# Quasi-static Limits in Nonrelativistic Quantum Electrodynamics

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# Outline



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# Main Motivation and Aims

- The interaction between charged particles is usually described by instantaneous pair potentials of Coulomb type.
- This is assumed (and it is an experimentally verified fact) to be a good approximation if the particles move sufficiently slowly.
- On a more fundamental level, the particles interact through the electromagnetic field they generate.
- The main aim of the talk is to illustrate the derivation of the Schrödinger equation with Coulomb potentials (and second order corrections to them) starting from nonrelativistic quantum electrodynamics.

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# Smeared Charges: Ultraviolet Cutoff

Ultraviolet cutoff, no infrared cutoff. The particles have a charge distribution:

$$\varrho_j = e_j \varphi(x), \quad j = 1, \dots, N; \quad x \in \mathbb{R}^3$$

with form factor

$$\hat{\varphi}(k) = \begin{cases} (2\pi)^{-3/2} & |k| \leq \Lambda\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

# Classical Equations of Motion

$$\frac{1}{c}\partial_t B(x,t) = -\nabla \times E(x,t),$$

$$\frac{1}{c}\partial_t E(x,t) = \nabla \times B(x,t) - \sum_{j=1}^{N} e_j \varphi(x - q_j(t)) \frac{\dot{q}_j(t)}{c}$$

$$\nabla \cdot E(x,t) = \sum_{j=1}^{N} e_j \varphi(x - q_j(t)), \qquad \nabla \cdot B(x,t) = 0,$$

$$m_l \ddot{q}_l(t) = e_l \left[ E_\varphi(q_l(t),t)) + \frac{\dot{q}_l(t)}{c} \times B_\varphi(q_l(t),t) \right], \quad l = 1, \dots, N$$

$$E_\varphi(x,t) := (E *_x \varphi)(x,t)$$

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# Formal Limit $c \to \infty$ for the Classical Model

$$\begin{aligned} 0 &= -\nabla \times E(x,t), \\ 0 &= \nabla \times B(x,t) - 0 \\ \nabla \cdot E(x,t) &= \sum_{j=1}^{N} e_j \varphi(x - q_j(t)), \qquad \nabla \cdot B(x,t) = 0, \\ m_l \ddot{q}_l(t) &= e_l E_\varphi(q_l(t),t)) + 0, \quad l = 1,\dots,N \end{aligned}$$

$\Rightarrow$The particles interact through a smeared Coulomb field

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# Two Interpretations of the Limit $c \to \infty$

- $c$ is a quantity with dimension $\Rightarrow$ one should actually say $|v|/c \to 0$, $v$ a typical velocity of the particles.
- The limit $|v|/c \to 0$ can be achieved in two ways: $v$ fixed and $c \to \infty$ or $c$ fixed and $v \to 0$.
- In the classical equations of motion this is reflected by the fact that the limit $c \to \infty$ is equivalent, up to rescaling of time, to the limit of heavy particles:

$$m_l \to \varepsilon^{-2} m_l, \qquad t \to \varepsilon^{-1} t$$

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# Rescaling of Mass and Time: $m \to \varepsilon^{-2} m, \quad t \to \varepsilon^{-1} t$

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# Quantum Case

- Hilbert Space: $\mathscr{H} = \mathscr{H}_{\mathrm{p}} \otimes \mathscr{F}$
  - $\mathscr{H}_{\mathrm{p}} = L^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes N}$; $\mathbb{Z}_2$ : spin of the electron
  - $\mathscr{F} = \oplus_{M=0}^{\infty} \otimes_{(s)}^{M} L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$; $\mathbb{Z}_2$ : helicity of the photon;
- Unscaled Hamiltonian:

$$H^c = \sum_{j=1}^{N} \frac{1}{2m_j} \left[ \sigma_j \cdot \left( -\mathrm{i}\nabla - \frac{1}{\sqrt{c}} e_j A_\varphi(x_j) \right) \right]^2 + V_{\varphi\,\mathrm{coul}}(x) + cH_{\mathrm{f}}$$

$\sigma_j$ : Pauli matrix for the $j$-th electron,

$A_\varphi$ : transverse vector potential in the Coulomb gauge,

$V_{\varphi\,\mathrm{coul}}$ : smeared Coulomb potential,

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# Rescaled Hamiltonian in the Quantum Case

- $H^{c} = \sum_{j=1}^{N} \frac{1}{2m_j}\left[\sigma_j \cdot \left(-\mathrm{i}\nabla - \frac{1}{\sqrt{c}} e_j A_\varphi(x_j)\right)\right]^2 + V_{\varphi\,\mathrm{coul}}(x) + cH_{\mathrm{f}}$

Scaling $m_j \to \varepsilon^{-2} m_j$, $t \to \varepsilon^{-1} t$ (units $c = 1$) the Hamiltonian becomes

- $H^{\varepsilon} = \sum_{j=1}^{N} \frac{\varepsilon^2}{2m_j}\left[\sigma_j \cdot \left(-\mathrm{i}\nabla - e_j A_\varphi(x_j)\right)\right]^2 + V_{\varphi\,\mathrm{coul}}(x) + H_{\mathrm{f}}$

Since we look at the dynamics for times of order $\varepsilon^{-1}$, the Schrödinger equation is

$$\mathrm{i}\varepsilon\partial_t \psi(t) = H^{\varepsilon}\psi(t)$$

# Rough Presentation of the Results

- Our goal is to approximate the time evolution of the reduced density matrix for the particles

$$\omega_{\mathrm{p}}(t) := \mathrm{Tr}_{\mathscr{F}}\omega(t); \qquad \omega(t) := \mathrm{e}^{-\mathrm{i}tH^{\varepsilon}/\varepsilon}\omega\,\mathrm{e}^{\mathrm{i}tH^{\varepsilon}/\varepsilon},$$

  in terms of an effective evolution for the particles alone.

- In a weak sense,

$$\omega_{\mathrm{p}}(t) \simeq \mathrm{e}^{-\mathrm{i}tH_{\mathrm{p}}^{\varepsilon}/\varepsilon}\omega_{\mathrm{p}}(0)\mathrm{e}^{\mathrm{i}tH_{\mathrm{p}}^{\varepsilon}/\varepsilon} \quad .$$

  $H_{\mathrm{p}}^{\varepsilon}$ contains the effective Coulomb interaction between the particles and second order corrections (Darwin term and effective mass for the electron).

# Existence of Adiabatically Decoupled Subspaces

**Theorem**

*There exist approximate $M$-photons dressed projectors which are almost invariant for the dynamics.*

$$\left\| [e^{-itH^\varepsilon/\varepsilon}, P_M^\varepsilon]\chi(H^\varepsilon) \right\|_{\mathcal{L}(\mathscr{H})} \leq C\sqrt{M+1}|t|\varepsilon\sqrt{\log(\varepsilon^{-1})}.$$

- The projectors $P_M^\varepsilon$ are associated to the $M$-photons subspaces of the Fock space by a unitary mapping.
- To have a uniform decoupling, we choose initial states of uniformly bounded energy, $\chi \in C_0^\infty(\mathbb{R})$.

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# Effective Equation for the Reduced Density Matrix

## Theorem

*Given an observable for the particles, $S \in \mathcal{L}(\mathscr{H}_{\mathrm{p}})$, and a density matrix $\omega \in \mathscr{I}_1(P^{\varepsilon}_{M}\chi(H^{\varepsilon})\mathscr{H})$ whose time evolution is defined by*

$$\omega(t) := e^{-\mathrm{i}tH^{\varepsilon}/\varepsilon}\omega e^{\mathrm{i}tH^{\varepsilon}/\varepsilon},$$

*then*

$$\mathrm{Tr}_{\mathscr{H}}\Big(\big(S \otimes \mathbf{1}_{\mathscr{F}}\big)\omega(t)\Big) = \mathrm{Tr}_{\mathscr{H}_{\mathrm{p}}}\Big(S e^{-\mathrm{i}tH^{\varepsilon}_{\mathrm{p}}/\varepsilon}\mathrm{Tr}_{\mathscr{F}}(\omega)e^{\mathrm{i}tH^{\varepsilon}_{\mathrm{p}}/\varepsilon}\Big) + \\ + \mathcal{O}(\varepsilon^{3/2}|t|)(1-\delta_{M0}) + \mathcal{O}\big(\varepsilon^{2}\log(\varepsilon^{-1})(|t|+|t|^{2})\big)$$

# The Effective Hamiltonian

$$H_{\mathrm{p}}^{\varepsilon} = \sum_{j=1}^{N} \frac{1}{2m_j}\hat{p}_j^2 + V_{\varphi\,\mathrm{coul}} + \varepsilon^2 V_{\mathrm{darw}},$$

$$V_{\mathrm{darw}} = -\sum_{l,j=1}^{N} \frac{e_j e_l}{m_j m_l} \int_{\mathbb{R}^3} dk\, \frac{|\hat{\varphi}(k)|^2}{2|k|^2} \mathrm{e}^{\mathrm{i}k\cdot x_j} \hat{p}_j \cdot (\mathbf{1} - \kappa \otimes \kappa) \hat{p}_l \mathrm{e}^{-\mathrm{i}k\cdot x_l}.$$

- Darwin potential: electromagnetic correction to the mass and velocity dependent term, due to retardation effects.
- No spin dependent term. The limit $c \to \infty$ gives also a spin dependent potential $V_{\mathrm{spin}}$.

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# Radiated Energy

- The subspaces $P^{\varepsilon}_{M}$ are only approximately invariant. A system starting in the dressed vacuum will make a transition emitting a free photon.
- The radiated power, for a system starting in the subspace $P^{\varepsilon}_{0}$ is given by

$$E_{\rm rad}(t) = \langle \Psi_{\rm rad}(t), H_{\rm f} \Psi_{\rm rad}(t) \rangle,$$

$$P_{\rm rad}(t) = \frac{d}{dt} E_{\rm rad}(t) \cong \frac{\varepsilon^3}{3\pi^2} \langle \psi, \mathrm{Op}^W_{\varepsilon}\left( |\ddot{D}(t)|^2 \right) \psi \rangle_{\mathscr{H}_{\rm p}}.$$

- $D(s;x,p) = \sum_{j=1}^{N} \frac{e_j}{m_j} x_j^{cl}(s;x,p)$
$m_j \ddot{x}_j^{cl}(s;x,p) = -\nabla_{x_j} V_{\varphi\,\rm coul}(x^{cl}(s;x,p))$,
$x_j^{cl}(0;x,p) = x_j, \qquad \dot{x}_j^{cl}(0;x,p) = p_j$

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# The Case $c \to \infty$

The limit $c \to \infty$ for the unscaled Hamiltonian $H^{\mathrm{c}}$ can be treated using methods of the weak coupling limit theory.

**Theorem (Spohn 2004)**

$$\lim_{c\to\infty} \|(\mathrm{e}^{-\mathrm{i}H^{c}c^{2}t} - \mathrm{e}^{-\mathrm{i}H_{\mathrm{darw}}c^{2}t})\psi \otimes \Omega_{\mathrm{F}}\|_{\mathscr{H}} = 0,$$

*where*

$$H_{\mathrm{darw}} = \sum_{j=1}^{N} \frac{1}{2m_j} p_j^2 + V_{\varphi\,\mathrm{coul}} + c^{-2} V_{\mathrm{darw}} + c^{-2} V_{\mathrm{spin}},$$

$$V_{\mathrm{spin}} = -\sum_{j,l=1}^{N} \frac{e_j e_l}{12 m_l m_j} \sigma_j \cdot \sigma_l \int_{\mathbb{R}^3} dk\, |\hat{\varphi}(k)|^2 e^{ik\cdot(x_j - x_l)},$$

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# Operator Valued Symbols

- $H^{\varepsilon} = \sum_{j=1}^{N} \frac{\varepsilon^2}{2m_j} \left[\sigma_j \cdot \left( -\mathrm{i}\nabla - e_j A_{\varphi}(x_j) \right)\right]^2 + V_{\varphi\,\mathrm{coul}}(x) + H_{\mathrm{f}}$
- $H^{\varepsilon}$ is the Weyl quantization of an operator valued semiclassical symbol, acting on the Fock space $\mathscr{F}$.

$$
\begin{aligned}
H^{\varepsilon} &= \mathrm{Op}^{W}_{\varepsilon}(h(p,q)) \\
h(p,q) &= h_0(p,q) + \varepsilon h_1(p,q) + \varepsilon^2 h_2(p,q) \\
h_0(p,q) &= \sum_{j=1}^{N} \frac{1}{2m_j} p_j^2 + V_{\varphi\,\mathrm{coul}}(q) + H_{\mathrm{f}} \\
h_1(p,q) &= -\sum_{j=1}^{N} \frac{e_j}{m_j} p_j \cdot A_{\varphi}(q_j)
\end{aligned}
$$

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# Absence of Spectral Gap

- The main problem is that the principal symbol

$$h_0(p,q) = \sum_{j=1}^{N} \frac{1}{2m_j} p_j^2 + V_{\varphi\,\mathrm{coul}}(q) + H_{\mathrm{f}}$$

has no spectral gap, due to the free field Hamiltonian $H_{\mathrm{f}}$.

- The strategy to cope with this problem is
  - Introduce an infrared cutoff $\sigma$, with plays the role of an effective gap.
  - Apply space-adiabatic perturbation theory to the infrared cutoff Hamiltonian $H^{\varepsilon,\sigma}$, which depends on two parameters.
  - Show that one can eliminate the cutoff without destroying the approximation.

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# Unitary Dressing Operator

- For $H^{\varepsilon,\sigma}$ one can construct a unitary dressing operator $\mathscr{U}$, which can be expanded in powers of $\varepsilon$ with $\sigma$-dependent coefficients, which are at most logarithmically divergent.

$$\mathscr{U} = \sum_{j=0}^{\infty} \varepsilon^j u_j(\sigma)$$

- The dressed Hamiltonian, $H_{\mathrm{dres}} := \mathscr{U} H^{\varepsilon,\sigma} \mathscr{U}^*$, can also be expanded in a convergent power series in $\varepsilon$ with logarithmically divergent coefficients.
- The different coefficients correspond to different physical effects, which are now clearly separated according to their magnitude in $\varepsilon$.

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# A Short List of References

H. Spohn.
*Dynamics of Charged Particles and Their Radiation Field*.
Cambridge University Press, 2004.

G. Panati, H. Spohn and S. Teufel.
*Space-adiabatic perturbation theory*
*Adv. Theor. Math. Phys.* **7**, 145–204, 2003.

L. Tenuta and S. Teufel.
*Effective dynamics for particles interacting with a quantized scalar field*
*arXiv: math-ph/0702067v3* (2007).
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