Analysis on Metric Graphs

What is it good for?

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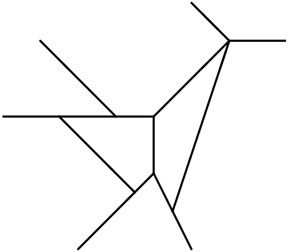
Contents[1]

[1] Surveys and original articles referring to the topic are contained in Refs. 1-3.

The Basic Objects :Metric Graphs

Definition: A metric graph $\mathcal{G}$ is a finite collection of halflines and intervals of given lengths with an identification of some of its endpoints (=vertices)



A graph with $n = 6$ external lines and $m = 8$ internal lines

$\mathcal{G}$ is a metric space:

There is the unique notion of a distance between two points

This additional metric structure makes metric graphs different from combinatorial graphs

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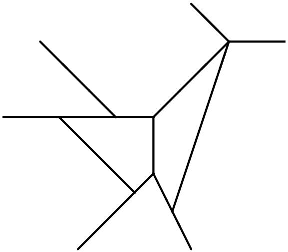

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Networks

Networks appear in many contexts

Mostly they are used to decribe

Information and data tranfer

and more generally

Transport Phenomena

Optical fibres

Question:

Is there a theory of solitary waves on networks?

In particular:

Can one split a solitary wave into several solitary waves at junctions?

Possible applications :

Information transmission in Optical Fibres

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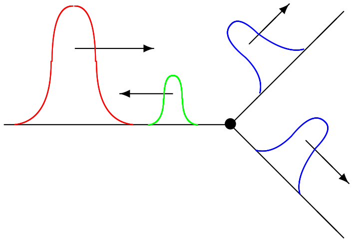
Signal Transport in Optical Fibres

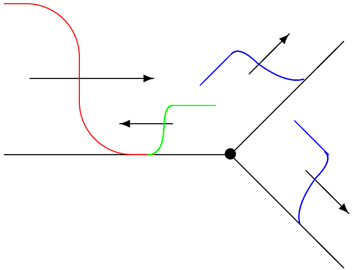
Illustration of signal splitting (A desideratum):



An incoming solitary wave from the left approaching a node
and then being split into 3 outgoing solitary waves
one of which is a reflected wave
and two of which are transmitted waves

Propagation of Shock Waves and Fire Fronts

Another desideratum : Propagation of shock waves or fire fronts through networks



An incoming kink from the left approaching a node and then being split into 3 outgoing kinks one of which is a reflected kink and two of which are transmitted kinks

Integrable Systems

The notion of solitary waves and kinks stems from the mathematical theory of Integrable Systems

Historical origin

In 1834 John Scott Russell followed on horseback a wave in a narrow water channel, which remained stable for several miles

Integrable Systems

In 1895 Diederik Korteweg and Gustav de Vries showed that this phenomenon could be described by a solution to the equation for the hight η of the wave

$$\partial_t \eta = \frac{3}{2}\sqrt{\frac{g}{l}}\partial_x \left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\partial_x^2\eta\right)$$

with

$$\sigma = \frac{l^3}{3} - \frac{Tl}{\rho g}$$

(t is time, x is the coordinate of the waterchannel, g is the graviational constant, l is the depth of the channel, T is the surface tension and ρ the density of the water)

Integrable Systems

The reason for the stability of the solitary wave is that the KdV system is a

completely integrable system

which means it has infinitely many

conserved quantities.

By now there is an enormous literature on completely integrable systems living on the real axis $\mathbb{R}$.

Except for some studies on intervals[2], so far a theory of integrable systems on graphs is lacking.

[2] see *e.g.* the articles in Ref. 4 and literature quoted there

Biological Networks: Blood flow

In computational and mathematical models of

Blood Flow

one discusses a wave equation of the form

$$\partial_t Z + \Xi \partial_x Z = 0, \quad Z(x,t) = \begin{pmatrix} z_1(x,t) \\ z_2(x,t) \end{pmatrix}, \quad \Xi(x) = \begin{pmatrix} c(x) & 0 \\ 0 & -c(x) \end{pmatrix}$$

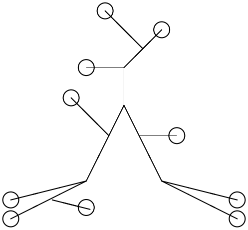
where x is a coordinate on and $c(x)$ represents the wave speed

AIM:

To improve the understanding of basic human physiology and diseases affecting the circulatory system

Parameters of interest are

Blood velocity, Blood Pressure



The arterial tree

Blood flow : Modelling

Difficulties in modelling:

1. partial occlusions, surgical interventions
2. spatial extent, shape and material properties of junctions

Difficulties in evaluating

1. the suitability of simplified graphs
2. the sensibility to geometric variations

Example:

There is one interesting model where the spectral properties of a nice class of differential operators can be studied [3]:

A metric tree graph with an infinite number of edges but of finite depth

[3] see Ref. 6

Biological Networks: Additional problems

Other possible problems:

1. Provide a graph model for the neural network of the brain
2. Describe propagation of diseases along networks
3. Describe spatial patterns of degeneration/tissue deaths caused by inadequate supply of essential chemicals distributed by networks

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Brownian Motion and Diffusion

Stochastic processes like Brownian motion have by now found a solid mathematical foundation with applications in relatively new fields of mathematics and physics like *e.g.* Financial Mathematics and Quantum Field Theory.

Except for some studies on intervals (or the half line) by Ito and MacKean [4] nothing seems to be known so far about stochastic processes on arbitrary graphs.

Possible applications of such a theory:

1. Stochastic processes on a decision tree with possible time delay or termination at the junctions.
2. Heat flow on graphs or more generally: Any diffusion process with possible losses at the junctions. This would involve the study of heat kernels.

[4] see Ref. 5

Quantum Systems: General considerations

General Idea

Quantum mechanical description of the motion of electrons on networks. One speaks of

Quantum wires or Quantum graphs

Applications

:

Nanotubes and magnetic rings

So far mostly one-particle theories have been studied

Quantum Systems: Nanotubes

Carbon nanotubes

:

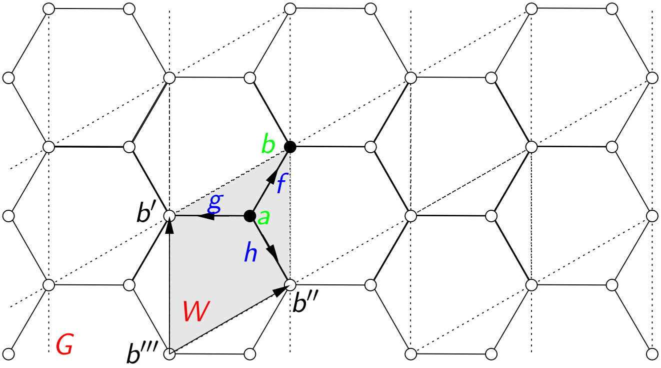


Figure: The hexagonal lattice G and a fundamental domain W together with its set of vertices $V(W) = \{a, b\}$ and set of edges $E(W) = \{f, g, h\}$.

Procedure

Cut out an infinite strip and fold it to an infinite cylinder

Two examples:

Armchair and Zig-Zag

The Quantum Dynamics for a particle of mass m is given by the Schrödinger operator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

So this is essentially the Laplace operator. It is neccessary to specify certain boundary conditions at the vertices ($\cong$ Carbon atoms). Physical interpretation: The electron moves freely away from the vertices.

Bloch Theory

Post, Kuchment; Korotyaev, Lobanov[5]

For certain boundary conditions , the so called standard boundary conditions, the *band structure* can be calculated analytically.

As a consequence other quantities like *e.g.* asymptotics of gap lengths and formulas for the density of states can be obtained. Thus the electronic properties of different types of nanotubes (armchair or zig-zag) can be predicted within this quantum wire model.

[5] see Ref. 5

Quantum Systems: Quantum transport in networks

There is by now an extensive literature on graph models of

mesoscopic systems and wave guides

dealing with

quantum transport problems.

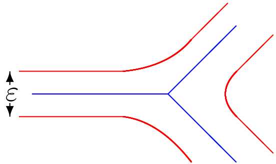
This is an important and extensive topic in itself and for lack of time we just refer to the survey talk by C. Texier and G. Montambaux at the Isaac Newton Institute [6]

[6] Quantum transport in networks of weakly disordered metallic wires, INI April 11 2007

Mathematical Problems

1. Nature is not strictly <u>one</u> dimensional

Any graph is can be viewed as an idealized version of a thickened graph



<u>Mathematical Problem</u>:

Given a modelling of a physical problem on the thickened graph, how well does a corresponding modelling on the graph (= deformation retract of the thickened graph) provide a reliable approximation when the thickness ε is small? [7]

[7] although some studies have been carried out, see e.g. Ref. 8, the status is at the moment quite unsatisfactory and mathematically not yet well understood

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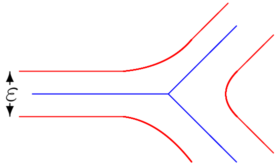

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Mathematical Problems

2. Sometimes the analytic results should be robust against minor changes of the graph

Example: Arterial trees

This leads to the notion of two graphs being close, a study of which has been intitiated recently [8].

[8] see the preprint Ref. 9

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