sqrt{z-\beta}}{4\pi i} + \alpha & 0 \\ 0 & \frac{\sqrt{z+\beta}}{4\pi i} + \alpha \end{pmatrix}$$

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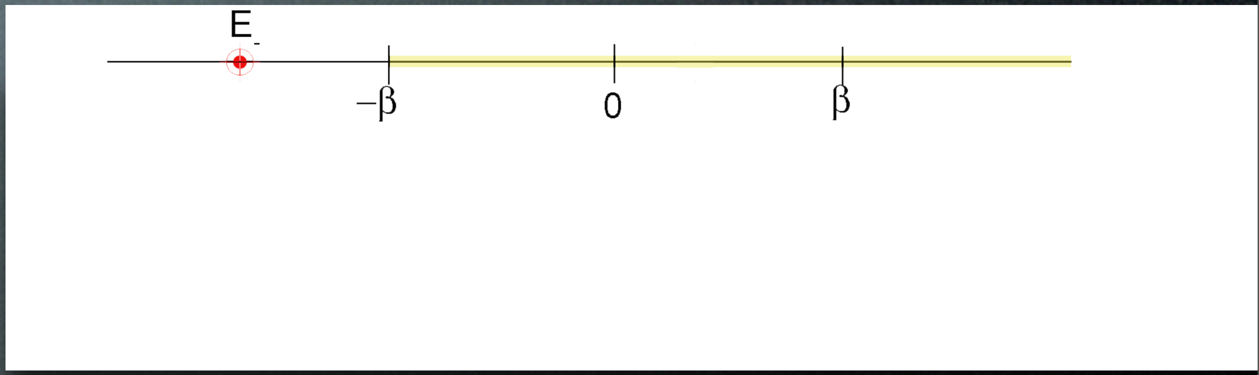

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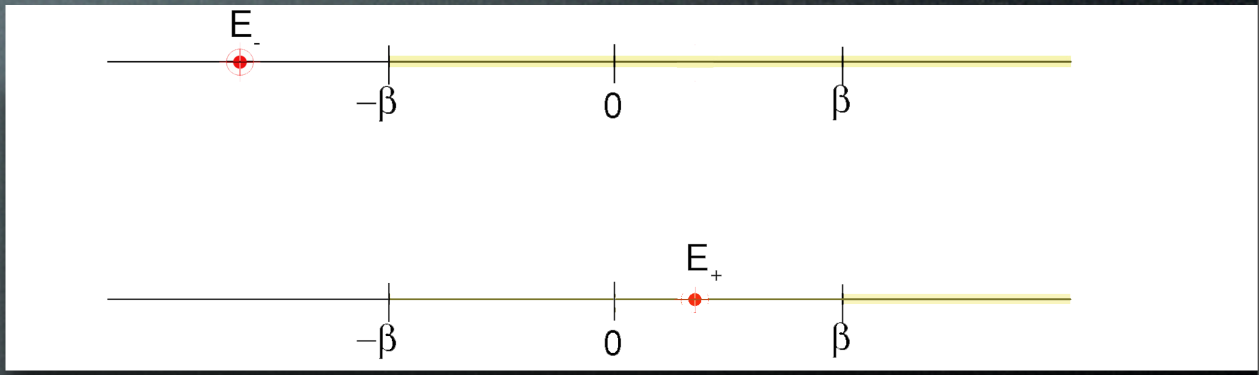

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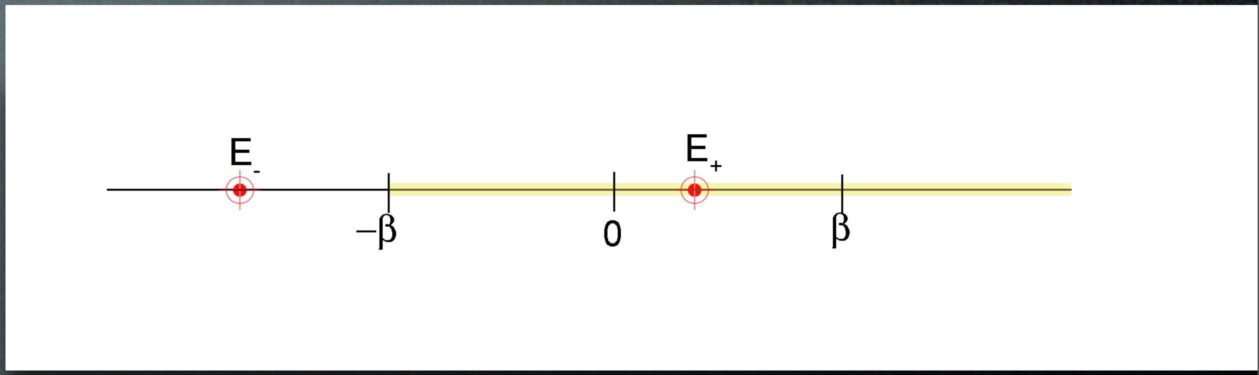
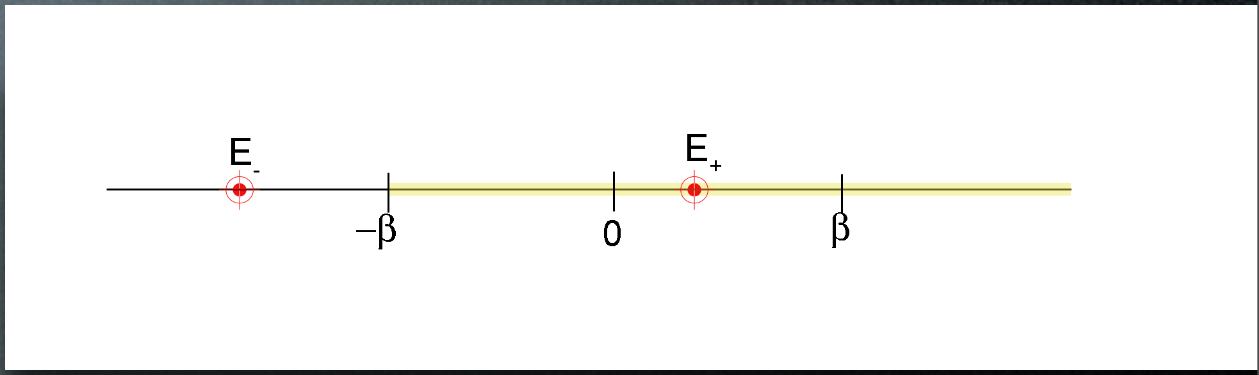


For $d = 1, 2, 3$ the essential spectrum is given by

$$\sigma_{ess}(\hat{H}) = [-\beta, +\infty)\,.$$



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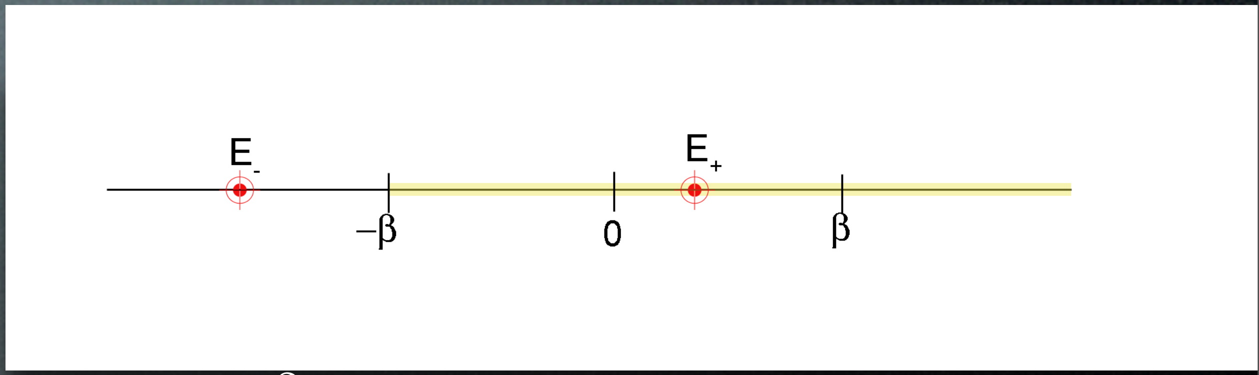

Consider $-\infty < \alpha < 0$

$d = 1.$

$$E_{0,-} = -\beta - \frac{\alpha^2}{4}\,; \qquad E_{0,+} = \beta - \frac{\alpha^2}{4}\,. \tag{1}$$

For all $-\infty < \alpha < 0$ the lowest eigenvalue, $E_{0,-}$, is below the threshold of essential spectrum and $-2\sqrt{2\beta} \leqslant \alpha < 0$ the second eigenvalue is embedded in the continuous spectrum, $-\beta \leqslant E_{0,+} < \beta$.

$d = 2.$

$$E_{0,-} = -\beta - 4e^{-2(2\pi\alpha+\gamma)}\,; \qquad E_{0,+} = \beta - 4e^{-2(2\pi\alpha+\gamma)}\,. \tag{2}$$

The lowest eigenvalue, $E_{0,-}$, is always below the threshold of essential spectrum if $-(\ln(\sqrt{\beta/2}) + \gamma)/(2\pi) \leqslant \alpha < \infty$ the second eigenvalue is embedded in the continuous spectrum, $-\beta \leqslant E_{0,+} < \beta$.

$d = 3.$

$$E_{0,-} = -\beta - (4\pi\alpha)^2\,; \qquad E_{0,+} = \beta - (4\pi\alpha)^2\,. \tag{3}$$

The lowest eigenvalue, $E_{0,-}$, is always below the threshold of essential spectrum and if $-\sqrt{2\beta}/(4\pi) \leqslant \alpha < 0$ the second eigenvalue is embedded in the continuous spectrum, $-\beta \leqslant E_{0,+} < \beta$.

Turning to Resonances

Let $-\infty < \alpha \leqslant \infty$ and $0 < \varepsilon \ll \alpha$, then

$$D(\hat{H}_\varepsilon) := \Big\{ \Psi \in \mathcal{H} \Big| \ \Psi = \Psi^z + \sum_\sigma q_\sigma \Phi_\sigma^z \, ; \ \Psi^z = \sum_\sigma \psi_\sigma^z \otimes \chi_\sigma \in D(H); \ z \in \rho(\hat{H}_\varepsilon)\, ;$$

$$\begin{aligned} q_\pm &= -\alpha f_\pm - \varepsilon f_\mp \qquad && d = 1\, ; \\ \alpha q_\pm + \varepsilon q_\mp &= f_\pm && d = 2\, ; \\ \alpha q_\pm + \varepsilon q_\mp &= f_\pm && d = 3 \ \Big\} \end{aligned}$$

$$\hat{H}_\varepsilon \Psi := H\Psi^z + z \sum_\sigma q_\sigma \Phi_\sigma^z \qquad \Psi \in D(\hat{H}_\varepsilon) \Big\}$$

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$$\begin{aligned} & q_\pm = -\alpha f_\pm - \varepsilon f_\mp && d = 1\,; \\ & \alpha q_\pm + \varepsilon q_\mp = f_\pm && d = 2\,; \\ & \alpha q_\pm + \varepsilon q_\mp = f_\pm && d = 3 \Big\} \end{aligned}$$

$$\hat{H}_\varepsilon \Psi := H\Psi^z + z \sum_\sigma q_\sigma \Phi_\sigma^z \qquad \Psi \in D(\hat{H}_\varepsilon) \Big\}$$

$$\hat{R}_\varepsilon(z) = R(z) + \sum_{\sigma,\sigma'} \big((\Gamma_\varepsilon(z))^{-1} \big)_{\sigma,\sigma'} \langle \Phi_{\sigma'}^{\bar{z}}, \cdot \rangle \Phi_\sigma^z \qquad z \in \mathbb{C} \backslash \mathbb{R}\,,$$

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$$\Gamma_\varepsilon(z) = \begin{pmatrix} -\dfrac{i}{2\sqrt{z-\beta}} - \dfrac{\alpha}{\alpha^2 - \varepsilon^2} & \dfrac{\varepsilon}{\alpha^2 - \varepsilon^2} \\ \dfrac{\varepsilon}{\alpha^2 - \varepsilon^2} & -\dfrac{i}{2\sqrt{z+\beta}} - \dfrac{\alpha}{\alpha^2 - \varepsilon^2} \end{pmatrix} \qquad d = 1$$

$$\Gamma_\varepsilon(z) = \begin{pmatrix} \dfrac{\ln(\sqrt{z-\beta}/2) + \gamma - i\pi/2}{2\pi} + \alpha & \varepsilon \\ \varepsilon & \dfrac{\ln(\sqrt{z+\beta}/2) + \gamma - i\pi/2}{2\pi} + \alpha \end{pmatrix} \qquad d = 2$$

$$\Gamma_\varepsilon(z) = \begin{pmatrix} -\dfrac{i\sqrt{z-\beta}}{4\pi} + \alpha & \varepsilon \\ \varepsilon & -\dfrac{i\sqrt{z+\beta}}{4\pi} + \alpha \end{pmatrix} \qquad d = 3$$

For $d = 1, 2, 3$ the point spectrum is given by real roots of equation $\det \Gamma_\varepsilon(z) = 0$. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$d = 1$. If $0 \leqslant \alpha \leqslant \infty$ the point spectrum is empty. If $-\infty < \alpha < 0$ the point spectrum consists of one simple eigenvalue $E_{\varepsilon,-} < -\beta$.

$d = 2$. For $\alpha = \infty$ the point spectrum is empty. For all $-\infty < \alpha < \infty$ the point spectrum consists of one simple eigenvalue $E_{\varepsilon,-} < -\beta$.

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Resonances are defined as zeroes in the unphysical sheet of $\det \Gamma_\varepsilon(z)$

$$\zeta_\varepsilon = \frac{\varepsilon^2\alpha^2}{16\beta}\left[1 - i\left(\frac{8\beta}{\alpha^2} - 1\right)^{1/2}\right] + \mathcal{O}\big((\varepsilon/\alpha)^4\big) \qquad d = 1$$

$$\zeta_\varepsilon = -\frac{(2\pi\varepsilon)^2}{\eta^2 + (\pi/2)^2}\left(\eta + i\frac{\pi}{2}\right) + \mathcal{O}\big((\varepsilon/\alpha)^4\big) \qquad d = 2$$

$$\zeta_\varepsilon = \frac{(4\pi)^4\varepsilon^2\alpha^2}{\beta}\left[1 - i\left(\frac{2\beta}{(4\pi\alpha)^2} - 1\right)^{1/2}\right] + \mathcal{O}\big((\varepsilon/\alpha)^4\big) \qquad d = 3$$

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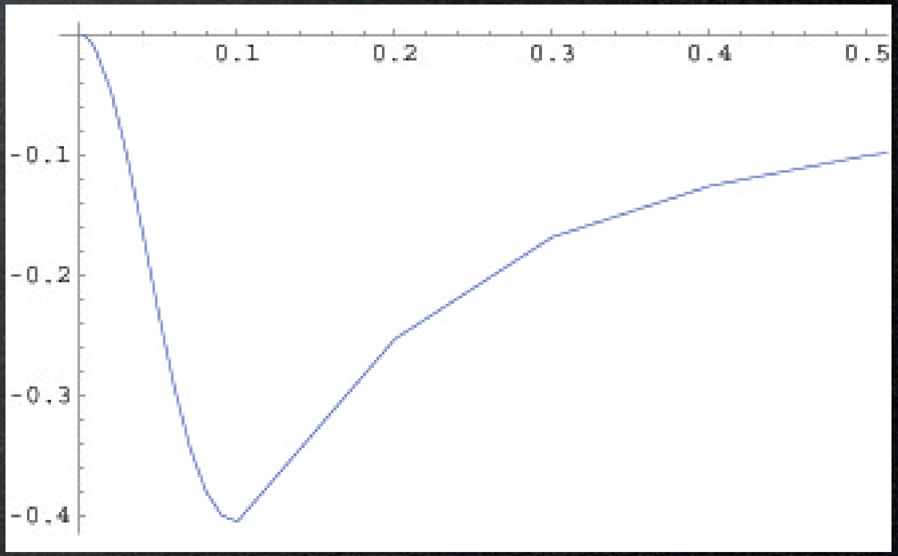


$$\Phi_i^{\epsilon,\lambda} = \Phi_i^{\lambda} + \sum_{\sigma'} \phi_i^{\lambda}(0) \left(\Gamma_\epsilon(\lambda)\right)^{-1}_{\sigma'\sigma} G_+^{\lambda-\sigma'\alpha}(\cdot) \otimes \chi_{\sigma'}$$

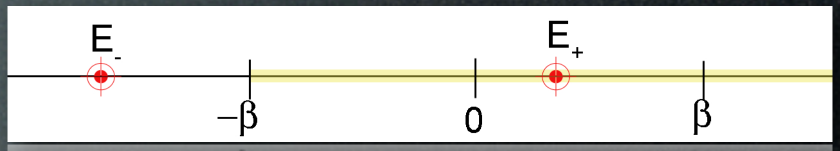
$$\Phi^{\lambda}_{(1)}(\omega) = \frac{\sqrt{\lambda-\beta}}{4\pi^{3/2}} \left[e^{i\omega\sqrt{\lambda-\beta}x} \otimes \chi_+ + \frac{\frac{\sqrt{\lambda+\beta}}{4\pi i} - \alpha}{\left(\frac{\sqrt{\lambda-\beta}}{4\pi i} - \alpha\right)\left(\frac{\sqrt{\lambda+\beta}}{4\pi i} - \alpha\right) - \epsilon^2} \frac{e^{i\sqrt{\lambda-\beta}|x|}}{4\pi|x|} \otimes \chi_+ + \right.$$

$$\left. + \frac{i\epsilon}{\left(\frac{\sqrt{\lambda-\beta}}{4\pi i} - \alpha\right)\left(\frac{\sqrt{\lambda+\beta}}{4\pi i} - \alpha\right) - \epsilon^2} \frac{e^{i\sqrt{\lambda+\beta}|x|}}{4\pi|x|} \otimes \chi_- \right] \qquad \lambda > \beta,\ \omega \in S^2$$

$$\Phi^{\lambda}_{(2)}(\omega) = \frac{\sqrt{\lambda+\beta}}{4\pi^{3/2}} \left[e^{i\omega\sqrt{\lambda+\beta}x} \otimes \chi_- + \frac{\frac{\sqrt{\lambda-\beta}}{4\pi i} - \alpha}{\left(\frac{\sqrt{\lambda-\beta}}{4\pi i} - \alpha\right)\left(\frac{\sqrt{\lambda+\beta}}{4\pi i} - \alpha\right) - \epsilon^2} \frac{e^{i\sqrt{\lambda+\beta}|x|}}{4\pi|x|} \otimes \chi_- + \right.$$

$$\left. + \frac{i\epsilon}{\left(\frac{\sqrt{\lambda-\beta}}{4\pi i} - \alpha\right)\left(\frac{\sqrt{\lambda+\beta}}{4\pi i} - \alpha\right) - \epsilon^2} \frac{e^{i\sqrt{\lambda-\beta}|x|}}{4\pi|x|} \otimes \chi_+ \right] \qquad \lambda > -\beta,\ \omega \in S^2$$

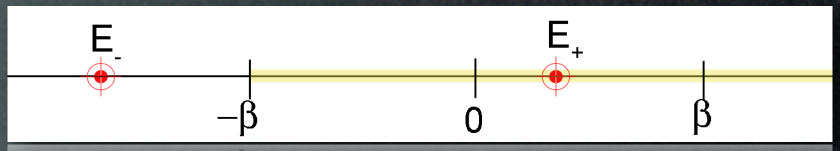
Time Decay



$$P^{\Delta}_{++}(t) = \int_{\Delta} \frac{\epsilon^2\, e^{-\imath\lambda t}}{\left|\left(\frac{\sqrt{\beta-\lambda}}{4\pi} - \alpha\right)\left(\frac{\sqrt{\beta+\lambda}}{4\pi\imath} - \alpha\right) - \epsilon^2\right|^2} \frac{e^{-2\sqrt{\beta-\lambda}|x|}}{16\pi^2|x|^2} d\lambda$$

$$\simeq \frac{e^{-8\pi\alpha|x|}}{16\pi^2|x|^2} \int_{\Delta} \frac{\epsilon^2\, e^{-\imath\lambda t}}{\left|\left(\frac{\sqrt{\beta-\lambda}}{4\pi} - \alpha\right)\left(\imath\frac{\sqrt{2\beta-(4\pi\alpha)^2}}{4\pi} + \alpha\right) + \epsilon^2\right|^2} d\lambda$$

$$= A\frac{e^{-8\pi\alpha|x|}}{|x|}\left(\frac{1}{\pi}\int_{\Delta} e^{-\imath\lambda t} L(\lambda; x_0)\, d\lambda\right)$$



$$P_{++}^{\Delta}(t) = \int_{\Delta} \frac{\epsilon^2\, e^{-\imath\lambda t}}{\left|\left(\frac{\sqrt{\beta-\lambda}}{4\pi} - \alpha\right)\left(\frac{\sqrt{\beta+\lambda}}{4\pi\imath} - \alpha\right) - \epsilon^2\right|^2} \frac{e^{-2\sqrt{\beta-\lambda}|x|}}{16\pi^2|x|^2} d\lambda$$

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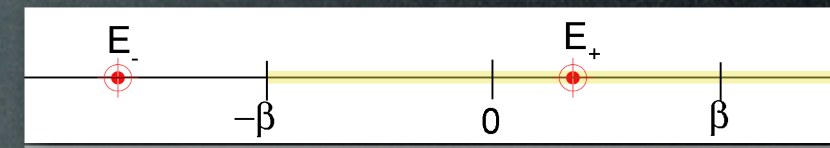
$L(\lambda; x_0) = \frac{\gamma}{(\lambda-x_0)^2+\gamma^2}$ is the Lorentzian distribution in the variable λ and

$$\Delta = [-a, a],\ x_0 = E_+ + \frac{(4\pi\epsilon)^2}{\beta}(4\pi\alpha)^2,\ A = \frac{8\pi^2\alpha}{\sqrt{2\beta - (4\pi\alpha)^2}}$$

$$\gamma = (4\pi\alpha)^3\epsilon^2\frac{\alpha}{\beta}\sqrt{2\beta - (4\pi\alpha)^2}$$



$$P^{\Delta}_{++}(t) = \int_{\Delta} \frac{\epsilon^2\, e^{-\imath\lambda t}}{\left|\left(\frac{\sqrt{\beta-\lambda}}{4\pi} - \alpha\right)\left(\frac{\sqrt{\beta+\lambda}}{4\pi\imath} - \alpha\right) - \epsilon^2\right|^2} \frac{e^{-2\sqrt{\beta-\lambda}|x|}}{16\pi^2|x|^2} d\lambda$$

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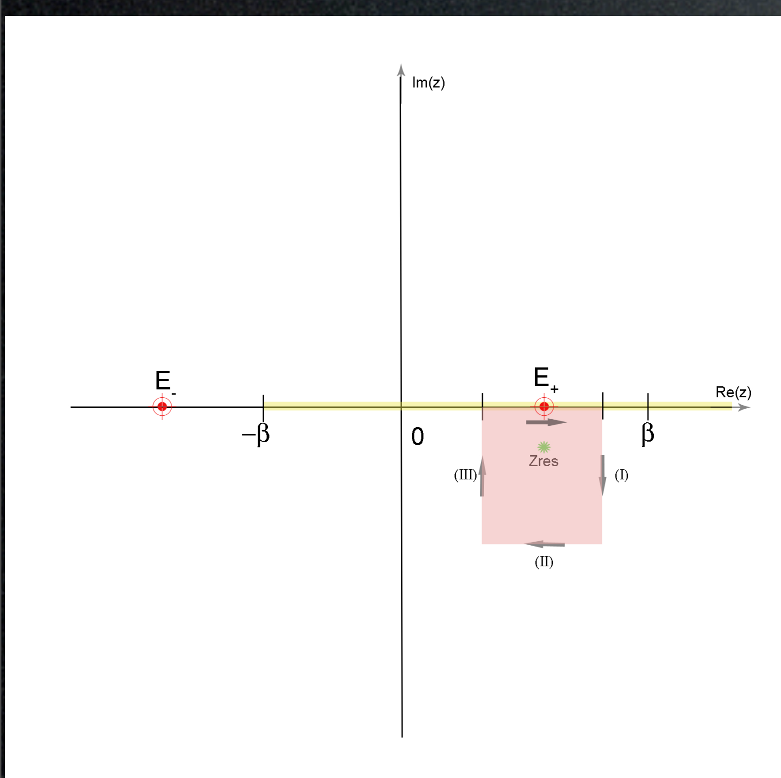
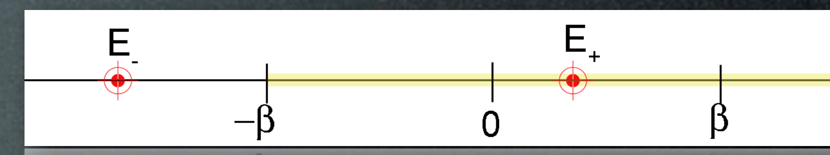


TIME DECAY



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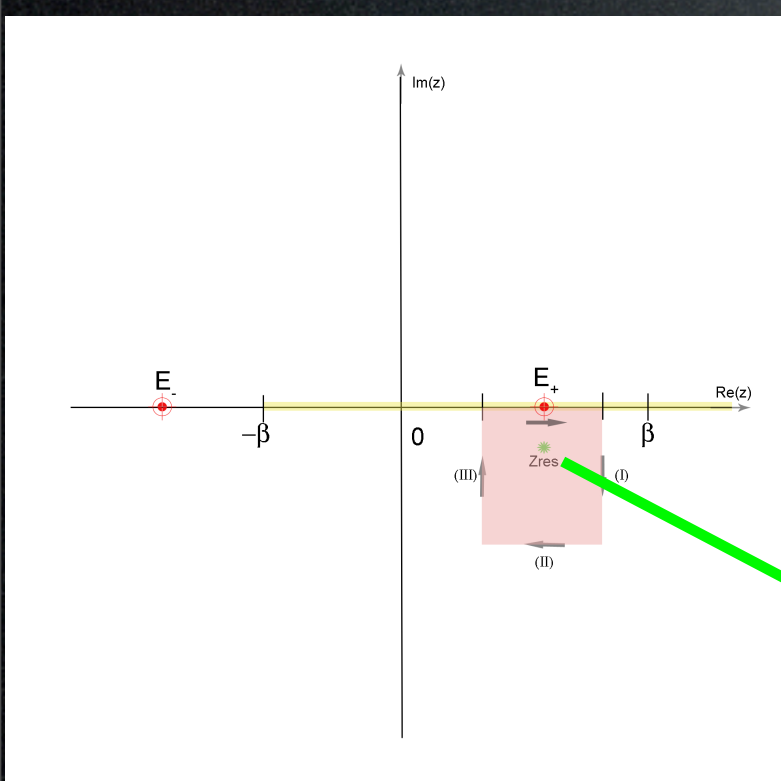
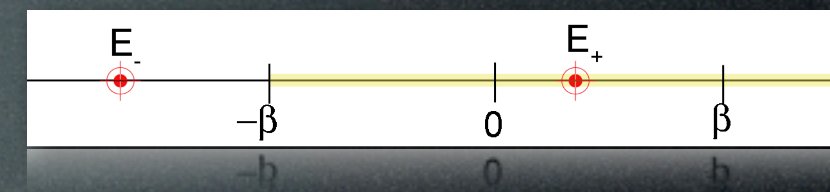


$$2\pi\imath\,\mathrm{Res}_{z=(x_0-i\gamma)}\left[e^{-\imath z t}L(z;x_0)\right] = \pi\, e^{-\imath\, x_0\, t}e^{-\gamma\, t}$$

TIME DECAY



$$P^{\Delta}_{++}(t) = \int_{\Delta} \frac{\epsilon^2\, e^{-\imath\lambda t}}{\left|\left(\frac{\sqrt{\beta-\lambda}}{4\pi}-\alpha\right)\left(\frac{\sqrt{\beta+\lambda}}{4\pi\imath}-\alpha\right)-\epsilon^2\right|^2}\,\frac{e^{-2\sqrt{\beta-\lambda}|x|}}{16\pi^2|x|^2}d\lambda$$

$$\simeq \frac{e^{-8\pi\alpha|x|}}{16\pi^2|x|^2}\int_{\Delta}\frac{\epsilon^2\, e^{-\imath\lambda t}}{\left|\left(\frac{\sqrt{\beta-\lambda}}{4\pi}-\alpha\right)\left(\imath\frac{\sqrt{2\beta-(4\pi\alpha)^2}}{4\pi}+\alpha\right)+\epsilon^2\right|^2}d\lambda$$

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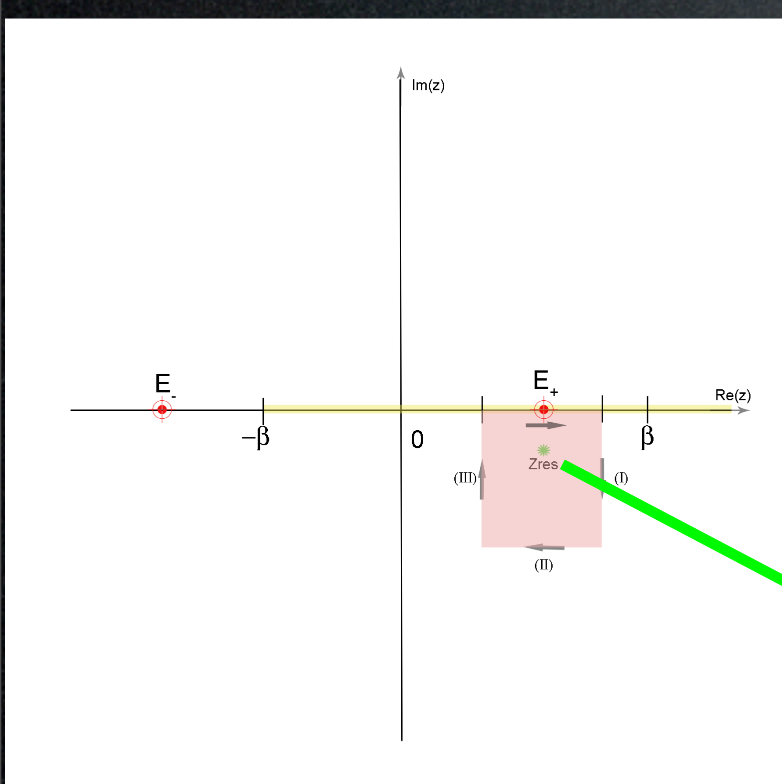

$$2\pi\imath\,\mathrm{Res}_{z=(x_0-i\gamma)}\left[e^{-\imath z t}L(z;x_0)\right] = \pi\, e^{-\imath\, x_0\, t}e^{-\gamma\, t}$$

Conclusions

The system we have analyzed is made of a localized q-bit, a two level model atom, and a particle, interacting with the q-bit via zero range forces.

We have reproduced in a solvable model the mechanism of formation of a resonance.

The generalization to an N-level atom is straightforward.

Model with a gas of non interacting particles are under study.