Analytic Aspects of Sasakian Geometry

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Kähler and Sasakian Geometry in Rome

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Aim: Establish a systematic way to find Sasaki-Einstein metrics.

1. Sasakian Geometry: Introduction

2. Sasakian-Ricci flow
(with K. Smoczyk (Hanover), Yongbing Zhang (Hefei) 2006)

3. Sasakian-Einstein metrics on Sasakian toric manifolds
(with A. Futaki, H. Ono (Tokyo)) arXiv:math/0607586, JDG

4. Sasakian-Ricci flow on 3-dimensional Sasakian manifolds
(with Yongbing Zhang (Hefei) 2009)

Contact manifolds

- **Contact manifold:** (M^{2n+1}, η), a 1-form η (*contact form*)

$$\eta \wedge (d\eta)^n \neq 0.$$

- **Characteristic vector field** or **Reeb vector field**: ξ

$$\eta(\xi) = 1 \qquad \text{and} \qquad d\eta(\xi, X) = 0$$

- **Almost contact manifold**: (M, η, Φ), Φ – $(1,1)$ tensor,

$$\Phi^2 = -I + \eta \otimes \xi.$$

- **Metric contact manifold**: (M, η, g, Φ), g – (*compatible*) metric,

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

Sasakian Manifolds

- **Sasakian Manifold:** (M, g, η, ξ, Φ), a metric contact manifold with

$$[\Phi, \Phi] = -2d\eta \otimes \xi,$$

where the *Nijenhuis bracket* is defined by

$$[\Phi, \Phi](X, Y) := [\Phi X, \Phi Y] + \Phi^2[X, Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y]$$

A Riemannian manifold (M, g) is Sasakian if and only if one of the following equivalent statements holds:

- $(C(M), \bar{g}) = (\mathbb{R}_+ \times M, dr^2 + r^2 g)$ is Kähler.
- $\exists$ a Killing field ξ of $|\xi| = 1$ with $R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi$.
- $\cdots$.

Examples

$\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ has a standard Sasakian structure (ξ, η, Φ, g)

$$\eta = \sum_{i=0}^{n} (x_i dy_i - y_i dx_i), \quad \xi = \sum_{i=0}^{n} (x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i})$$

Φ is a “restriction” of J_0 and g the standard metric.

Other Sasakian structures $(\xi_w, \eta_w, \Phi_w, g_w)$ on $\mathbb{S}^{2n+1}$:
for $w = (w_0, w_1, \cdots, w_n) \in \mathbb{R}_+^{n+1}$

$$\eta_w = \frac{\sum_{i=0}^{n}(x_i dy_i - y_i dx_i)}{\sum_{i=0}^{n} w_i(x_i^2 + y_i^2)}, \quad \xi_w = \sum_{i=0}^{n} w_i (x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}),$$

$\Phi_w = \Phi - \Phi\xi_w \otimes \eta_w$ and g_w.

Characteristic Foliation

Reeb field ξ generates a foliation $\mathcal{F}_\xi$. Let $\mathcal{Z}$ be the space of leaves.

A foliation $\mathcal{F}_\xi$ is quasi-regular if there is integer k such that in a foliated chart U each leaf passing U at most k times. Otherwise $\mathcal{F}_\xi$ is irregular. When $k = 1$, $\mathcal{F}_\xi$ is regular.

When $\mathcal{F}_\xi$ is regular, $\mathcal{Z}$ is a (Kähler) manifold.
(The standard Sasakian Structure on $\mathbb{S}^{2n+1}$ is regular, and $\mathcal{Z} = \mathbb{C}P^n$.)

When $\mathcal{F}_\xi$ is quasi-regular, $\mathcal{Z}$ is an (Kähler) orbifold.
(If w_i are integers, then $(\mathbb{S}^{2n+1}, \xi_w, \eta_w, \Phi_w, g_w)$ is quasi-regular. $\mathcal{Z}$ is a *weighted* complex projective space.)

Other cases are irregular.

Sasaki-Einstein manifolds

A **Sasaki-Einstein** manifold is a Sasakian manifold with

$$Ric = \lambda g.$$

- λ is always positive, since $Ric(\xi, \xi) = 2n$.

(M, g) is SE $\Leftrightarrow C(M) = (\mathbb{R}_+ \times M, dr^2 + r^2 g_M)$ is Calabi-Yau.

Boyer, Galicki, Kollár, $\cdots$ constructed many new Einstein metrics on $\mathbb{S}^{2n+1}$, $\#k(\mathbb{S}^2 \times \mathbb{S}^3)$. **Quasi-regular examples**. (Kähler case: Yau, Aubin-Yau, Tian, Tian-Yau, $\cdots$,)

Gauntlett, Martelli, Sparks, Waldram, $\cdots$ new Einstein metrics on $\mathbb{S}^2 \times \mathbb{S}^3$, inspired by supergravity theory. **Irregular examples**.

η-Einstein Manifolds

Sasakian η-Einstein manifold: A Sasakian manifold M^{2n+1} with (constants λ,ν)

$$Ric = \lambda g + \nu \eta \otimes \eta.$$

M^{2n+1} is a Sasakian manifold with $Ric = \lambda(x) g + \nu(x) \eta \otimes \eta$ and $n \geq 2$, then λ, ν are constants (this is not true for $n = 1$) and $\lambda + \nu = 2n$.

- $\lambda < -2$.

- $\lambda = -2$.

- $\lambda > -2$. $\Rightarrow$ a Sasakian-Einstein manifold by $\mathcal{D}$-homothety:

$\mathcal{D}$-**homothety**(Tanno): $(a^{-1}\xi, a\eta, \Phi, ag + a(a-1)\eta \otimes \eta)$

$$Ric_{g'} = \lambda' g' + \nu' \eta' \otimes \eta', \quad \nu' = 2n - \frac{\lambda + 2 - 2a}{a}.$$

Transverse Geometry

Reeb vector field: ξ, $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$.

$\mathcal{D} := \ker \eta$, rank $2n$ vector bundle, contact bundle (contact distribution)

$$TM = \mathcal{D} \oplus L_\xi,$$

where L_ξ is the line bundle generated by ξ.
- $\mathcal{F}_\xi$: characteristic foliation generated by ξ.
- transverse metric: $g^T : g^T(X, Y) = d\eta(X, JY)$ for $X, Y \in \mathcal{D}$ $(g = g^T \oplus (\eta \otimes \eta))$.
- **transverse Levi-Civita connection** ∇^T w.r.t. g^T

$$\nabla^T_X V = \begin{cases} [\xi, V]^p & \text{if } X = \xi \\ (\nabla_X V)^p & \text{if } X \in \Gamma(\mathcal{D}). \end{cases}$$

and the **transverse Ricci tensor** Ric^T of ∇^T.
Transverse Einstein metric: $Ric^T = (\lambda + 2) g^T$.

On a Sasakian manifold, η-Einstein $\Leftrightarrow$ transverse Einstein.

Transverse Kähler Geometry

Reeb vector field: ξ, $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$.

$\mathcal{D} := \ker \eta$, rank $2n$ vector bundle, contact bundle (contact distribution, contact strcture)

$$TM = \mathcal{D} \oplus L_\xi,$$

where L_ξ is the line bundle generated by ξ.

- $\mathcal{F}_\xi$: characteristic foliation generated by ξ.
- $\mathcal{D} \sim$ the normal bundle of $\mathcal{F}_\xi$, $\nu(\mathcal{F}_\xi) = TM \backslash L_\xi$

$$0 \to L_\xi \to TM \to \nu(\mathcal{F}_\xi) \to 0$$

- $J = \Phi_{|\mathcal{D}}$ a complex structure on $\mathcal{D}$, $J^2 = -I$.
- $d\eta_{|\mathcal{D}}$ a symplectic form.
- g^T : $g^T(X, Y) = d\eta(X, JY)$ for $X, Y \in \mathcal{D}$ $(g = g^T \oplus (\eta \otimes \eta))$
- $(\mathcal{F}_\xi, \mathcal{D}, J, d\eta_\mathcal{D}, g^T)$ gives $\mathcal{F}_\xi$ a **transverse Kähler structure**.

Basic forms

p-form α is called **basic** if

$$i_\xi \alpha = \mathcal{L}_\xi \alpha = 0.$$

Examples: $d\eta$ is basic, η is not basic. ($\eta(\xi) = 1$, $d\eta(\xi, X) = 0$.)

basic function: $\xi(f) = 0$.

Λ^p_B: Sheaf of germs of basic p-forms
Ω^p_B: Set of section of Λ^p_B. $C^\infty_B(M) = \Omega^0_B$.

d preserves the basic forms $\Rightarrow$
Basic cohomology of $(M, \mathcal{F}_\xi)$, $H^*_B(\mathcal{F}_\xi) = \mathrm{Ker}d/\mathrm{Im}d$.

By transverse Kähler structure of $(M, \mathcal{F}_\xi)$, one consider the complexifaction $\mathcal{D}_\mathbb{C}$ of $\mathcal{D}$ and decompose it *w.r.t* J, $\mathcal{D}_\mathbb{C} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. Similarly, we have $\Lambda^1_B \otimes \mathbb{C} = \Lambda^{1,0}_B \oplus \Lambda^{0,1}_B$

Basic Chern forms

Basic first Chern Form: $c_1^B = c_1(\mathcal{D}^{1,0})$. c_1^B can be represented by a basic real closed (1,1) form ρ_B .

A Sasakian structure (M, ξ, η, Φ, g) is $c_1^B > 0$ ($c_1^B < 0$, $c_1^B = 0$), if ρ_B is **positive** (**negative, null**).

transverse Ricci form: $\rho_g^T(X,Y) = Ric^T(X, \Phi Y)$. It is closed

$$c_1^B = [\rho^T]_B \in H_B^{1,1}(\mathcal{F}_\xi)$$

$Ric^T = \lambda g^T \Leftrightarrow \rho_g^T = \lambda d\eta$ (transverse Kähler-Einstein)

Deformations of Sasakian structures

Decompose $d = \partial_B + \bar{\partial}_B$ by $\partial_B : \Lambda_B^{p,q} \to \Lambda_B^{p+1,q}$, $\bar{\partial}_B : \Lambda_B^{p,q} \to \Lambda_B^{p,q+1}$ and $d_B^c = \frac{\sqrt{-1}}{2}(\bar{\partial}_B - \partial_B)$. We have $d_B d_B^c = \sqrt{-1}\partial_B \bar{\partial}_B$.

• **Deformations preserving** ξ: Fix a Sasakian structure (ξ, η, Φ, g). $\forall$ basic function φ, Then $(\xi, \bar{\eta}, \bar{\Phi}, \bar{g})$ is also a Sasakian structure, where $\bar{\eta} = \eta + 2d_B^c\varphi$, $\bar{\Phi} = \Phi - \xi \otimes (2d_B^c\varphi) \circ \Phi$ and $\bar{g} = d\bar{\eta} \circ (\bar{\Phi} \otimes \bar{I}) + \bar{\eta} \otimes \bar{\eta}$. (Boyer-Galicki)

$d\bar{\eta} = d\eta + 2dd_B^c\varphi$

$[d\eta]_B = [d\bar{\eta}]_B$ and c_1^B is invariant under such deformations.

• $\mathcal{D}$**-homothetic deformation**: $(a^{-1}\xi, a\eta, \Phi, ag + a(a-1)\eta \otimes \eta)$

• $(-\xi, -\eta, -\Phi, g)$.

• **Deformations preserving the contact structure** $\{\mathcal{D}\}$: $\tilde{\eta} = f\eta$ with $f > 0$ & other conditions.

Sasakian Calabi Problem

Sasakian Calabi Problem (Boyer-Galicki): Give a manifold M with Sasakian structure (ξ, η, Φ, g) and c_1^B is positive, negative or null, can one deform it to another Sasakian structure (ξ, η', Φ', g') with an η-Einstein metric g'?

Recall $Ric_g = \lambda g + \nu \eta \otimes \eta$. $Ric^T = Ric + 2g$. Hence, η-Einstein $\Leftrightarrow$ **transverse Einstein metric**

$$Ric^T = (\lambda + 2) g^T, \qquad \text{or} \qquad \rho_g^T = (\lambda + 2) d\eta$$

$c_1^B > 0$ ($c_1^B < 0$ and $c_1^B = 0$) $\Leftrightarrow \lambda > -2$ ($\lambda < -2$ and $\lambda = -2$).

The existence of η-Einstein metric implies $c_1^B = \kappa [d\eta]_B$.

Local coordinates and Deformations

One can choose local coordinates $(x, z^1, z^2, \cdots, z^n)$ on a small neighborhood U such that

- $\xi = \frac{\partial}{\partial x}$,
- $\eta = dx + \sqrt{-1}\sum_{j=1}^n h_j dz^j - \sqrt{-1}\sum_{j=1}^n h_{\bar{j}} d\bar{z}^j$,
- $\Phi = \sqrt{-1}\{\sum_{j=1}^n \{(\frac{\partial}{\partial z^j} - \sqrt{-1}h_j \frac{\partial}{\partial x}\} \otimes dz^j - c.c$
- $g = \eta \otimes \eta + 2\sum_{j,l=1}^n h_{j,\bar{l}} dz^j d\bar{z}^l =: \eta \otimes \eta + g^T$,
- $d\eta = 2\sqrt{-1}\sum_{j,l=1}^n h_{j\bar{l}} dz^j \wedge d\bar{z}^l$,

where $h : U \to \mathbb{R}$ is a (local) **basic** function, i.e. $\frac{\partial}{\partial x} h = 0$ and $h_j = \frac{\partial}{\partial z^j} h$ and $h_{j\bar{l}} = \frac{\partial^2}{\partial z^j \partial \bar{z}^l} h$.

- $\mathcal{D}^{\mathbb{C}}$ is spanned by $X_j := \frac{\partial}{\partial z^j} - \sqrt{-1}h_j \frac{\partial}{\partial x}$, $X_{\bar{j}} := \frac{\partial}{\partial \bar{z}^j} + \sqrt{-1}h_{\bar{j}} \frac{\partial}{\partial x}$.

$JX_j = \sqrt{-1}X_j$ and $JX_{\bar{j}} = -\sqrt{-1}X_{\bar{j}}$.

Deformation. If (ξ, η, Φ, g) is Sasakian, so is $(\xi, \eta + d_B^c\varphi, \Phi_\varphi, g_\varphi)$ for a **basic** function ϕ, i.e., $\xi(\varphi) = 0$. Locally, $h \Rightarrow h + \varphi$.

$$R^T_{j\bar{l}} = -\frac{\partial^2}{\partial z^j \partial z^{\bar{l}}} \log\det(g^T_{m\bar{n}}) = -\frac{\partial^2}{\partial z^j \partial z^{\bar{l}}} \log\det(h_{m\bar{n}})$$

Transverse Monge-Ampere equation

Assuming $c_1^B = \kappa[d\eta]_B$, there is a basic function F (El Kacimi-Alaoui)

$$\rho^T - \kappa d\eta = \sqrt{-1}\partial_B \bar{\partial}_B F.$$

Transverse Kähler-Einstein equation

$$\frac{\det(g^T_{i\bar{j}} + \phi_{i\bar{j}})}{\det(g^T_{i\bar{j}})} = e^{-\kappa\phi + F}, \quad g^T_{i\bar{j}} + \phi_{i\bar{j}} > 0.$$

Here ϕ is basic, i.e., $\xi(\phi) = 0$.

- It is not elliptic, but transversal elliptic.

Sasakian Ricci flow

Sasakian Ricci flow:(*Smoczyk, W., Y. Zhang (2006)*) On a compact manifold with Sasakian structure (M, ξ, η, Φ, g), $c_1^B = \kappa[d\eta]_B$. There is a smooth family of Sasakian structures $(\xi, \eta(t), \Phi(t), g(t))$ satisfying $(\xi, \eta(0), \Phi(0), g(0)) = (\xi, \eta, \Phi, g)$ and

$$\frac{d}{dt} g^T(t) = -(Ric^T_{g(t)} - \kappa g^T(t)).$$

$$\frac{d}{dt}\varphi = \log\det(g^T_{i\bar{j}} + \varphi_{i\bar{j}}) - \log(\det g^T_{i\bar{j}}) + \kappa\varphi - F.$$

(Transverse Ricci flow for Riemannian foliations studied by *Lovric, Min-Oo, Ruh*)

• When c_1^B is negative or null, then the flow converges to η-Einstein metric. (El Kacimi-Alaoui, Boyer-Galicki) (Cao, Kähler case) Maximum principle holds.

• When c_1^B is positive, it is a difficult problem. $\Rightarrow$ Sasaki-Ricci solitons

Sasakian Ricci solitons

Let (ξ, η, Φ, g) be a Sasakian manifold. If there is a transverse (Hamiltonian) holomorphic vector field X on M with

$$Ric_g^T - g^T = \mathcal{L}_X(g^T),$$

then (ξ, η, Φ, g, X) is called **Sasakian Ricci soliton**.

A *transverse (Hamiltonian) holomorphic vector field* X on M can be local expressed as

$$X = \eta(X)\frac{\partial}{\partial x} + \sum_{i=1}^{m} X^i \frac{\partial}{\partial z^i} - \eta(\sum_{i=1}^{m} X^i \frac{\partial}{\partial z^i})\frac{\partial}{\partial x},$$

where X^i are local holomorphic and $u_X := \sqrt{-1}\eta(X)$ satisfies

$$\bar{\partial}_B u_X = -\frac{\sqrt{-1}}{2} i(X) d\eta.$$

Existence of Solitons

Assume $c_1^B = [d\eta]_B$ ($c_1(\mathcal{D}) = 0$ and $c_1^B > 0$) $\exists$ a basic function h such that $\rho^T - d\eta = \sqrt{-1}\partial_B\bar{\partial}_B h$.

Sasaki Futaki invariant: (Boyer-Galicki-Simanca, Futaki-Ono-W.)

$$SF(X) = \int X(h)\eta \wedge (d\eta)^n.$$

This is an invariant.

Obstruction. If SF does not vanish, then there is no η-Einstein metrics.

• (Futaki, Ono, W.(2006)) Let M be a compact toric Sasaki manifold with $c_1^B > 0$ and $c_1(\mathcal{D}) = 0$. Then there exists a Sasaki metric which is a Sasaki-Ricci soliton. In particular M admits a Sasaki-Einstein metric if and only if the Sasaki Futaki invariant vanishes.

(X.-J. Wang and X. Zhu, Kähler)

Sasakian-Einstein metrics

• (Futaki, Ono, W.(2006)) Let M be a compact toric Sasakian manifold with $c_1^B > 0$. Then by deforming the Sasakian structure varying the contact structure we get a Sasakian structure with vanishing SF invariant. Hence, there is a Sasaki-Einstein metric.

$\mathbb{S}^2 \times \mathbb{S}^3$ admits **irregular** Sasaki-Einstein metrics (*Gauntlett, Martelli, Sparks and Waldrum (2004)*).

$2\#\mathbb{S}^2 \times \mathbb{S}^3$ ($k = 2$) is a toric Sasakian manifold, there is a Sasakian-Einstein structure, which is irregular

Toric Sasakian manifolds

M is Sasakian toric $\Longleftrightarrow C(M)$ is Kähler toric, ie, the product metric $\bar{g}$ is invariant under a holomorphic action of the $n+1$-torus $\mathbb{T}^{n+1}$.

• **Momemt map** $\mu : C(M) \to \mathbb{R}^{n+1} \cong \mathfrak{t}^*$, $\langle \mu, X \rangle = r^2 \eta(X)$ and

• its image $\mathcal{C} := \mu(C(M))$ is a strictly convex rational polyhedral cone of the form

$$\mathcal{C} = \{ y \in \mathbb{R}^{n+1} = \mathfrak{t}^* \,|\, \langle\, y, v_a \,\rangle \geq 0, a = 1, 2, \cdots d \}.$$

$\xi = b \in \mathcal{C}^* := \{ x \in \mathbb{R}^{n+1} \cong \mathfrak{t} \,|\, \langle x, y \rangle \geq 0, \forall y \in \mathcal{C} \}$ dual cone of $\mathcal{C}$.

$\mu(M) = \mathcal{C} \cap H_\xi$, where $H_\xi := \{ y \in \mathbb{R}^{n+1} = \mathfrak{t}^* \,|\, \langle y, b \rangle = 1 \}$, the charateristic plane. $\mathcal{C} \cap H_\xi$ is a compact polytope.

• $\mathcal{C} \cap H_\xi$ is rational iff M is quasi-regular.

A volume functional

• The set of Sasakian structure preserving $\mathcal{D} \cong \{\xi \in \mathcal{C}^*\}$.

(Martelli-Sparks-Yau) Volume functional $V : \mathcal{C}^* \to \mathbb{R}$:

$$V(\xi) = c(n)vol(\mathcal{C} \cap H_\xi) = c(n)vol(\mathcal{C} \cap \{y \in \mathbb{R}^{n+1} \mid \langle y, b\rangle = 1\}).$$

• V is convex and $V(\xi) \to \infty$ as $\xi \to \partial\mathcal{C}^*$, hence V has a (unique) minimizer.

• (Martelli-Sparks-Yau, FOW) ξ is a critical points of V iff its Sasaki-Futaki invariant is 0.

3-dimensional Sasaki Ricci flow

On a 3-dimensional Sasakian manifold, $R^T_{ij} = \frac{1}{2} R^T g^T_{ij}$

$$\frac{d}{dt} g^T_{ij} = (r - R^T) g^T_{ij}, \quad \frac{d}{dt} d\eta = (r - R^T) d\eta.$$

Here r is the average of the transverse scalar curvature.

$\frac{d}{dt} \int_M d\eta \wedge \eta = 0.$

$\frac{d}{dt} R^T = \Delta_B R^T + R^T (R^T - r)$

- Entropy (Hamilton) $\int R^T \log R^T \eta \wedge d\eta$ is non-increasing.

Convergence

(Zhang-W.) The Sasaki Ricci flow converges to a (gradient) Sasaki Ricci soliton. The soliton is unique.

$X = -\frac{1}{2}\nabla f$ with $\xi(f) = 0$ and $\nabla_i^T \nabla_j^T f - \frac{1}{2}\Delta_B f g_{ij}^T = 0$.

- 1. Regular case (Hamilton), 2-sphere
- 2. Quasi-regualr case (Langfang Wu, B. Chow-L.F. Wu) 2-orbifold

Uniqueness: weighted structures on $\mathbb{S}^3$ (Gauduchon-Ornea, Belgun). We find the same ODE as Hamilton and Wu did.

Convergence: Idea of Proof for the irregular case:

Proof 1. Approximated by (2)

Proof 2. Direct proof (using methods in (1))

Comparison Theorem

Transverse distance between x and y

$$d^T(x,y) := \inf_\gamma \int_\gamma |\frac{d}{ds}\gamma(s)|_{g^T} ds,$$

where $\gamma(s)$ are curves joining x to y

Harnack inquality (with d^T).

On the Sasakian 3-sphere M^3 of positive transverse scalar curvature R^T

(1) $diam^T \leq c\frac{\pi}{\sqrt{R^T_{\min}}}$

(2) $Vol(T(p, \frac{\pi}{\sqrt{R^T_{\max}/2}})) \geq C/R^T_{\max}$,

where $T(p,r) := \{x \in M \,|\, d^T(x,p) < r\}$.

Thank You!