Model Reduction Methods

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Lecture 2

- **Lecture 2: Elliptic Problems II, Parabolic Problems**

1. Elliptic Problems II
   (e) A Posteriori Error Estimation (elements)
   (f) General Outputs (non-compliant), Non-symmetric Forms (Dual Problem, A Posteriori Error Estimation)
2. Parabolic Problems
   (a) Problem Statement, Truth Approximation
   (b) Reduced Basis Approximation
   (c) Offline-Online Computational Procedures
   (d) A Posteriori Error Estimation

A posteriori Error Estimation: Role

OFFLINE

Error bound permits “large” $\Xi_{\mathbf{train}} \subset \mathcal{D}$,

$\Rightarrow$ rapidly convergent $W_N^{\mathcal{N}}$,

$\Rightarrow$ small $\partial t_{\mathbf{comp}}(\mu \longrightarrow s_N^{\mathcal{N}}(\mu))$; and

rigorous assessment $|s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu)|$, $\forall\, \mu \in \mathcal{D}$.

ONLINE

A posteriori Error Estimation: Preliminaries

Residual

Define $\boldsymbol{r}\colon \boldsymbol{\mathcal{D}} \rightarrow (\boldsymbol{X}^{\boldsymbol{\mathcal{N}}})'$ and $\hat{e}\colon \boldsymbol{\mathcal{D}} \rightarrow \boldsymbol{X}^{\boldsymbol{\mathcal{N}}}$

$$\begin{aligned} \boldsymbol{r}(\boldsymbol{v};\boldsymbol{\mu}) &\equiv \boldsymbol{f}(\boldsymbol{v}) - \boldsymbol{a}(\boldsymbol{u}_{\boldsymbol{N}}^{\boldsymbol{\mathcal{N}}}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}) \ , \\ (\hat{e}(\boldsymbol{\mu}), \boldsymbol{v})_{\boldsymbol{X}} &= \boldsymbol{r}(\boldsymbol{v};\boldsymbol{\mu}), \ \forall \boldsymbol{v} \in \boldsymbol{X}^{\boldsymbol{\mathcal{N}}} \ ; \end{aligned}$$

then dual norm given by

$$\begin{aligned} \|\boldsymbol{r}(\,\cdot\,;\boldsymbol{\mu})\|_{(\boldsymbol{X}^{\boldsymbol{\mathcal{N}}})'} &= \sup_{\boldsymbol{v}\in\boldsymbol{X}^{\boldsymbol{\mathcal{N}}}} \frac{\boldsymbol{r}(\boldsymbol{v};\boldsymbol{\mu})}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} \\ &= \|\hat{e}(\boldsymbol{\mu})\|_{\boldsymbol{X}} \ . \end{aligned}$$

A posteriori Error Estimation: Preliminaries

Coercivity, Continuity Constants

Introduce coercivity "constants"

$$\alpha^{\mathrm{e}}(\mu) \equiv \inf_{w \in X^{\mathrm{e}}} \frac{a(w, w; \mu)}{\|w\|_X^2}, \quad \alpha^{\mathcal{N}}(\mu) \equiv \inf_{w \in X^{\mathcal{N}}} \frac{a(w, w; \mu)}{\|w\|_X^2} ;$$

for our *coercive* problems,

$$\alpha^{\mathcal{N}}(\mu) \geq \alpha^{\mathrm{e}}(\mu) \geq \alpha_0^{\mathrm{e}} > 0, \quad \forall\, \mu \in \mathcal{D} .$$

Also define continuity "constant,"

$$\gamma^{\mathrm{e}}(\mu) = \sup_{w \in X^{\mathrm{e}}} \sup_{v \in X^{\mathrm{e}}} \frac{a(w, v; \mu)}{\|w\|_X \, \|v\|_X} .$$

A posteriori Error Estimation: Preliminaries

Coercivity Lower Bound

Require

$$\alpha_{\mathrm{LB}}^{\mathcal{N}} : \mathcal{D} \to \mathbb{R}$$

such that

$$0 < \alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu) \leq \alpha^{\mathcal{N}}(\mu), \quad \forall\, \mu \in \mathcal{D} \ ,$$

and $\partial t_{\mathrm{comp}}(\mu \to \alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu))$

is $[O(1)]$ independent of $\mathcal{N}$.

† A prescription can be found in [ARCME (Sec.10)].

A posteriori Error Estimators: Error Bounds

Error estimators:

$$\begin{aligned} \Delta_N^{\mathrm{en}}(\mu) &\equiv \|\hat{e}(\mu)\|_X / (\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu))^{1/2} , \\ \Delta_N^{s}(\mu) &\equiv \|\hat{e}(\mu)\|_X^2 / \alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu) ; \end{aligned}$$

Effectivities:

$$\begin{aligned} \eta_N^{\mathrm{en}}(\mu) &\equiv \Delta_N^{\mathrm{en}}(\mu) / |||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu , \\ \eta_N^{s}(\mu) &\equiv \Delta_N^{s}(\mu) / (s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu)) . \end{aligned}$$

A posteriori Error Estimators: Error Bounds

Effectivity Results

Proposition 2

For $N = 1, \ldots$ †

$$1 \leq \eta_N^{\mathrm{en}} \leq \sqrt{\frac{\gamma^{\mathrm{e}}(\mu)}{\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu)}}, \quad \forall\, \mu \in \mathcal{D}\,,$$

(rigor) (sharpness)

$$1 \leq \eta_N^{s}(\mu) \leq \frac{\gamma^{\mathrm{e}}(\mu)}{\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu)}, \quad \forall\, \mu \in \mathcal{D}\,;$$

recall a is symmetric and s is "compliant" ($\ell = f$). □

† Similar results obtain for $\Delta_N(\mu)$, the error bound in the X norm.

A posteriori Error Estimators: Error Bounds

Proofs

It follows from $a(e(\mu), v; \mu) = (\hat{e}(\mu), v)_X$ for $v = e(\mu)$ and the Cauchy-Schwarz inequality that

$$|||e(\mu)|||^2_\mu \le \|\hat{e}(\mu)\|_X \, \|e(\mu)\|_X \ , \qquad (1)$$

but $(\alpha^{\mathcal{N}}(\mu))^{\frac{1}{2}} \, \|e(\mu)\|_X \le a^{\frac{1}{2}}(e(\mu), e(\mu); \mu) \equiv |||e(\mu)|||_\mu$, and hence from (1) we obtain

$$(\alpha^{\mathcal{N}}(\mu))^{\frac{1}{2}} \frac{|||e(\mu)|||^2_\mu}{\|\hat{e}(\mu)\|_X} \le |||e(\mu)|||_\mu$$

s.t. $|||e(\mu)|||_\mu \le \Delta^{\text{en}}_N(\mu)$ or $\eta^{\text{en}}_N(\mu) \ge 1$.

A posteriori Error Estimators: Error Bounds

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s.t. $|||e(\mu)|||_{\mu} \leq \Delta_N^{\mathrm{en}}(\mu)$ or $\eta_N^{\mathrm{en}}(\mu) \geq 1$.

A posteriori Error Estimators: Error Bounds

Proofs: Again for $v = \hat{e}(\mu)$ in $a(e(\mu), v; \mu) = (\hat{e}(\mu), v)_X$ and the Cauchy-Schwarz inequality we have

$$\|\hat{e}(\mu)\|_X^2 \leq |||\hat{e}(\mu)|||_\mu \, |||e(\mu)|||_\mu \, . \qquad (2)$$

But from continuity $|||\hat{e}(\mu)|||_\mu \leq (\gamma^e(\mu))^{\frac{1}{2}} \|\hat{e}(\mu)\|_X$, and hence from (2)

$$\eta_N^{en} = \frac{\Delta_N^{en}(\mu)}{|||e(\mu)|||_\mu} = \frac{\alpha_{LB}^{\mathcal{N}}(\mu))^{-\frac{1}{2}} \|\hat{e}(\mu)\|_X}{|||e(\mu)|||_\mu} =$$

$$\frac{\alpha_{LB}^{\mathcal{N}}(\mu))^{-\frac{1}{2}} \|\hat{e}(\mu)\|_X^2}{|||e(\mu)|||_\mu \|\hat{e}(\mu)\|_X} \leq \frac{\alpha_{LB}^{\mathcal{N}}(\mu))^{-\frac{1}{2}} |||\hat{e}(\mu)|||_\mu \, |||e(\mu)|||_\mu}{|||e(\mu)|||_\mu \|\hat{e}(\mu)\|_X} \leq$$

$$(\alpha_{LB}^{\mathcal{N}}(\mu))^{-\frac{1}{2}} (\gamma^e(\mu))^{\frac{1}{2}}, \quad \text{or}$$

$$\eta_N^{en}(\mu) \leq \sqrt{\frac{\gamma^e(\mu)}{\alpha_{LB}^{\mathcal{N}}(\mu)}}$$

A posteriori Error Estimators: Error Bounds

Proofs: Again for $v = \hat{e}(\mu)$ in $a(e(\mu), v; \mu) = (\hat{e}(\mu), v)_X$ and the Cauchy-Schwarz inequality we have

$$\|\hat{e}(\mu)\|_X^2 \leq |||\hat{e}(\mu)|||_\mu \, |||e(\mu)|||_\mu \, . \tag{2}$$

But from continuity $|||\hat{e}(\mu)|||_\mu \leq (\gamma^{\mathrm{e}}(\mu))^{\frac{1}{2}} \, \|\hat{e}(\mu)\|_X$, and hence from (2)

$$\eta_N^{\mathrm{en}} = \frac{\Delta_N^{\mathrm{en}}(\mu)}{|||e(\mu)|||_\mu} \equiv \frac{\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu))^{-\frac{1}{2}} \|\hat{e}(\mu)\|_X}{|||e(\mu)|||_\mu} \equiv$$

$$\frac{\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu))^{-\frac{1}{2}} \|\hat{e}(\mu)\|_X^2}{|||e(\mu)|||_\mu \|\hat{e}(\mu)\|_X} \leq \frac{\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu))^{-\frac{1}{2}} |||\hat{e}(\mu)|||_\mu \, |||e(\mu)|||_\mu}{|||e(\mu)|||_\mu \|\hat{e}(\mu)\|_X} \leq$$

$$(\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu))^{-\frac{1}{2}} (\gamma^{\mathrm{e}}(\mu))^{\frac{1}{2}}, \quad \text{or}$$

$$\eta_N^{\mathrm{en}}(\mu) \leq \sqrt{\frac{\gamma^{\mathrm{e}}(\mu)}{\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu)}}$$

A posteriori Error Estimators: Error Bounds

Since $s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu) = |||e(\mu)|||_{\mu}^2$, and hence since $\Delta_N^s(\mu) = \left(\Delta_N^{\mathrm{en}}(\mu)\right)^2$

$$\eta_N^s(\mu) \equiv \frac{\Delta_N^s(\mu)}{s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu)} = \frac{\left(\Delta_N^{\mathrm{en}}(\mu)\right)^2}{|||e(\mu)|||_{\mu}^2} = \left(\eta_N^{\mathrm{en}}(\mu)\right)^2. \quad (3)$$

Offline-Online: $||\hat{e}(\mu)||_X$

Ingredients: 1. Affine Parameter Dependence

$$\begin{aligned} r(v;\mu) &\equiv f(v) - a(u_N(\mu), v; \mu) \\ &= f(v) - a\Big(\sum_{n=1}^{N} u_{Nn}(\mu)\, \zeta^n, v; \mu\Big) \\ &= f(v) - \sum_{n=1}^{N} u_{Nn}(\mu)\, a(\zeta^n, v; \mu) \\ &= f(v) - \sum_{n=1}^{N} u_{Nn}(\mu) \sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(\zeta^n, v)\ . \end{aligned}$$

Offline-Online: $\|\hat{e}(\mu)\|_X$

Ingredients: 2. Linear Superposition

$$
\begin{aligned}
(\hat{e}(\mu), v)_X &= f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\, u_{Nn}(\mu)\, a^q(\zeta^n, v), \\
\Rightarrow \qquad \hat{e}(\mu) &= \mathcal{C} + \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\, u_{Nn}(\mu)\, \mathcal{L}_n^q \, ,
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad (\mathcal{C}, v)_X &= f(v), && \forall\, v \in X^{\mathcal{N}}; \\
(\mathcal{L}_n^q, v)_X &= -a^q(\zeta^n, v), && \forall\, v \in X^{\mathcal{N}}, \\
& && 1 \le n \le N,\ 1 \le q \le Q.
\end{aligned}
$$

Offline-Online: $\|\hat{e}(\mu)\|_X$

Ingredients: 2. Linear Superposition

Thus $\|\hat{e}(\mu)\|_X^2$

$$= \left(\mathcal{C} + \sum_{q=1}^{Q}\sum_{n=1}^{N} \Theta^q(\mu)\, u_{Nn}(\mu)\, \mathcal{L}_n^q\,, \bullet\right)_X$$

$$= (\mathcal{C},\mathcal{C})_X + \sum_{q=1}^{Q}\sum_{n=1}^{N} \Theta^q(\mu)\, u_{Nn}(\mu) \Big\{$$

$$2(\mathcal{C},\mathcal{L}_n^q)_X + \sum_{q'=1}^{Q}\sum_{n'=1}^{N} \Theta^{q'}(\mu)\, u_{Nn'}(\mu)\, (\mathcal{L}_n^q,\mathcal{L}_{n'}^{q'})_X \Big\}.$$

Offline-Online: $\|\hat{e}(\mu)\|_X$

Computational Procedure

Offline: once, parameter independent

Compute $\mathcal{C},\ \mathcal{L}_n^q,\quad 1 \le n \le N_{\max},\ 1 \le q \le Q.$

Form/Store $(\mathcal{C},\mathcal{C})_X,\ (\mathcal{C},\mathcal{L}_n^q)_X,\ (\mathcal{L}_n^q,\mathcal{L}_{n'}^{q'})_X,$

$$1 \le n, n' \le N_{\max},$$
$$1 \le q, q' \le Q.$$

Complexity depends on N, Q, *and* $\mathcal{N}$.

Offline-Online: $\|\hat{e}(\mu)\|_X$

Computational Procedure

Online: many times, for each μ *deployed*

Evaluate $\|\hat{e}(\mu)\|_X^2 =$

$$\left[(\mathcal{C},\mathcal{C})_X + \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\, u_{Nn}(\mu) \Big\{ 2(\mathcal{C},\mathcal{L}_n^q)_X + \sum_{q'=1}^{Q} \sum_{n'=1}^{N} \Theta^{q'}(\mu)\, u_{Nn'}(\mu)\, (\mathcal{L}_n^q, \mathcal{L}_{n'}^{q'})_X \Big\} \right]$$

$$— O(Q^2 N^2).$$

Complexity depends on N, Q, *but not* $\mathcal{N}$.

A Posteriori Error Estimation: Numerical Example

TBlock-$(3,3)$: Metrics

Define

$$\Delta^s_{\max} = \max_{\mu \in \Xi_{\text{test}}} \Delta^s_N(\mu) \ ,$$

$$\eta^s_{N,\ {\text{ave} \atop \max}} = {\text{ave} \atop \max_{\mu \in \Xi_{\text{test}}}} \eta^s_N(\mu) \ ;$$

recall from *Proposition 2* $\mu_r = 100$

$$1 \leq \eta^s_{N,\max} \leq \max_{\mu \in \Xi_{\text{test}}} \frac{\gamma^{\mathrm{e}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)} \leq 100 \ .^\dagger$$

† Result for $\overline{\mu} = (1, \ldots, 1)$; improvement for "*multi*-inner product."

A Posteriori Error Estimation: Numerical Example

TBlock-$(3,3)$: Metrics

Effectivities †

N	$\Delta^s_{N,\max}$	$\eta^s_{N,\text{ave}}$	$\eta^s_{N,\max}$
10	$2.2036\text{E}+00$	6.7067	31.2850
20	$2.0020\text{E}-01$	7.5587	37.3024
30	$1.5100\text{E}-02$	12.1138	62.2537
40	$1.2000\text{E}-03$	14.4598	73.1151
50	$1.0000\text{E}-04$	10.2566	57.5113

† Note penalty for η^s_N "large" mitigated by rapid convergence $\Delta^s_N \to 0$.

Coercivity Lower Bound: Parametric Coercivity

If $\Theta^q(\mu) > 0$, $\forall \mu \in \mathcal{D}$ and $a^q(w,w) \geq 0$, $\forall w \in X^{\mathrm{e}}$, $1 \leq q \leq Q$, $a(\cdot,\cdot;\mu)$ is said to be *parametrically coercive*.

In this case:

$$a(w,w;\mu) = \sum_{q=1}^{Q} \Theta^q(\mu) a^q(w,w)$$

$$= \sum_{q=1}^{Q} \frac{\Theta^q(\mu)}{\Theta^q(\mu')} \Theta^q(\mu') a^q(w,w)$$

$$\geq \min_{q\in[1,Q]} \frac{\Theta^q(\mu)}{\Theta^q(\mu')} \sum_{q=1}^{Q} \Theta^q(\mu') a^q(w,w)$$

$$\geq \min_{q\in[1,Q]} \frac{\Theta^q(\mu)}{\Theta^q(\mu')} a(w,w;\mu') = \min_{q\in[1,Q]} \frac{\Theta^q(\mu)}{\Theta^q(\mu')} ||w||_X^2$$

so also

$$\alpha^{\mathcal{N}}(\mu) \equiv \inf_{w\in X^{\mathcal{N}}} \frac{a(w,w;\mu)}{||w||_X^2} \geq \min_{q\in[1,Q]} \frac{\Theta^q(\mu)}{\Theta^q(\mu')} = \alpha_{LB}^{\mathcal{N}}(\mu).$$

Problem Generalization: General Output

Consider $u \in X$

$$a(u, v; \mu) = f(v), \quad \forall v \in X,$$

and

$$s(\mu) = \ell(u(\mu))$$

If a is symmetric and $\ell = f$ we revert to compliant case. If not, with Primal only, we find $u_N \in W_n$ (Lagrange RB space)

$$a(u_N(\mu), v; \mu) = f(v), \quad \forall v \in W_N,$$

Then we need

$$s_N(\mu) = \ell(u_N(\mu)).$$

We can readily develop an a posteriori error bound for $s_N(\mu)$:

$$|s(\mu) - s_N(\mu)| \leq ||\ell||_{(X^{\mathcal{N}})'} \Delta_N(\mu)$$

where

$$||u(\mu) - u_N(\mu)||_X \leq \Delta_N(\mu) = \frac{||\hat{e}(\mu)||_X}{\alpha_{LB}(\mu)}.$$

Problem Generalization: General Output

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$$a(u, v; \mu) = f(v), \quad \forall v \in X,$$

and

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where

$$||u(\mu) - u_N(\mu)||_X \leq \Delta_N(\mu) = \frac{||\hat{e}(\mu)||_X}{\alpha_{LB}(\mu)}.$$

Problem Generalization: Output Error Bounds

Proof: First

$$\begin{aligned} a(e(\mu), v; \mu) =& r(v;\mu) = f(v) - a(u_N, v, \mu) \\ =& a(u, v, \mu) - a(u_N, v, \mu) = (\hat{e}(\mu), v)_X \end{aligned}$$

hence for $v = e \quad \alpha_{LB}(\mu)||e(\mu)||_X^2 \leq ||\hat{e}(\mu)||_X ||e||_X$, or

$$||e(\mu)||_X \leq \frac{||\hat{e}(\mu)||}{\alpha_{LB}(\mu)}.$$

But then

$$\begin{aligned} |s(\mu) - s_N(\mu)| =& |\ell(u(\mu)) - \ell(u_N(\mu))| = |\ell(e(\mu))| \\ =& \frac{|\ell(e(\mu))|}{||e(\mu)||_X} ||e(\mu)||_X \leq \underbrace{\left(\sup_{v \in X^{\mathcal{N}}} \frac{\ell(v)}{||v||_X} \right)}_{||\ell||_{(X^{\mathcal{N}})'}} ||e(\mu)||_X \\ \leq& \; ||\ell||_{(X^{\mathcal{N}})'} \Delta_N(\mu) = \Delta_N^{s,nc}(\mu) \end{aligned}$$

Problem Generalization: Primal-Dual

Dual Problem

Find $\Psi \in X$ such that

$$a(v, \Psi; \mu) = -\ell(v), \quad \forall v \in X.$$

Note we no longer assume that a is symmetric, and hence $\Psi \neq -u$ necessarily even if $\ell = f$.

(We still assume that a is coercive and affine, but this case considers transport-advection-convection terms.)

Problem Generalization: Primal-Dual

RB Approach Galerkin

Introduce

$$W^{pr}_{N_{pr}} = span\left\{u(\mu^k_{pr}) \equiv \zeta^k, 1 \le k \le N_{pr}\right\},$$
$$W^{du}_{N_{du}} = span\left\{\Psi(\mu^k_{du}), 1 \le k \le N_{du}\right\},$$
$$1 \le N_{pr} \le N_{pr,max}, \qquad 1 \le N_{du} \le N_{du,max}$$

Then $u_{N_{pr}} \in W^{pr}_{N_{pr}}, \Psi_{N_{du}} \in W^{du}_{N_{du}}$ satisfy

$$a(u_{N_{pr}}(\mu), v; \mu) = f(v), \quad \forall v \in W^{pr}_{N_{pr}},$$
$$a(v, \Psi_{N_{du}}(\mu); \mu) = -\ell(v), \quad \forall v \in W^{du}_{N_{du}},$$

Problem Generalization: Primal-Dual

And [Patera & Ronquist, Giles & Pierce]

$$s_{N_{pr},N_{du}}(\mu) = \ell(u_{N_{pr}}) - r^{pr}(\Psi_{N_{du}};\mu)$$

where

$$r^{pr}(v;\mu) = f(v) - a(u_{N_{pr}}, v;\mu)$$
$$r^{du}(v;\mu) = -\ell(v) - a(v, \Psi_{N_{du}};\mu)$$

Offline-Online is similar to Primal-only, but now we need to do everything both for Primal and Dual (see following Sampling).

Problem Generalization: Primal-Dual

A Priori Theory

It is standard that

$$|s - s_{N_{pr},N_{du}}| \leq$$

$$C \left(\inf_{w^{pr} \in W^{pr}_{N_{pr}}} ||u - w^{pr}||_X \right) \left(\inf_{w^{du} \in W^{du}_{N_{du}}} ||\Psi - w^{du}||_X \right)$$

Proof:

$$\begin{aligned} |s - s_{N_{pr},N_{du}}| &= \underbrace{\ell(u - u_{N_{pr}})}_{e^{pr}} + r^{pr}(\Psi_{N_{du}}; \mu) \\ &= -a(e^{pr}, \Psi; \mu) + a(e^{pr}, \Psi_{N_{du}}; \mu) \\ &= -a(e^{pr}, e^{du}; \mu). \end{aligned}$$

Then apply continuity and Galerkin optimality to Primal and Dual.

Problem Generalization: A Posteriori Output Bounds

We can readily derive that

$$|s^{\mathcal{N}} - s^{\mathcal{N}}_{N_{pr},N_{du}}| \leq \Delta_N^{s(,nc)} \qquad \text{“}N \equiv N_{pr}, N_{du}\text{”}$$

where

$$\Delta_N^s(\mu) = ||r_N^{du}(\cdot;\mu)||_{(X^{\mathcal{N}})'} \Delta_N(\mu).$$

Proof: we know that

$$\begin{aligned} s - s_{N_{pr},N_{du}} &= \ell(u - u_{N_{pr}}) + r^{pr}(\Psi_{N_{du}};\mu) \\ &= \ell(e^{pr}) + a(e^{pr}, \Psi_{N_{du}};\mu) = -r^{du}(e^{pr};\mu) \end{aligned}$$

So

$$\begin{aligned} |s - s_{N_{pr},N_{du}}| &\leq ||r^{du}(\cdot;\mu)||_{(X^{\mathcal{N}})'} ||e^{pr}||_X \\ &\leq ||r^{du}(\cdot;\mu)||_{(X^{\mathcal{N}})'} \Delta_N(\mu) \end{aligned}$$

where recall that $||e^{pr}||_X \leq \Delta_N(\mu)$.

Problem Generalization: A Posteriori Output Bounds

We can readily derive that

$$|s^{\mathcal{N}} - s^{\mathcal{N}}_{N_{pr},N_{du}}| \leq \Delta_N^{s(,nc)} \qquad \text{“}N \equiv N_{pr}, N_{du}\text{”}$$

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So

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where recall that $||e^{pr}||_X \leq \Delta_N(\mu)$.

Problem Generalization: Sampling

The Offline-Online procedure is very similar to before, but now we evaluate both a Primal and a Dual residual dual norm. We have

$$|s - s_{N_{pr},N_{du}}| \leq \left(\frac{||r^{du}(\cdot;\mu)||_{(X^{\mathcal{N}})'}}{\alpha_{LB}^{1/2}(\mu)} \right) \left(\frac{||r^{pr}(\cdot;\mu)||_{(X^{\mathcal{N}})'}}{\alpha_{LB}^{1/2}(\mu)} \right)$$

Hence if ϵ^s_{max} is the smallest output error desired, we perform a Primal greedy until ($\Rightarrow N_{pr,max}$)

$$\frac{||r^{pr}(\cdot;\mu)||_{(X^{\mathcal{N}})'}}{\alpha_{LB}^{1/2}(\mu)} \leq \sqrt{\epsilon^s_{max}}, \quad \text{over } \Xi^{pr}_{train}.$$

and a Dual greedy until ($\Rightarrow N_{du,max}$)

$$\frac{||r^{du}(\cdot;\mu)||_{(X^{\mathcal{N}})'}}{\alpha_{LB}^{1/2}(\mu)} \leq \sqrt{\epsilon^s_{max}}, \quad \text{over } \Xi^{du}_{train}.$$

If (say) the Dual converges much more quickly than the Primal, it would be more efficient to choose "$N_{pr,max} = 0$" and let the Dual do all the work.

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