Model Reduction Methods

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Focus

Model Order Reduction by Reduced Basis Method
for the efficient resolution of parametrized PDEs

Examples in heat and mass transfer, linear elasticity,
potential and viscous flows

Some Pre-requisites

Numerical Analysis, FEM, PDEs, Physical Mathematics

References and materials
(http://www.mat.uniroma1.it/cortona10/courses.html):

- Rozza G., Huynh D.B.P., Patera A.T., Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations, Arch Comput Methods Eng (2008) 15: 229-275
- Patera A.T., Rozza G., Reduced Basis Approximation and A Posteriori Error Estimation for Parametrized Partial Differential Equations, to appear in (tentative rubric) MIT Pappalardo Graduate Monographs in Mechanical Engineering, 2006–2010
- Rozza G., Nguyen N.C., Huynh D.B.P., Patera A.T., Real-Time Reliable Simulation of Heat Transfer Phenomena, Proceedings of HT2009 ASME Summer Heat Transfer Conference, paper HT2009-88212

Links:

- http://augustine.mit.edu/methodology/methodology/.._rbMIT_System.htm, *Matlab Software, rbMIT (C)MIT Library*
- http://augustine.mit.edu/methodology/methodology_book.htm *First part of RB book, A.T. Patera, G. Rozza (C)MIT*
- http://augustine.mit.edu/workedProblems.htm *Worked problems, examples, Webserver (C)MIT*

Outline

- **Lecture 1: Motivation, Coercive Elliptic Problems**
  1. Introduction/Motivation
     (a) Notation and Examples
     (b) Goal/Relevance
  2. Elliptic Problems I (coercive, affine, compliant)
     (a) Problem Statement, Truth Approximation, Affine Representation
     (b) Reduced Basis Approximation
     (c) Offline-Online Computational Procedures
     (d) Sampling/Spaces Strategies: POD, Greedy, ...

Outline

- **Lecture 2: Elliptic Problems II, Parabolic Problems**
  1. Elliptic Problems II
     - (e) A Posteriori Error Estimation (elements)
     - (f) General Outputs (non-compliant), Non-symmetric Forms (Dual Problem, A Posteriori Error Estimation)
  2. Parabolic Problems
     - (a) Problem Statement, Truth Approximation
     - (b) Reduced Basis Approximation
     - (c) Offline-Online Computational Procedures
     - (d) A Posteriori Error Estimation

Outline

- **Lecture 3: Parabolic problems, Non-affine Problems, Software**
  1. Parabolic Problems
     (d) A Posteriori Error Estimation
     (e) Offline-Online Decomposition
     (f) POD/Greedy Sampling
     (g) NOn-symmetric problems
  2. Non-Affine Problems
     (a) Empirical Interpolation Method
     (b) EIM + RB
  3. Summary on Software: RB@MIT
- **Lecture 4: Applied Talk**
  1. Optimization & Optimal Control
     - Parameter Optimization, GMA Welding Process, Advection-Diffusion (Environmental and thermal)
  2. Shape Optimization
     - Potential, thermal, (Navier)-Stokes flows

Lecture 1

- **Lecture 1: Motivation, Coercive Elliptic Problems**

  1. Introduction/Motivation
     (a) Notation and Examples
     (b) Goal/Relevance
  2. Elliptic Problems I (coercive, affine, compliant)
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     (d) Sampling/Spaces Strategies: POD, Greedy, ...

Statement: simple elliptic μPDEs

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate $s^{\mathrm{e}}(\mu) = \ell(u^{\mathrm{e}}(\mu))$ †

where $u^{\mathrm{e}}(\mu) \in X^{\mathrm{e}}$ satisfies

$$a(u^{\mathrm{e}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathrm{e}} \, .$$

μ:	input parameter;	P-tuple
$\mathcal{D}$:	input domain;	
s^{e}:	output;	
ℓ:	linear bounded output functional;	
u^{e}:	field variable;	
X^{e}:	function space $(H_0^1(\Omega))^\nu \subset X^{\mathrm{e}} \subset (H^1(\Omega))^\nu$;	

† Here $^{\mathrm{e}}$ refers to "exact."

Statement: hypotheses and definitions

$$\left.\begin{array}{ll} a(\,\cdot\,,\cdot\,;\mu): & \text{bilinear,} \\ & \text{continuous,} \\ & \text{symmetric,} \\ & \text{coercive form, } \forall\, \mu \in \mathcal{D}; \\ f: & \text{linear bounded functional.} \end{array}\right\} \mu\text{PDE}$$

Compliant case: $l = f$,
$a(\,\cdot\,,\cdot\,;\mu)$ symmetric

Statement: hypotheses and definitions

- a symmetric: $a(u,v;\mu) = a(v,u;\mu)$,
- a bilinear: $a(\lambda u + \gamma v, w;\mu) = \lambda a(u,w;\mu) + \gamma a(v,w;\mu),\ \forall \lambda,\gamma \in \mathbb{R},\ \forall u,v,w \in X^{\mathrm{e}}$,
  or $a(u, \lambda v + \gamma w;\mu) = \lambda a(u,v;\mu) + \gamma a(u,w;\mu),\ \forall \lambda,\gamma \in \mathbb{R},\ \forall u,v,w \in X^{\mathrm{e}}$,
- a continuous:
  $|a(u,v;\mu)| \le M ||u||_{X^{\mathrm{e}}} ||v||_{X^{\mathrm{e}}}, \forall u,v \in X^{\mathrm{e}}$,
- a coercive: $\exists \alpha > 0 : a(u,u;\mu) \ge \alpha^{\mathrm{e}} ||u||^2_{X^{\mathrm{e}}},\ \forall u \in X^{\mathrm{e}}$,
- f (and l) bounded/continuous:
  $|f(v)| \le C ||v||_{X^{\mathrm{e}}},\ \forall v \in X^{\mathrm{e}}$,
- f linear:
  $f(\gamma v + \eta w) = \gamma f(v) + \eta f(w),\ \forall \gamma, \eta \in \mathbb{R}, v \in X^{\mathrm{e}}$.

Statement: affine parameter dependence †

Definition:

$$a(w,v;\mu) = \sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(w,v)$$

for $q = 1,\ldots,Q$

μ-dependent functions $\Theta^q: \quad \mathcal{D} \rightarrow \mathbb{R}$,

μ-*independent* forms $a^q: \quad X^e \times X^e \rightarrow \mathbb{R}$.

Stiffness matrix:

$$a(\underset{\zeta_j}{w}, \underset{\zeta_i}{v}; \mu) = \sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(\underset{\zeta_j}{w}, \underset{\zeta_i}{v})$$

for $q = 1,\ldots,Q$

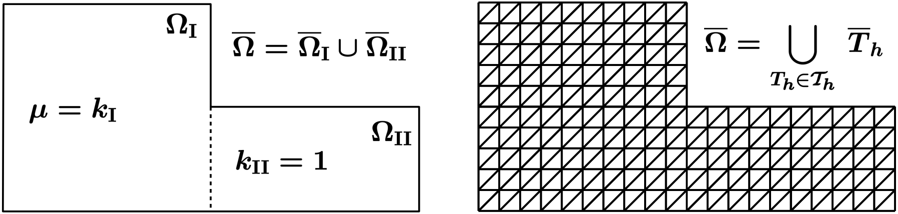
† In fact, *broadly applicable* to many instances of geometry and property parametric variation.

Little example: heat conduction

Given $k_{\mathrm{I}} \in [0.1, 10]$, evaluate $\overline{u}^{\mathrm{e}}_{\mathrm{I}}(k_{\mathrm{I}}) = \frac{1}{|\Omega_{\mathrm{I}}|} \int_{\Omega_{\mathrm{I}}} u^{\mathrm{e}}$

where $u^{\mathrm{e}}(k_{\mathrm{I}}) \in H^1_0(\Omega)$ satisfies ($Q = 2$)

$$k_{\mathrm{I}} \int_{\Omega_{\mathrm{I}}} \nabla u^{\mathrm{e}} \cdot \nabla v + \int_{\Omega_{\mathrm{II}}} \nabla u^{\mathrm{e}} \cdot \nabla v = \int_{\Omega} v, \ \forall\, v \in H^1_0(\Omega) \ .$$



$$X^{\mathcal{N}} \equiv \{v|_{T_h} \in \mathbb{P}_{\mathrm{I}}(T_h), \ \forall\, T_h \in \mathcal{T}_h\} \cap X^{\mathrm{e}}; \ \dim(X^{\mathcal{N}}) \equiv \mathcal{N} \ .$$

Classical Approximation: FEM (Galerkin projection)

Given $\mu \in \mathcal{D}$,

$$\text{evaluate } s^{\mathcal{N}} = \ell(u^{\mathcal{N}}(\mu)),$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}} .$$

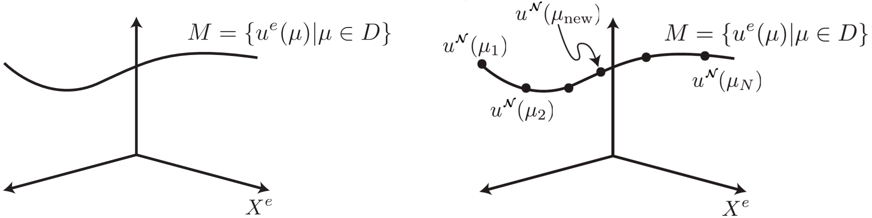
Typically: $|s^{\mathrm{e}}(\mu) - s^{\mathcal{N}}(\mu)|$ *small* $\Rightarrow \mathcal{N}$ *large*.

Surrogate for $s^{\mathrm{e}}(\mu)$, $u^{\mathrm{e}}(\mu)$: "truth"

- upon which we *build* reduced-basis *approximation*;[†]
- relative to which we *measure* reduced-basis *error*.[†]

[†] Require *stability* and *efficiency* as $\mathcal{N} \to \infty$.

Reduced-Basis Approximation: basic idea †



$\mathcal{M}^{\mathrm{e}}$ = parameter-induced manifold
(low-Dimensional ($\mathcal{D} \subset \mathbb{R}^P$), very smooth)

Classical Approach

$$X^{\mathcal{N}} \equiv \{v|_{T_h} \in \mathbb{P}_{\mathrm{I}}(T_h), \; \forall \, T_h \in \mathcal{T}_h\} \cap X^{\mathrm{e}}; \; \dim(X^{\mathcal{N}}) \equiv \mathcal{N} \, .$$

Reduced Basis Approach

$$W_N \equiv \mathrm{span}\{\zeta_n \equiv u^{\mathcal{N}}(\mu_n), \; 1 \leq n \leq N\}$$

† Pioneering works by Almroth, Stern& Brogan (1978), Noor & Peters (1980).

Reduced-Basis Approximation: formulation

Samples: $S_N = \{\mu_1 \in \mathcal{D}, \ldots, \mu_N \in \mathcal{D}\}$

Spaces: $W_N = \text{span}\{\zeta_n \equiv u^{\mathcal{N}}(\mu_n),\ 1 \leq n \leq N\}$

Given $\mu \in \mathcal{D}$,

evaluate $s_N(\mu) = \ell(u_N(\mu))$,

where $u_N(\mu) \in W_N$ satisfies

$$a(u_N(\mu), v; \mu) = f(v),\ \forall\, v \in W_N \,.$$

Reduced-Basis Approximation: convergence

Classical arguments yield

$$a(u^{\mathcal{N}}(\mu) - u_N(\mu), u^{\mathcal{N}}(\mu) - u_N(\mu); \mu) = \inf_{w_N \in W_N} a(u^{\mathcal{N}}(\mu) - w_N, u^{\mathcal{N}}(\mu) - w_N; \mu)$$

Properties of $\mathcal{M}^{\mathrm{e}}$ suggest

$$\inf_{w_N \in W_N} a(u^{\mathcal{N}}(\mu) - w_N, u^{\mathcal{N}}(\mu) - w_N; \mu) \to 0$$

rapidly (exponentially): N *small*.

Reduced-Basis Approxmation: discrete equations

Given $\mu \in \mathcal{D}$,

evaluate $s_N(\mu) = \ell(u_N(\mu))$,

where $u_N(\mu) \in W_N$ satisfies

$a(u_N(\mu), v; \mu) = f(v), \; \forall \, v \in W_N$.

Express

$$u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\, \zeta_j;$$

then

$$s_N(\mu) \equiv \ell(u_N(\mu)) = \sum_{j=1}^{N} u_{N\,j}(\mu)\, \ell(\zeta_j)$$

where

$$\sum_{j=1}^{N} a(\zeta_j, \zeta_i; \mu)\, u_{N\,j} = f(\zeta_i), \quad 1 \le i \le N .$$

RB Approximation: Offline-Online Procedure

Evaluation of $s_N(\mu)$ — *GIVEN* $u_{N\,j}$, $1 \le j \le N$

OFFLINE: Compute ζ_j, $1 \le j \le N$; $O(\mathcal{N})$

Form/Store $\ell(\zeta_j)$, $1 \le j \le N$.

ONLINE: Perform sum

$$s_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\ \ell(\zeta_j) \qquad O(N)$$

RB Approximation: Offline-Online Procedure

Evaluation of $u_{N\,j}(\mu),\ 1 \le j \le N$

IF $a(w, v; \mu)$ is affine,

$$\sum_{j=1}^{N} a(\zeta_j, \zeta_i; \mu)\, u_{N\,j} = f(\zeta_i), \quad 1 \le i \le N\ .$$

$$\Downarrow$$

$$\sum_{j=1}^{N} \left(\sum_{q=1}^{Q} \Theta^q(\mu)\ a^q(\zeta_j, \zeta_i) \right) u_{N\,j} = f(\zeta_i),\ 1 \le i \le N\ .$$

RB Approximation: Offline-Online Procedure

Evaluation of $u_{N\,j}(\mu),\ 1 \le j \le N$

OFFLINE: Form/Store $a^q(\zeta_j, \zeta_i),\quad 1 \le i, j \le N,$
$1 \le q \le Q. \qquad O(\mathcal{N})$

ONLINE: Form $\sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(\zeta_j, \zeta_i),\ 1 \le i, j \le N$
$— \quad O(QN^2)\ ;$

Solve for $u_{N\,j}(\mu),\ 1 \le j \le N \quad — \quad O(N^3)\ .$

RB Approximation: Offline-Online Procedure

. . . Evaluation of $u_{N\,j}(\mu),\ 1 \leq j \leq N$

Note $a^q(\zeta^j, \zeta^i)$ $\qquad 1 \leq i, j \leq N_{\max}$

$$= a^q\Big(\sum_{k=1}^{\mathcal{N}} \zeta_k^j \, \phi_k^{\mathrm{FE}}, \sum_{k'=1}^{\mathcal{N}} \zeta_{k'}^i \, \phi_{k'}^{\mathrm{FE}} \Big)$$

$$= \sum_{k=1}^{\mathcal{N}} \sum_{k'=1}^{\mathcal{N}} \zeta_k^j \, a^q(\phi_k^{\mathrm{FE}}, \phi_{k'}^{\mathrm{FE}}) \, \zeta_{k'}^i$$

$$= \underline{Z}_{N_{\max}}^T \, \underline{A}^{\mathrm{FE}\,q} \, \underline{Z}_{N_{\max}} \, .$$

RB Approximation: Goal

For *any* $\varepsilon_{\mathbf{des}} > \mathbf{0}$, evaluate — ACCURACY

$$\mu \in \mathcal{D} \to s_N^{\mathcal{N}}(\mu) \;\; (\approx s^{\mathcal{N}}(\mu))$$

that *provably* achieves desired accuracy — RELIABILITY

$$|s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu)| \leq \varepsilon_{\mathbf{des}}$$

but at (very low) marginal cost $\partial t_{\mathbf{comp}}$[†] — EFFICIENCY

independent of $\mathcal{N}$ as $\mathcal{N} \to \infty$.

[†] $\partial t_{\mathbf{comp}}$: time to perform *one additional certified* evaluation $\mu \to s_N^{\mathcal{N}}(\mu)$.

RB Approximation: Goal

Real-Time Context (parameter estimation, ...):

$$t_0 : \quad \mu \quad \longrightarrow \quad t_0 + \partial t_{\text{comp}} : \quad s_N^{\mathcal{N}}(\mu) \; .$$

"need" "response"

Many-Query Context (dynamic simulation, ...):

$$t_{\text{comp}}(\mu_j \rightarrow s_N^{\mathcal{N}}(\mu_j), \; j = 1, \ldots, J)$$
$$= \partial t_{\text{comp}} \; J \text{ as } J \rightarrow \infty \; .$$

If we require

real-time evaluation $\mu \rightarrow s^{\mathcal{N}}(\mu)$

or

many evaluations $\mu^k \rightarrow s^{\mathcal{N}}(\mu^k), \; k = 1, \ldots, \infty$

OFFLINE-ONLINE reduced-basis approximation offers *order-of-magnitude* — N vs. $\mathcal{N}$ — advantage.

Questions: Approximation

Can we develop *stable approximations*
for noncoercive and nonlinear problems? Y [ST, HZ; NS]

Can we choose our *parameter samples*
S_N ($\Rightarrow W_N$) wisely? . . . adaptively? Y [Greedies]

Can we prove *exponential convergence*
$u_N \longrightarrow u^{\mathcal{N}}$ *uniformly*† for all $\mu \in \mathcal{D}$? Y, BUT

ST = Stokes, HZ = Helmholtz, NS = Navier-Stokes problem
Greedy Algorithm: see ARCME, sect. 7

† *Uniform* (sharp) proofs available only for $P = 1$ parameter [MPT].

Questions: *A Posteriori* Error Estimation

Can we develop real-time $\times \mathbf{2}$

rigorous, *sharp*, *efficient*†

(Output) Error Bounds for

coercive problems? Y[SCM, α]

noncoercive problems? Y[SCM, β]

(quadratically) nonlinear problems? Y

SCM = Successive Constraints Method (see ARCME, sect. 10)

†Efficiency equates to online complexity *independent of* $\mathcal{N}$.

Questions: Efficiency

Can we develop efficient

OFFLINE ($\mathcal{N}$) — ONLINE (N) Procedures

even for problems with

non-affine parameter (μ) dependence? Y, [EIM]†

non-polynomial "state" (u) dependence? Y, BUT†

EIM = Empirical Interpolation Method

† In general, there will be some *loss of rigor* in our *a posteriori* error bounds.

Questions: Many Parameters

Can we consider *many* $(P \gg 1)$

"*correlated*" parameters†? Y

independent parameters

of small variation? Y

of large variation? N

† For example, as in smooth shape/boundary optimization.

Questions: Domain Decomposition

Can we consider various

Domain Decomposition

approaches to improve

efficiency?	Y
generality?	Y [RBEM]

RBEM = Reduced Basis Element method
(Maday, Ronquist, Lovgren, ...)

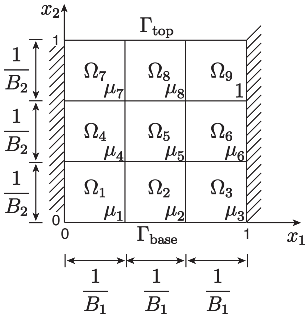
Example: Thermal Block

Given $\mu \equiv (\mu_1, \ldots, \mu_P) \in \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$ †

evaluate $s^{\mathrm{e}}(\mu) = f(u^{\mathrm{e}}(\mu))$

where $u^{\mathrm{e}}(\mu) \in X^{\mathrm{e}} \equiv \{v \in H^1(\Omega) \mid v|_{\Gamma_{\mathrm{top}}} = 0\}$

satisfies $a(u^{\mathrm{e}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathrm{e}}$.



$$\overline{\Omega} = \cup_{i=1}^{B_1 B_2} \overline{\Omega}_i$$

† Here $P = B_1 B_2 - 1$; we require $0 < \mu^{\min} < \mu^{\max} < \infty$.

Example: Thermal Block

Here

$$f(v) \equiv f^{\mathrm{Neu}}(v) \equiv \int_{\Gamma_{\mathrm{base}}} v \ ,$$

and symmetric, coercive

$$a(w, v; \mu) = \sum_{i=1}^{P} \mu_i \int_{\Omega_i} \nabla w \cdot \nabla v + \int_{\Omega_{P+1}} \nabla w \cdot \nabla v \ ,$$

where $\overline{\Omega} = \cup_{i=1}^{P+1} \overline{\Omega}_i$.

Example: Thermal Block

We obtain $P = B_1 B_2 - 1$

$$a(w, v; \mu) = \sum_{q=1}^{Q=P+1} \Theta^q(\mu)\, a^q(w, v)$$

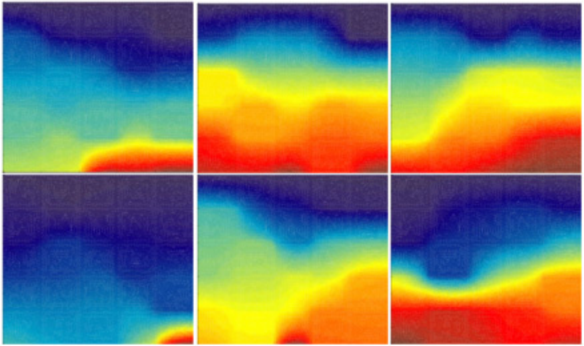
for

$$\Theta^q(\mu) = \mu_q,\ 1 \le q \le P, \quad \text{and} \quad \Theta^{P+1} = 1\ ,$$

and

$$a^q(w, v) = \int_{\Omega_q} \nabla w \cdot \nabla v,\ 1 \le q \le P + 1\ .$$

Example: Thermal Block



Representative Solutions

Sampling/Spaces Strategies: Preliminaries

Inner Products and Norms

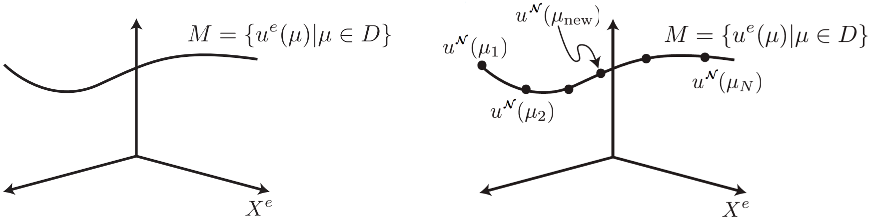
Define, $\forall\, w, v \in X^{\mathrm{e}}$ $\qquad X^{\mathcal{N}} \subset X^{\mathrm{e}}$

$$\left.\begin{aligned} ((w,v))_{\mu} &\equiv a(w,v;\mu) \\ |||w|||_{\mu} &\equiv ((w,w))_{\mu}^{1/2} \end{aligned}\right\} \text{energy}$$

and, given $\overline{\mu} \in \mathcal{D}$

$$\left.\begin{aligned} (w,v)_X &\equiv ((w,v))_{\overline{\mu}} + \tau (w,v)_{L^2(\Omega)} \\ \|w\|_X &\equiv (w,w)_X^{1/2} \end{aligned}\right\} X\,.$$

Sampling/Spaces Strategies: Preliminaries



$\mathcal{M}^{\mathrm{e}}$ = parameter-induced manifold
(low-Dimensional ($\mathcal{D} \subset \mathbb{R}^P$), very smooth)

Classical Approach

$$X^{\mathcal{N}} \equiv \{v|_{T_h} \in \mathbb{P}_{\mathrm{I}}(T_h),\ \forall\, T_h \in \mathcal{T}_h\} \cap X^{\mathrm{e}};\ \dim(X^{\mathcal{N}}) \equiv \mathcal{N}\ .$$

Reduced Basis Approach

$$W_N \equiv \mathrm{span}\{\zeta_n \equiv u^{\mathcal{N}}(\mu_n),\ 1 \le n \le N\}$$

Sampling/Spaces Strategies: Spaces

Nested Samples:

$$S_N = \{\mu^1 \in \mathcal{D}, \ldots, \mu^N \in \mathcal{D}\},\ 1 \le N \le N_{\max}\ .$$

Hierarchical Spaces: Lagrange

$$W_N^{\mathcal{N}} = \mathrm{span}\{u^{\mathcal{N}}(\mu^n),\ 1 \le n \le N\},\ 1 \le N \le N_{\max}\ .$$

Orthonormal Basis:

$$\{\zeta^{\mathcal{N}\,n}\}_{1 \le n \le N_{\max}} = \text{G-S}\left(\{u^{\mathcal{N}}(\mu^n)\}_{1 \le n \le N_{\max}}; (\,\cdot\,,\,\cdot\,)_X\right).$$

Sampling/Spaces Strategies: Orthogonalization

Given $u^{\mathcal{N}}(\mu^n)$, $1 \leq n \leq N\max$:

Form $\{\zeta^n\}$, $1 \leq n \leq N\max$ given as

$n = 1, \zeta^1 = u^{\mathcal{N}}(\mu^1))/\|u^{\mathcal{N}}(\mu^1)\|_X;$
for $n = 2: N_{\max}$

$$z^n = u^{\mathcal{N}}(\mu^n) - \sum_{m=1}^{n-1} (u^{\mathcal{N}}(\mu^n), \zeta^m)_X \, \zeta^m;$$

$$\zeta^n = z^n/\|z^n\|_X;$$

end.

As a result of this process we obtain the orthogonality condition

$$(\zeta^n, \zeta^m)_X = \delta_{n\,m}, \qquad 1 \leq n, m \leq N_{\max} \ , \qquad (1)$$

where $\delta_{n\,m}$ is the Kronecker-delta symbol

Sampling/Spaces Strategies: RB Galerkin Projection

Optimality:

$$|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu \leq \inf_{w \in W_N^{\mathcal{N}}} |||u^{\mathcal{N}}(\mu) - w|||_\mu \ ;$$

combination of snapshots.

Note also:

$$s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu) \equiv |||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu^2 \ ;$$

output converges as square.

Sampling/Spaces Strategies: RB Galerkin Projection

$$s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu) \equiv |||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu^2\ ;$$

$$s^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu)); s_N^{\mathcal{N}}(\mu) = f(u_N^{\mathcal{N}}(\mu));$$

$$s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu)) - f(u_N^{\mathcal{N}}(\mu)) =$$

$$= a(v, u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu); \mu);$$

$$e(\mu) = u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu);$$

$$a(v, e(\mu); \mu) = a(e(\mu), v; \mu) = a(e(\mu), e(\mu); \mu);$$

$$a(e(\mu), e(\mu); \mu) = |||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu^2.$$

Sampling/Spaces Strategies: General "Reduced Model"

Given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\text{evaluate } s_N^{\mathcal{N}}(\boldsymbol{\mu}) = f(u_N^{\mathcal{N}}(\boldsymbol{\mu})) \ ,$$

where $u_N^{\mathcal{N}}(\boldsymbol{\mu}) \in X_N^{\mathcal{N}} \subset X^{\mathcal{N}}$ satisfies $\qquad \dim(X_N^{\mathcal{N}}) = N$ †

$$a(u_N^{\mathcal{N}}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v), \ \forall\, v \in X_N^{\mathcal{N}} \ .$$

"Train" sample:

$$\Xi_{\text{train}} \subset \mathcal{D} \subset \mathbb{R}^P; \qquad |\Xi_{\text{train}}| = n_{\text{train}} \ (\gg 1) \ .$$

"Test" sample:

$$\Xi_{\text{test}} \subset \mathcal{D} \subset \mathbb{R}^P; \qquad |\Xi_{\text{test}}| = n_{\text{test}} \ (\gg 1) \ .$$

† Here $X_N^{\mathcal{N}}$ may be a hierarchical or non-hierarchical Lagrange ($W_N^{\mathcal{N}}$) or non-Lagrange RB space (Taylor, Hermite), or even a "non-RB" (non-$\mathcal{M}^{\mathcal{N}}$) space (Kolmogorov).

Sampling/Spaces Strategies: Norms

Given $\Xi \subset \mathcal{D}$, $y \colon \mathcal{D} \rightarrow \mathbb{R}$,

$$\|y\|_{L^\infty(\Xi)} \equiv \operatorname*{ess\,sup}_{\mu \in \Xi} |y(\mu)| ,$$

$$\|y\|_{L^2(\Xi)} \equiv \left(|\Xi|^{-1} \sum_{\mu \in \Xi} y^2(\mu) \right)^{1/2} .$$

Given $z \colon \mathcal{D} \rightarrow X^{\mathcal{N}}$ (or X^{e})

$$\|z\|_{L^\infty(\Xi;X)} \equiv \operatorname*{ess\,sup}_{\mu \in \Xi} \|z(\mu)\|_X ,$$

$$\|z\|_{L^2(\Xi;X)} \equiv \left(|\Xi|^{-1} \sum_{\mu \in \Xi} \|z(\mu)\|_X^2 \right)^{1/2} .$$

Sampling/Spaces Strategies: 1. Lagrange “à la main”

Example: $P = 1$ $\qquad\qquad \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]$

$$S_N^{\mathrm{nh,ln}} = \{\mu_N^n,\ 1 \le n \le N\},\ 1 \le N \le N_{\max}\ ,$$

$$\mu_N^n = \mu^{\min} \exp\left\{\frac{n-1}{N-1}\ \ln\left(\frac{\mu^{\max}}{\mu^{\min}}\right)\right\},\ 1 \le n \le N\ ;$$

$$W_N^{\mathcal{N}\ \mathrm{nh,ln}} = \mathrm{span}\{u^{\mathcal{N}}(\mu_N^n),\ 1 \le n \le N\}\ ,$$

$$1 \le N \le N_{\max}\ .$$

† Note this Lagrange space is not hierarchical, and hence *not very practical*; we denote non-hierarchical by “**nh**.”

Sampling/Spaces Strategies: 2. POD

Given Ξ_{train},

$$X_N^{\mathcal{N}\,\text{POD}} = \arg \inf_{X_N^{\mathcal{N}} \subset \text{span}\{u^{\mathcal{N}}(\mu) \mid \mu \in \Xi_{\text{train}}\}} \|u^{\mathcal{N}} - \Pi_{X_N^{\mathcal{N}}} u^{\mathcal{N}}\|_{L^2(\Xi_{\text{train}};X)} ;$$

eigenproblem interpretation demonstrates

hierarchical property.

Issues: $$ — n_{train} FE solutions,
$n_{\text{train}} \times n_{\text{train}}$ eigenproblem;
weaker norm over Ξ_{train}.

Sampling/Spaces Strategies: 2. POD

$$\underline{C}^{POD} \in \mathbb{R}^{n_{\text{train}} \times n_{\text{train}}} : \quad for 1 \leq i, j \leq n_{\text{train}} \quad ,$$

$$C_{ij}^{POD} = \frac{1}{n_{\text{train}}} \left(u(\mu_{\text{train}}^{i}), u(\mu_{\text{train}}^{j}) \right)_{X} ,$$

Eigenpairs:
$(\underline{\psi}^{POD,k} \in \mathbb{R}^{n_{\text{train}}}, \lambda^{POD,k} \in \mathbb{R}_{+0})$, $1 \leq k \leq n_{\text{train}}$,

$$\underline{C}^{POD}\, \underline{\psi}^{POD,k} = \lambda^{POD,k} \underline{\psi}^{POD,k} \; .$$

Arranging eigenvalues in *descending* order:

$$\lambda^{POD,1} \geq \lambda^{POD,2} \geq \cdots \lambda^{POD,n_{\text{train}}} \geq 0.$$

We now identify $\Psi^{POD,k} \in X$, $1 \leq k \leq n_{\text{train}}$, as
$\Psi^{POD,k} \equiv \sum_{m=1}^{n_{\text{train}}} \psi_m^{POD,k}\, u(\mu_{\text{train}}^{m})$;

Sampling/Spaces Strategies: 2. POD

Define $N_{\max}$ as the smallest N such that

$$\left(\overline{\overline{\varepsilon}}_N^{POD} \equiv\right) \sqrt{\sum_{k=N+1}^{n_{\text{train}}} \lambda^{POD,k}} \leq \varepsilon_{tol,\min} \ .$$

POD RB spaces

$$X_N^{POD} = \text{span}\{\Psi^{POD,n},\ 1 \leq n \leq N\}, \quad 1 \leq N \leq N_{\max} \ ;$$

$$(\Psi^{POD,n}, \Psi^{POD,m})_X = \delta_{n\,m}, 1 \leq n, m \leq n_{\text{train}}$$

and hence $(\Psi^{POD,n} \equiv)\ \xi^n = \zeta^n$, $1 \leq n \leq N_{\max}$.

Sampling/Spaces Strategies: 3. Greedy Procedure

Given Ξ_{train}, $S_1 = \{\mu^1\}$, $W_1^{\mathcal{N}} = \text{span}\{u^{\mathcal{N}}(\mu^1)\}$,

[for $N = 2, \ldots, N_{\max}$:

$$\begin{aligned} \mu^N &= \arg \max_{\mu \in \Xi_{\text{train}}} \Delta_{N-1}(\mu) \\ S_N &= S_{N-1} \cup \mu^N; \\ W_N^{\mathcal{N}} &= W_{N-1}^{\mathcal{N}} + \text{span}\{u^{\mathcal{N}}(\mu^N)\}.] \end{aligned}$$

Issue: suboptimal (heuristic).

Here, for $N = 1, \ldots$

$$\|u^{\mathcal{N}}(\mu) - u^{\mathcal{N}}_{W_N^{\mathcal{N}}}(\mu)\|_X \leq \Delta_N(\mu), \quad \forall\, \mu \in \mathcal{D}:$$

$\Delta_N(\mu)$ is a sharp, *inexpensive*† *a posteriori* error *bound* for $\|u^{\mathcal{N}}(\mu) - u^{\mathcal{N}}_{W_N^{\mathcal{N}}}(\mu)\|_X$.

Greedy only computes actual (*winning* candidate) snapshots.

† Marginal cost (= average asymptotic cost) is *independent* of $\mathcal{N}$.

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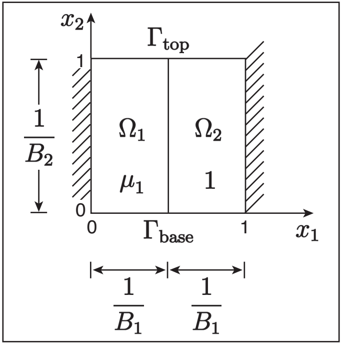
Convergence: $P = 1$

Example: Thermal Block — $(\mathbf{2}, \mathbf{1})$



Geometry

Convergence: $P = 1$

Problem statement

Given $\mu = (\mu_1) \in \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]$,

evaluate $s^{\mathrm{e}}(\mu) = f(u^{\mathrm{e}}(\mu))$

where $u^{\mathrm{e}}(\mu) \in X^{\mathrm{e}} \equiv \{v \in H^1(\Omega) \mid v|_{\Gamma_{\mathrm{top}}} = 0\}$

satisfies $a(u^{\mathrm{e}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathrm{e}}$.

Choose $\mu_{\min} = \frac{1}{\sqrt{\mu_r}}$, $\mu_{\max} = \sqrt{\mu_r}$ $(\frac{\mu_{\max}}{\mu_{\min}} = \mu_r)$; $\mu_r = 100$.

Convergence: $P = 1$

Problem statement

Here $(Q = 2)$

$$f \left(= f^{\mathrm{Neu}}(v) = \int_{\Gamma_{\mathrm{base}}} v, \ \forall v \in X^{\mathrm{e}} \right) \in (X^{\mathrm{e}})'$$

and

$$a(w, v; \mu) = \mu \int_{\Omega_1} \nabla w \cdot \nabla v + \int_{\Omega_2} \nabla w \cdot \nabla v \ ,$$

where $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$.

Convergence: $P = 1$

FE Approximation

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, $\qquad\qquad \mathcal{N} = 1024^\dagger$

$$\text{evaluate } s^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu)) \ ,$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}} \subset X^{\mathrm{e}}$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}} \ .$$

†Note here $\mathbf{span}\{\mathcal{M}^{\mathcal{N}}\}$ is of dimension $\approx 4\sqrt{\mathcal{N}}$.

Convergence: $P = 1$

RB Approximation

Given $\mu \in \mathcal{D}$,

evaluate $s_N^{\mathcal{N}}(\mu) = f(u_N^{\mathcal{N}}(\mu))$,

where $u_N^{\mathcal{N}}(\mu) \in X_N^{\mathcal{N}} \subset X^{\mathcal{N}}$ satisfies $\quad \dim(X_N^{\mathcal{N}}) = N$ †

$$a(u_N^{\mathcal{N}}(\mu), v; \mu) = f(v), \ \forall\, v \in X_N^{\mathcal{N}} \, .$$

† In Lagrange case, $X_N^{\mathcal{N}} = W_N^{\mathcal{N}}$ (hierarchical) or $W_N^{\mathcal{N}\ \mathrm{nh}}$ (non-hierarchical).

Convergence: $P = 1$

A *Priori* Theory

Choose (non-hierarchical)

$$\begin{aligned} S_N^{\text{nh,ln}} &= \{\mu_N^n,\ 1 \le n \le N\},\ 1 \le N \le N_{\max}\ , \\ \mu_N^n &= \mu^{\min} \exp\left\{\frac{n-1}{N-1}\ \ln\left(\frac{\mu^{\max}}{\mu^{\min}}\right)\right\},\ 1 \le n \le N\ ; \\ W_N^{\mathcal{N}\ \text{nh,ln}} &= \text{span}\{u^{\mathcal{N}}(\mu_N^n),\ 1 \le n \le N\}\ , \\ & \qquad 1 \le N \le N_{\max}\ . \end{aligned}$$

Proposition 1

For any $f \in (X^{\mathrm{e}})'$, $N \ge 2$

$$\frac{|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu}{|||u^{\mathcal{N}}(\mu)|||_\mu} \le \exp\left\{-\frac{N-1}{N_{\text{crit}}-1}\right\}, \quad \forall\, \mu \in \mathcal{D}$$

for $N \ge N_{\text{crit}} = 1 + [2e \ln \mu_r]_+$. □

Note "no" dependence on spatial regularity; "no" dependence on $\mathcal{N}$; weak dependence on μ_r.

Convergence: $P = 1$

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Note "no" dependence on spatial regularity; "no" dependence on $\mathcal{N}$; weak dependence on μ_r.

Convergence: $P = 1$

Algorithms

Recall Greedy *heuristically* minimizes

RB error *bound* in $L^\infty(\Xi_{\text{train}}; X)$,

while POD *truly* minimizes

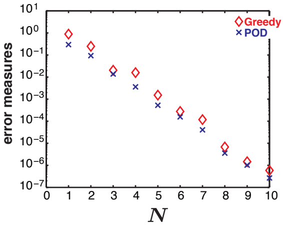
projection error in $L^2(\Xi_{\text{train}}; X)$;

further recall

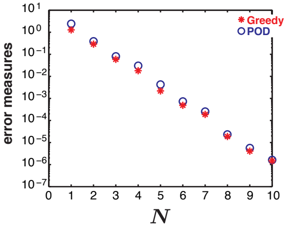
Cost (Greedy) $\ll$ Cost (POD) for large n_{train}.

Convergence: $P = 1$

Numerical Results



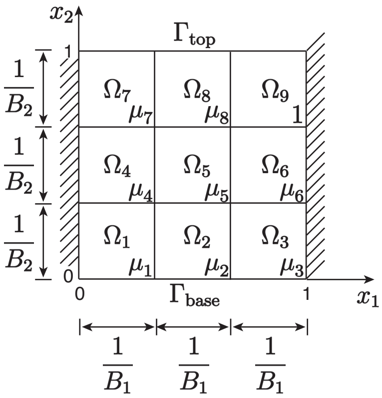
$$||u^{\mathcal{N}} - u_N^{\mathcal{N}}||_{L^2(\Xi;X)}$$



$$||u^{\mathcal{N}} - u_N^{\mathcal{N}}||_{L^\infty(\Xi;X)}$$

Convergence: $P > 1$

Example: Thermal Block — $(\mathbf{3}, \mathbf{3})$

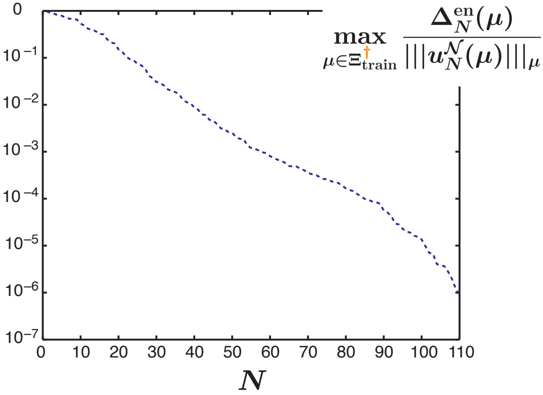


$$\overline{\boldsymbol{\Omega}} = \cup_{i=1}^{B_1 B_2} \overline{\boldsymbol{\Omega}}_i$$

Geometry

Convergence: $P > 1$

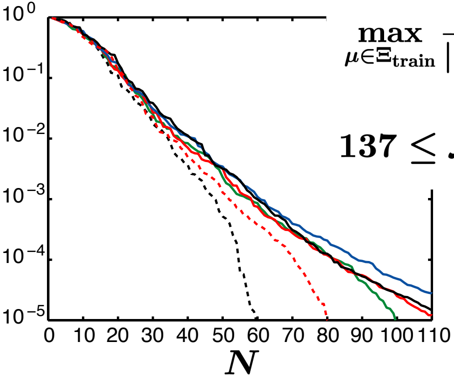
Example: Thermal Block — $(3,3)$
Greedy: RB Energy Error



† Here $\Xi_{\mathbf{train}}$ is a Monte Carlo sample in $\ln \mu$ of size $n_{\text{train}} = 5000$ ($\gg N$); note $|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu \le \Delta_N^{\text{en}}(\mu)$, $|||u_N^{\mathcal{N}}(\mu)|||_\mu \le |||u^{\mathcal{N}}(\mu)|||_\mu$.

Convergence: $P > 1$

Example: Thermal Block — $(3,3)$
Effect of $X^{\mathcal{N}}$



$$\max_{\mu \in \Xi_{\text{train}}} \frac{\Delta_N^{\text{en}}(\mu)}{|||u_N^{\mathcal{N}}(\mu)|||_\mu}$$

$$137 \le \mathcal{N} \le 6808$$

Problem "Scope": Geometry

Domain decomposition: definition

Original Domain $\boldsymbol{\Omega_o(\mu)}$, $\boldsymbol{u_o^e \in X_o^e(\Omega_o(\mu))}$

$$\overline{\boldsymbol{\Omega}}_o(\boldsymbol{\mu}) = \bigcup_{k=1}^{K_{\rm dom}} \overline{\boldsymbol{\Omega}}_o^k(\boldsymbol{\mu}) \; ;$$

Reference domain $\boldsymbol{\Omega}$, $\boldsymbol{u^e \in X^e(\Omega)}$

$$\overline{\boldsymbol{\Omega}} = \bigcup_{k=1}^{K_{\rm dom}} \overline{\boldsymbol{\Omega}}^k \; ,$$

common configuration

where $\boldsymbol{\Omega = \Omega_o(\mu_{\rm ref})}$ for $\boldsymbol{\mu_{\rm ref} \subset \mathcal{D}}$†.

For Ω^k, $\Omega_o^k(\mu)$ we choose in $\mathbb{R}^2$ triangles, elliptical triangles and curvy triangles. In $\mathbb{R}^3$ we choose parallelepipeds (and in theory tetrahedra).

† Connectivity requirement: subdomain intersections must be an entire edge, a vertex, or null.

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Affine Mappings

Require $\forall \mu \in \mathcal{D}$

$$\overline{\Omega}_o^k(\mu) = \mathcal{T}^{\text{aff},k}(\overline{\Omega}^k;\mu)\ , 1 \le k \le K_{\text{dom}}\ ,$$

where

$$\mathcal{T}^{\text{aff},k}(x;\mu) = C^{\text{aff},k}(\mu) + G^{\text{aff},k}(\mu)x\ ,$$

is an invertible affine mapping from $\overline{\Omega}^k$ onto $\overline{\Omega}_o^k(\mu)$.

Further require $\forall \mu \in \mathcal{D}$

$$\mathcal{T}^{\text{aff},k}(x;\mu) = \mathcal{T}^{\text{aff},k'}(x;\mu), \quad \forall\, x \in \overline{\Omega}^k \cap \overline{\Omega}^{k'}, \quad 1 \le k, k' \le K_{\text{dom}}\ ,$$

to ensure a *continuous* piecewise-affine global mapping $\mathcal{T}^{\text{aff}}(\,\cdot\,;\mu)$ from $\overline{\Omega}$ onto $\overline{\Omega}_o(\mu)^\dagger$.

† It follows that for $w_o \in H^1(\Omega_o(\mu))$, $w_o \circ \mathcal{T}^{\text{aff}} = H^1(\Omega)$.

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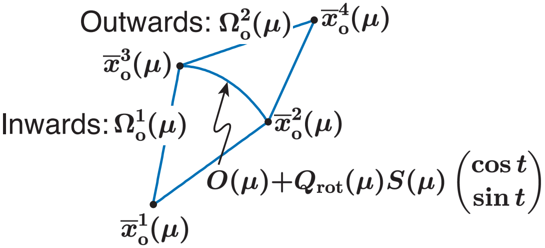
[†] It follows that for $w_{\mathrm{o}} \in H^1(\Omega_{\mathrm{o}}(\mu))$, $w_{\mathrm{o}} \circ \mathcal{T}^{\mathrm{aff}} = H^1(\Omega)$.

Affine Mappings

Elliptical Triangles: definition



$$\begin{aligned} O(\mu) &= (x_{o1}^{\rm cen}, x_{o2}^{\rm cen})^{\rm T} \\ Q_{\rm rot}(\mu) &= \begin{pmatrix} \cos\phi(\mu) & -\sin\phi(\mu) \\ \sin\phi(\mu) & \cos\phi(\mu) \end{pmatrix} \\ S(\mu) &= \mathrm{diag}(\rho_1(\mu), \rho_2(\mu)) \end{aligned}$$

Affine Mappings

Elliptical Triangles: constraints

Given $\overline{\boldsymbol{x}}_{\mathbf{o}}^{\mathbf{2}}(\boldsymbol{\mu}), \overline{\boldsymbol{x}}_{\mathbf{o}}^{\mathbf{3}}(\boldsymbol{\mu})$, find $\overline{\boldsymbol{x}}_{\mathbf{o}}^{\mathbf{1}}(\boldsymbol{\mu}), \overline{\boldsymbol{x}}_{\mathbf{o}}^{\mathbf{4}}(\boldsymbol{\mu})$ $(\Rightarrow \boldsymbol{\mathcal{T}}^{\mathbf{aff},\mathbf{1\&2}})$

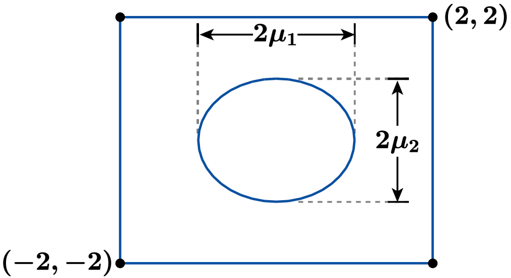
$$\left.\begin{array}{ll} (i) & \text{produce desired elliptical arc} \\ (ii) & \text{satisfy internal angle criterion} \end{array}\right\} \forall \boldsymbol{\mu} \in \boldsymbol{\mathcal{D}};$$

these conditions ensure *continuous invertible* mappings.

† Explicit recipes for admissible $x_{\mathrm{o}}^{1}(\boldsymbol{\mu})$ (Inwards case) and $x_{\mathrm{o}}^{4}(\boldsymbol{\mu})$ (Outwards case) are readily obtained.

Affine Mappings

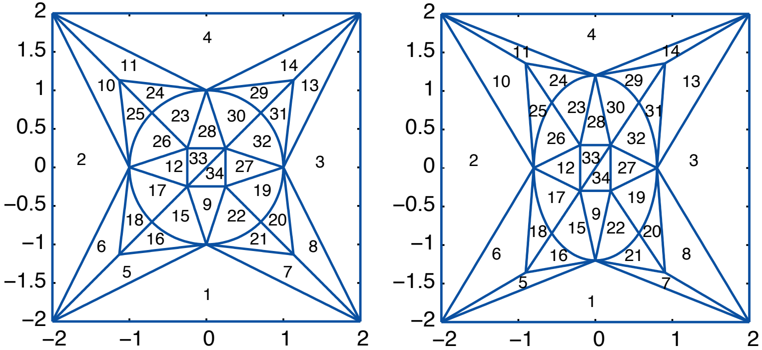
Elliptical Triangles: example (CinS triangulation)



$$\Omega_o(\mu): \; \mu = (\mu_1, \mu_2, \ldots) \subset \mathcal{D} \equiv [0.8, 1.2]^2 \times \ldots$$

Affine Mappings

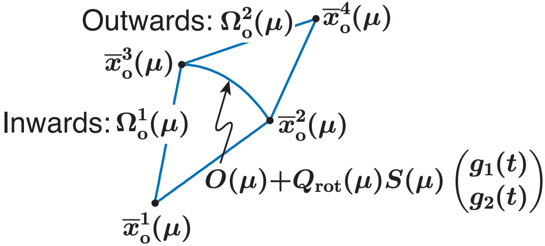
Elliptical Triangles: example (CinS triangulation)



$\Omega = \Omega_o(\mu_{\text{ref}} = (1, 1))$ $\Omega_o(\mu = (0.8, 1.2))$

Affine Mappings

Curvy Triangles: definition



$$
\begin{aligned}
O(\mu) &= (x_{o1}^{\text{cen}}, x_{o2}^{\text{cen}})^{\text{T}} \\
Q_{\text{rot}}(\mu) &= \begin{pmatrix} \cos\phi(\mu) & -\sin\phi(\mu) \\ \sin\phi(\mu) & \cos\phi(\mu) \end{pmatrix} \\
S(\mu) &= \text{diag}(\rho_1(\mu), \rho_2(\mu))
\end{aligned}
$$

Affine Mappings

Curvy Triangles: constraints

Given $\overline{\boldsymbol{x}}_{\mathbf{o}}^{2}(\boldsymbol{\mu}), \overline{\boldsymbol{x}}_{\mathbf{o}}^{3}(\boldsymbol{\mu})$, find $\overline{\boldsymbol{x}}_{\mathbf{o}}^{1}(\boldsymbol{\mu}), \overline{\boldsymbol{x}}_{\mathbf{o}}^{4}(\boldsymbol{\mu})$ $(\Rightarrow \mathcal{T}^{\mathbf{aff},\mathbf{1\&2}})$

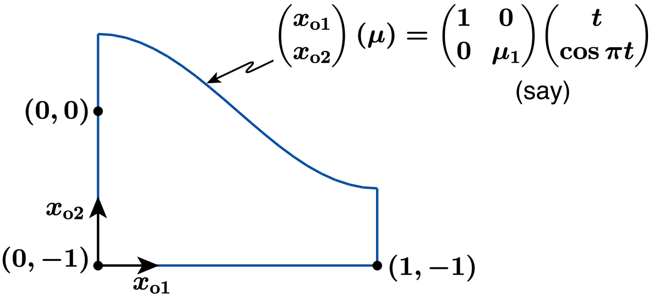
$$\left.\begin{array}{ll} (i) & \text{produce desired curvy arc} \\ (ii) & \text{satisfy internal angle criterion} \end{array}\right\} \forall \boldsymbol{\mu} \in \boldsymbol{\mathcal{D}};$$

these conditions ensure *continuous invertible* mappings.

† Quasi-explicit recipes for admissible $\overline{\boldsymbol{x}}_{\mathbf{o}}^{1}(\boldsymbol{\mu})$ and $\overline{\boldsymbol{x}}_{\mathbf{o}}^{4}(\boldsymbol{\mu})$ can (sometimes) be obtained in the convex/concave case.

Affine Mappings

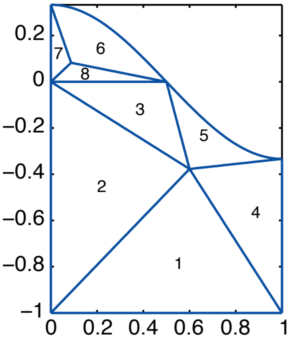
Elliptical Triangles: example (Cosine triangulation)



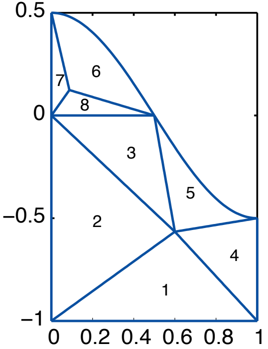
$$\Omega_o(\mu): \ \mu = (\mu_1, \ldots) \subset \mathcal{D} \equiv [\tfrac{1}{6}, \tfrac{1}{2}] \times \ldots$$

Affine Mappings

Elliptical Triangles: example (Cosine triangulation)



$\Omega = \Omega_o(\mu_{ref} = \frac{1}{3})$



$\Omega_o(\mu = \frac{1}{2})$

Problem Scope: Bilinear Form

Transformation: Formulation on original domain ($\mathbb{R}^2$)

For $w, v \in H^1(\Omega_o(\mu))^\dagger$ $\qquad u_o^e(\mu) \in H_0^1(\Omega_o(\mu))$

$$a_o(w, v; \mu) = \sum_{k=1}^{K_{dom}} \int_{\Omega_o^k(\mu)} \begin{bmatrix} \frac{\partial w}{\partial x_{o1}} & \frac{\partial w}{\partial x_{o2}} & w \end{bmatrix} \mathcal{K}_{oij}^k(\mu) \begin{bmatrix} \frac{\partial v}{\partial x_{o1}} \\ \frac{\partial v}{\partial x_{o2}} \\ v \end{bmatrix}$$

where $\mathcal{K}_o^k \colon \mathcal{D} \to \mathrm{R}^{3\times 3}$, SPD for $1 \leq k \leq K_{dom}$

(note $\mathcal{K}_o^k$ affine in x_o is also permissible).

† We consider the scalar case; the vector case (linear elasticity) admits an analogous treatment.

Problem Scope: Bilinear Form

Transformation: Formulation on reference domain

For $w, v \in H^1(\Omega)$ $\qquad u^e(\mu) \in H_0^1(\Omega)$

$$a(w, v; \mu) = \sum_{k=1}^{K_{\text{dom}}} \int_{\Omega^k} \begin{bmatrix} \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & w \end{bmatrix} \mathcal{K}_{ij}^k(\mu) \begin{bmatrix} \frac{\partial v}{\partial x_1} \\ \frac{\partial v}{\partial x_2} \\ v \end{bmatrix}$$

$$\mathcal{K}^k(\mu) = |\det G^{\text{aff},k}(\mu)| D(\mu) \mathcal{K}_o^k(\mu) D^T(\mu), \text{ and}$$

$$D(\mu) = \begin{pmatrix} (G^{\text{aff},k})^{-1} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem Scope: Bilinear Form

Transformation: Affine form

Expand

$$a(w, v; \mu) = \underbrace{\mathcal{K}^1_{11}(\mu)}_{\Theta^1(\mu)} \underbrace{\int_{\Omega^1} \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_1}}_{a^1(w,v)} + \dots$$

with as many as $Q = 4K$ terms.

We can often greatly reduce the requisite Q.

Achtung! Many interesting problems are **not** affine (or require Q very large).

For example, $\mathcal{K}^k_{\text{o}}(x; \mu)$ for general x dependence; and nonzero Neumann conditions on curvy $\partial\Omega$ yield non-affine $a(\,\cdot\,, \,\cdot\,; \mu)$.

Problem Scope: Bilinear Form

Transformation: Affine form

Expand

$$a(w,v;\mu) = \underbrace{\mathcal{K}^1_{11}(\mu)}_{\Theta^1(\mu)} \underbrace{\int_{\Omega^1} \frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_1}}_{a^1(w,v)} + \dots$$

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