# Homogenization of the dislocation dynamics

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Joint work with Cyril Imbert and Régis Monneau

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# Plan

1. Physical motivations
2. Homogenization of a particle system
3. Homogenization of the dynamics of dislocation lines
4. Qualitative properties of the effective Hamiltonian

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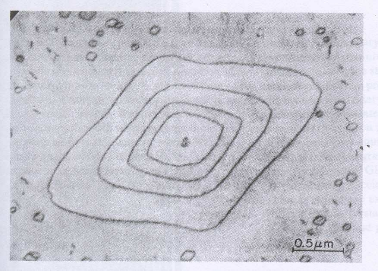
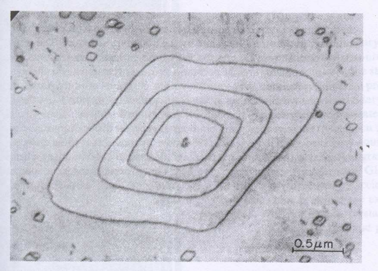
# Observation of dislocations




Definition : a dislocation is a line of crystal defects.

Goal: modeling of plastic behaviour of crystal.

# Observation of dislocations

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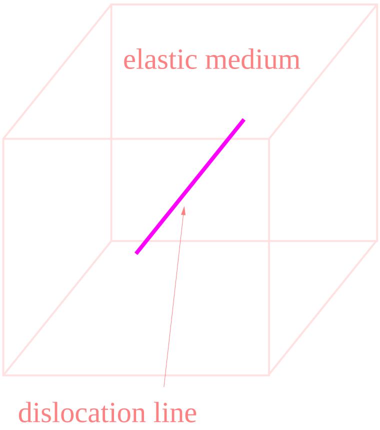
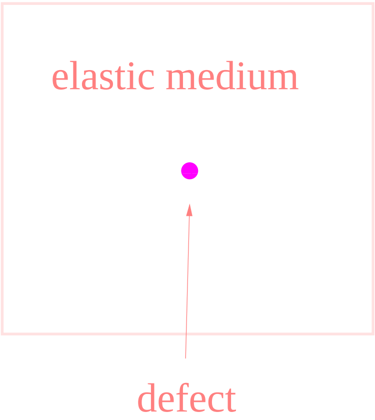
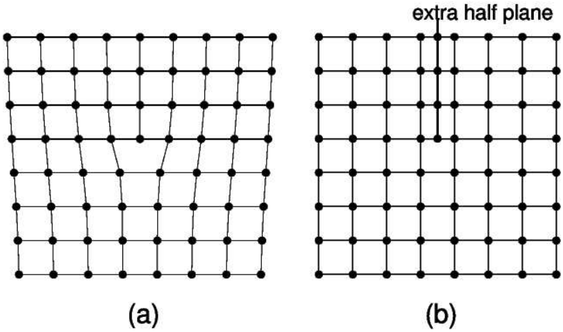
# Continous 3D model



# Continuous 2D model

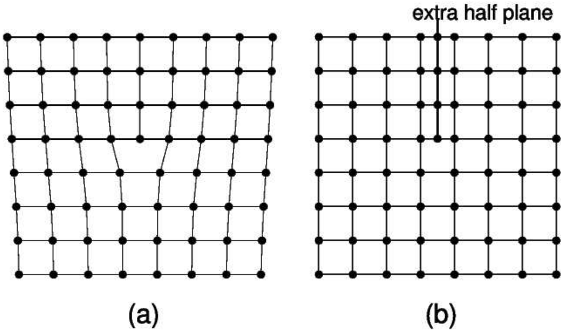


# Atomic structure of a dislocation




topological defect = dislocation

# Atomic structure of a dislocation

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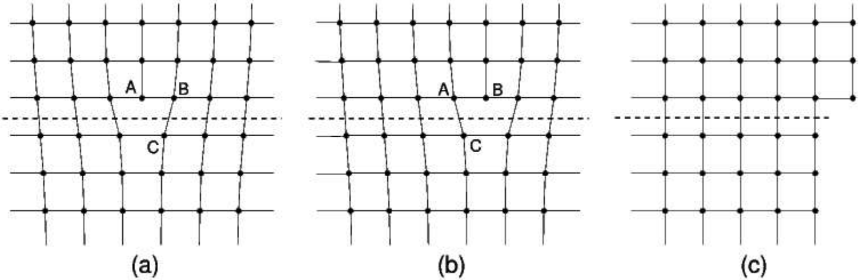
# Displacement of a dislocation



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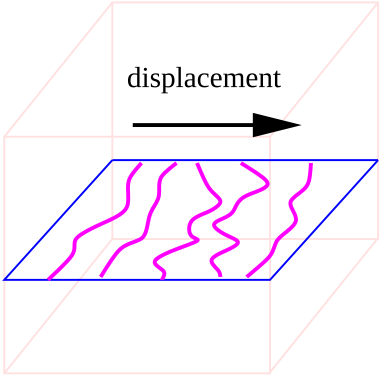
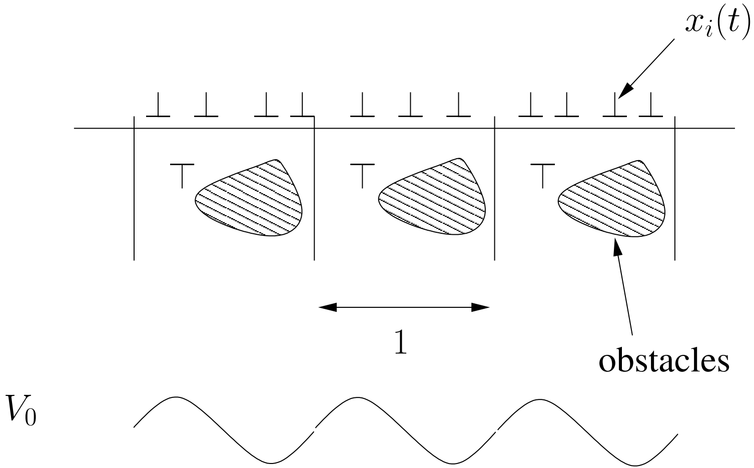
# Homogenization



# Homogenization 1D



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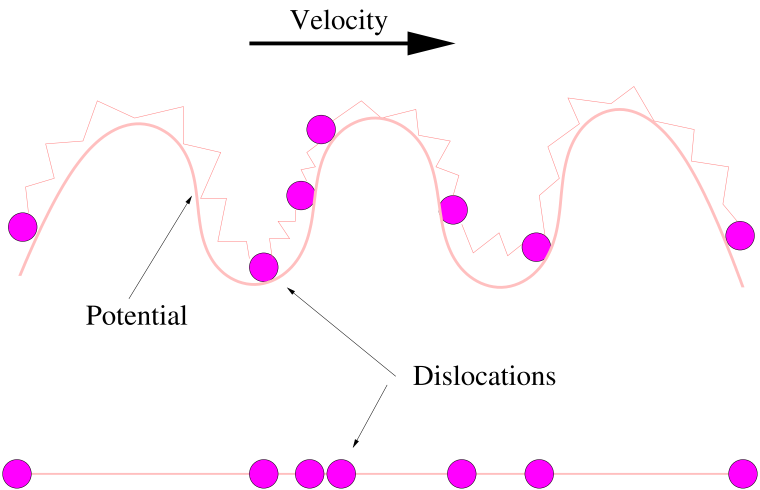

# Modelling

$$\dot{x}_i = -\nabla_{x_i} E$$

with

$$E = \sum_i V_0(x_i) + \sum_{i<j} V(x_i - x_j)$$

and

$$V_0(x+1) = V_0(x)$$

$$V(x) = \ln |x|$$

$$\dot{x}_i = -V_0(x_i) - \sum_{j \neq i} V'(x_i - x_j).$$

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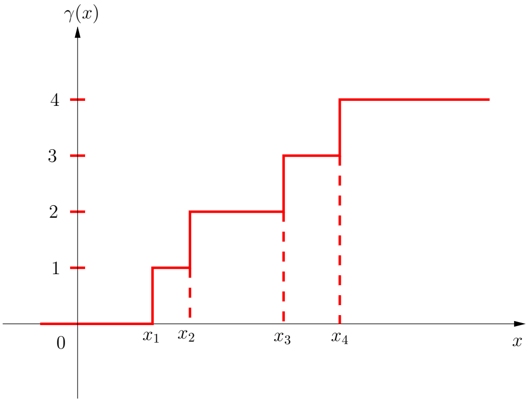
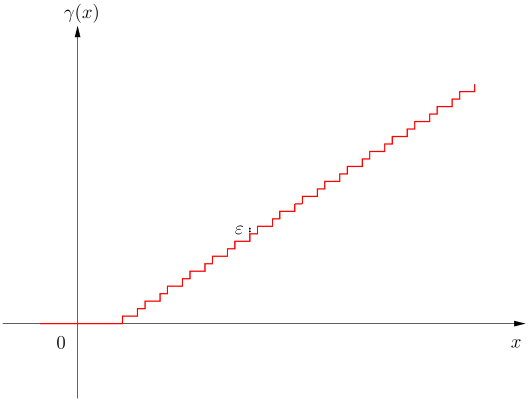



Plastic deformation: $\gamma(x,t) = \sum_i H(x - x_i(t))$

# Rescalling




Plastic deformation: $\gamma^\varepsilon(x,t) = \varepsilon\gamma\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$

# Limit $\varepsilon = 0$ : Homogenization result

## Theorem ( NF,Imbert, Monneau)

*Under certain assumptions on the initial data and on* $V_0$*, there exists* $\overline{H}$ *such that* $\gamma^\varepsilon(x,t)$ *converges to the solution* $u^0(x,t)$ *of*

$$\left\{ \begin{array}{l} \dfrac{\partial u^0}{\partial t} = \overline{H}(\mathcal{I}_1[u^0(\cdot,t)], Du_0) \\ +I.C. \end{array} \right. \quad (1)$$

*where*

$$\mathcal{I}_1[U](x) = \int_{\mathbb{R}} (U(x+z) - U(x) - \nabla_x U(x) \cdot z 1_B(z)) \frac{1}{|z|^2} dz$$

# Idea of the proof

$\gamma^\varepsilon$ is a (discontinuous) solution of

$$\begin{cases} \dfrac{\partial \gamma^\varepsilon}{\partial t} = \left( c\left(\dfrac{x}{\varepsilon}\right) + M^\varepsilon \left[ \dfrac{\gamma^\varepsilon(\cdot, t)}{\varepsilon} \right](x) \right) |D\gamma^\varepsilon| \\ +I.C. \end{cases} \quad (2)$$

where $M^\varepsilon$ is a non-local operator of order $0$ defined by

$$M^\varepsilon [U](x) = \int_{\mathbb{R}} dz\, J(z) E\left( U(x + \varepsilon z) - U(x) \right)$$

# Assumptions on $V$

We make the following assumptions on $V$:

- $V \in W^{1,\infty}_{\mathrm{Loc}}(\mathbb{R})$ and $V'' \in W^{1,1}(\mathbb{R}\backslash\{0\})$,
- $V$ is symmetric *i.e.* $V(-y) = V(y)$,
- $V$ is non-increasing and convex on $(0, +\infty)$,
- $V'(y) \to 0$ as $|y| \to +\infty$,
- there exists a constant $g_0$ such that $V''(y)y^2 = g_0$ for $|y| \geq 1$.

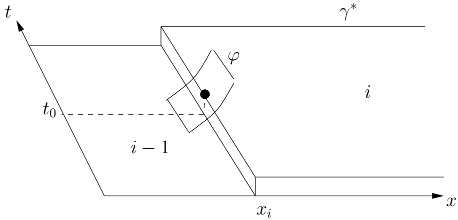
# Idea of the proof

**Lemma**

*We have*

$$\sum_{j\neq i} V'(x_i - x_j) = J \star E(\gamma^*(\cdot) - \gamma^*(x_i))(x_i).$$

# Proof that $\gamma^*$ is a sub-solution




$t \mapsto \varphi(x_i(t), t)$ reaches a local minimum in $t_0$

$$\Longrightarrow \quad \varphi_t(x_i(t_0), t_0) + \dot{x}_i(t_0) D\varphi(x_i(t_0), t_0) = 0.$$

$$\Longrightarrow \quad \varphi_t(x_i(t_0), t_0) = \Big( J \star E(\gamma^*(\cdot) - \gamma^*(x_i))(x_i) + c(x) \Big) |D\varphi(x_i(t_0), t_0)|.$$

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# Model

We consider the following model

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \left( c\left(\dfrac{x}{\varepsilon}\right) + M^\varepsilon \left[ \dfrac{u^\varepsilon(\cdot,t)}{\varepsilon} \right](x) \right) |Du^\varepsilon| & \text{in } \mathbb{R}^N \times [0,+\infty) \\ u^\varepsilon(\cdot, t=0) = u_0 & \text{on } \mathbb{R}^N \end{cases}$$

where $M^\varepsilon$ is a non-local operator of order $0$ defined by

$$M^\varepsilon[U](x) = \int_{\mathbb{R}^N} dz\, J(z) E\left(U(x+\varepsilon z) - U(x)\right)$$

# Assumptions on $J$

We make the following assumptions on $J$:

- $J \in W^{1,1}(\mathbb{R}^N)$ and $J \geq 0$,
- $J$ is symmetric, *i.e.* $J(-y) = J(y)$,
- there exists a function $g \in C^0(\mathbf{S}^{N-1})$, $g \geq 0$ such that $J(z) = \frac{1}{|z|^{N+1}} \; g\left(\frac{z}{|z|}\right)$ for $|z| \geq 1$,

# Cell problem

- Cell problem:

$$\lambda = \Big(c(y) + L + M_p[v(\tau,\cdot)](y)\Big)|p + Dv| \qquad (3)$$

where

$$M_p[U](y) = \int dz\, J(z) \left\{E\left(U(y+z) - U(y) + p\cdot z\right) - p\cdot z\right\}.$$

- $\lambda = \overline{H}(L,p)$.

# Cell problem

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$$\lambda = \Big( c(y) + L + M_p[v(\tau,\cdot)](y) \Big) |p + Dv| \tag{3}$$

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- $\lambda = \overline{H}(L,p)$.

# Ergodicity

**Theorem (NF, Imbert, Monneau)**

*Under certain assumptions on $J$ and $c$, there exists a unique $\lambda$ such that there exists a solution $v$ of (3). Moreover, the oscillation of $v$ is bounded.*

# Convergence result

## Theorem (NF, Imbert, Monneau)

*Under certain assumptions on $J$, $c$ and $u_0$, $u^\varepsilon$ converges to the unique viscosity solution $u^0$ of*

$$\begin{cases} \dfrac{\partial u^0}{\partial t} = \overline{H}(\mathcal{I}_1[u^0(\cdot,t)], Du_0) \\ \\ u^0(\cdot, t=0) = u_0 \end{cases} \tag{4}$$

*where*

$$\mathcal{I}_1[U](x) = \int_{\mathbb{R}^N} \frac{g\left(\frac{z}{|z|}\right)}{|z|^{N+1}} \left(U(x+z) - U(x) - \nabla_x U(x) \cdot z 1_B(z)\right) dz$$

# Some works on dislocation density dynamics

- [Groma, Balogh]
- [El Hajj]
- [Ibrahim]
- [Monneau]

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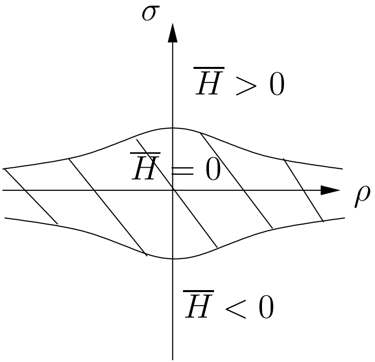
# Qualitative properties

**Theorem (NF, Imbert, Monneau)**

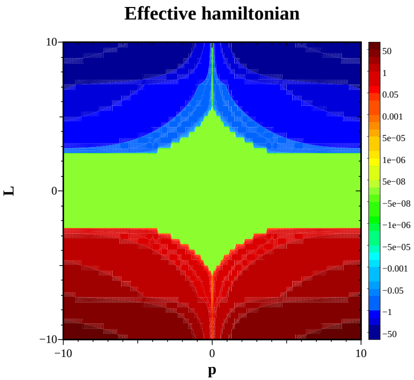
*We assume that $N = 1$ and that $\int c = 0$. Then we have the following properties:*

1. *If $c \equiv 0$ then $\overline{H}(L,p) = L|p|$*
2. $L\overline{H}(L,p) \geq 0$.
3. *If $c \not\equiv 0$ then*
$$\overline{H}(L,p) = 0 \quad \text{for } (L,p) \in B_\delta(0).$$
4. $\overline{H}(L,p) = 0$ *if* $L = 0$.
5. *For all $L > 0$,*
$$\overline{H}(L,p) > 0 \quad \text{for } p \text{ big enough}$$

# Schematic representation of the effective Hamiltonian



# Simulation of the effective Hamiltonian (Amin Ghorbel)



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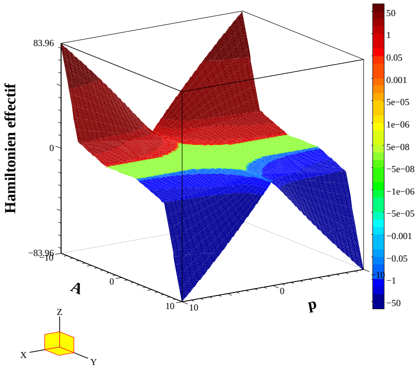

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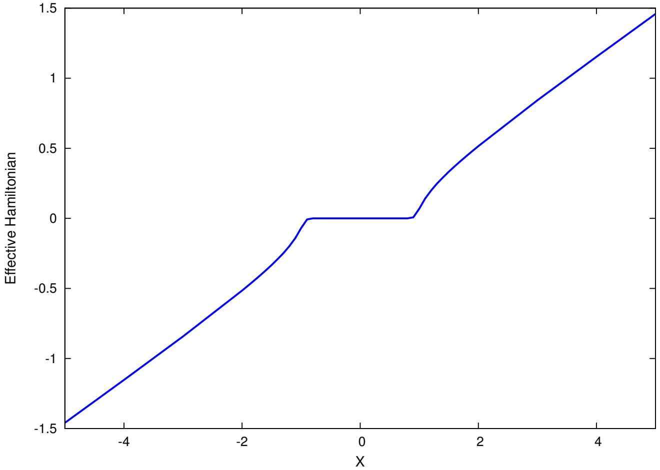
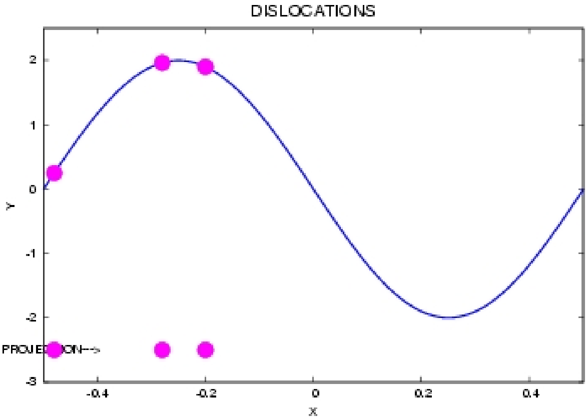


# Simulation of the dislocation dynamics




Animation

# Perspectives

- Homogenization of Frenkel-Kontorova model (joint work with C. Imbert and R. Monneau)
- Homogenization of more realistic models
- Numerical analysis