Shape optimization (Lectures 1 & 2)

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Plan

- Lecture 1: Introduction to shape optimization. Homogenization method I
- Lecture 2: Homogenization method II. Algorithm and numerical issues

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- Lecture 3: Domain variation. Level set method I
- Lecture 4: Level set method II and applications

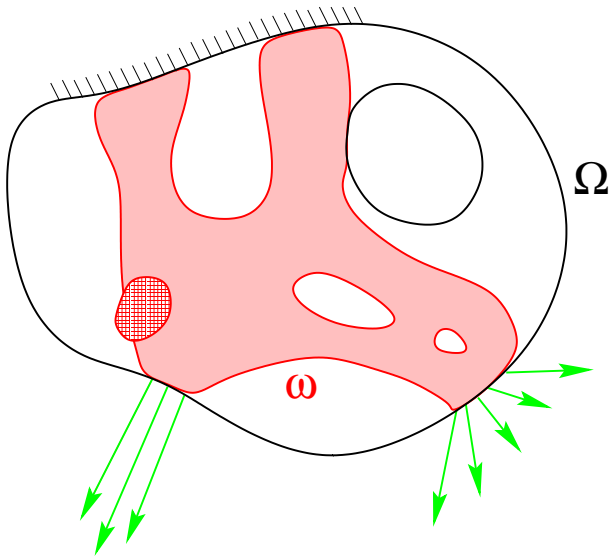
References

Allaire G., *Shape optimization by the homogenization method, Springer Verlag, New York* (2001).

Allaire G., *Conception optimale de structures, Springer Verlag, New York* (2007).

Papers available on `www.cmap.polytechnique.fr/~optopo/publi.html`

Model problem



Find a structure ω included into a fixed bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ such that it minimizes a cost function W

$$\inf_{\omega \subset \Omega, |\omega| \leq V} W(\omega)$$

Cost functions (objective functions)

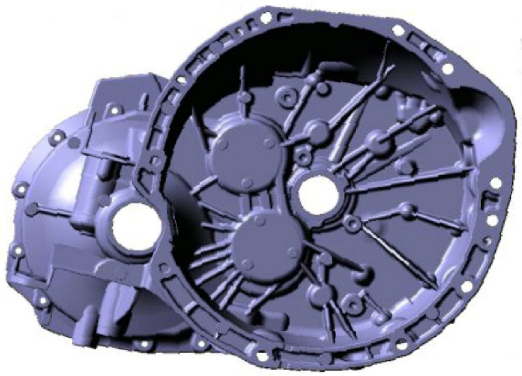
• Usually W dépends on ω through the resolution of a PDE (or a system of PDE's) posed on ω.

• Mathematicians use classical objective functions (sometimes because of theirs good properties rather than their relevance for applications).

• Real applications have sometimes "fuzzy" needs...

• Example of a popular objective function: compliance for elastic solids $\longrightarrow$ "find the most rigid structure for a given set of external forces and a given volume of constitutive material".

• Example of "fuzzy" needs: Optimize the structure for various solicitations with molding constraints, minimal size of small parts, minimal radii of curvature everywhere, cooling of the molded piece in less than a given time, etc...



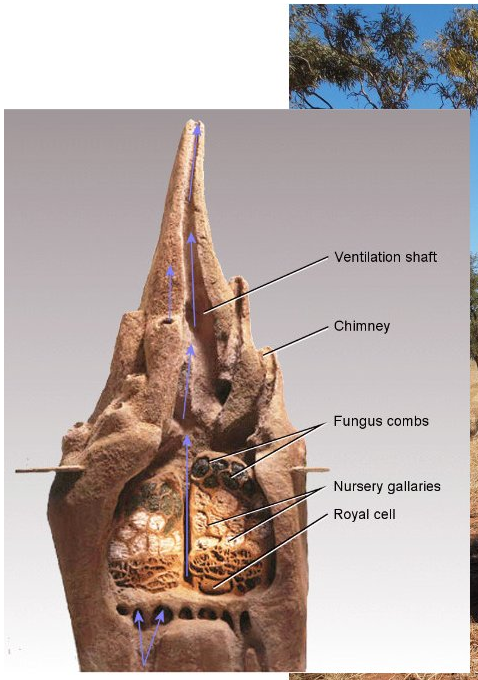
Shape representation

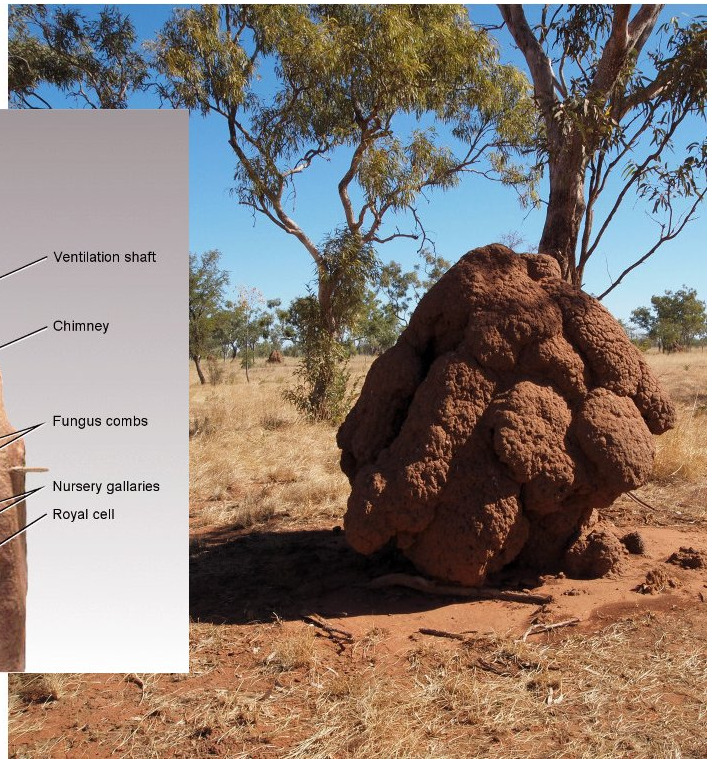
The problem can be considers in two different ways, according to the shape representation chosen:

• Reduce the number of parameters: parametrize the boundary with a (small) number of points (e.g. using splines) and optimize the control points with classical optimization algorithms
$\longrightarrow$ parametric optimization

• Search for ω in its full generality
$\longrightarrow$ topology optimization

Example of natural optimized structure (termite mounds)

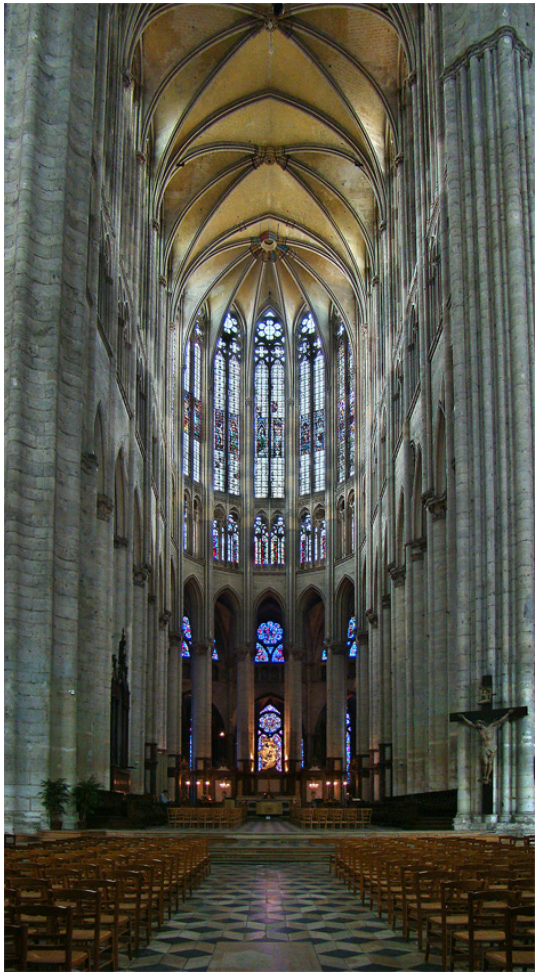




Shape optimization in history (trial-error 1)

Bent pyramid in Dashur (Saqqara, Egypt)

Shape optimization in history (trial-error 2)

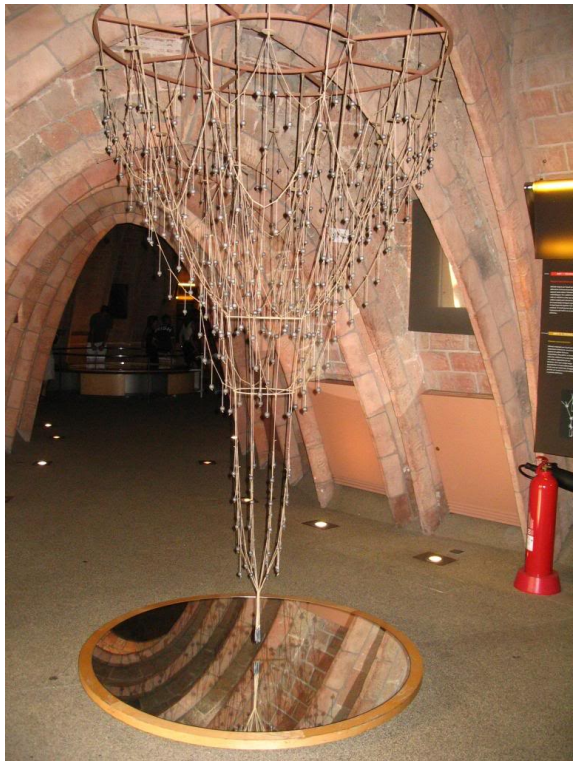


Beauvais Cathedral (France) (built between 1247 and 1569 !)

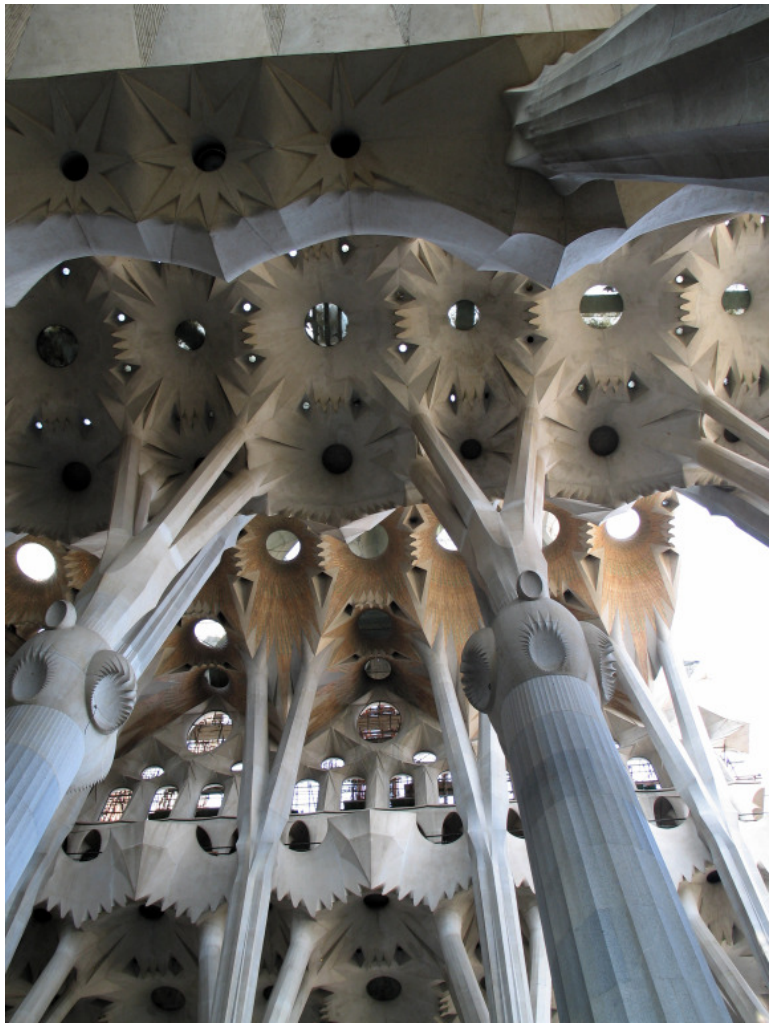
Bar trusses

Forth bridge (England) (1890)

Experimental shape optimization (Antoni Gaudì)



Experimental shape optimization (Antoni Gaudì)



Sagrada Familia (Barcelona, Spain) (1884-1926-20XX)

Observing natural shapes: a (more complex) related problem






What are the "natural" objective functions ?

Ill posed problem

- Not always existence of solutions (counter examples available)
- Local minima
- Numerical instabilities. No convergence under mesh refinement.

Model problem: optimal radiator (Tartar)

- Two conducting materials $0 < \alpha < \beta$
- Optimization domain: $\Omega =$ unit disc in $\mathbb{R}^2$
- Uniformly heated domain with prescribed temperature $u = 0$ on the boundary.
- $\chi(x) \in \{0, 1\}$ characterizes the good (and expensive) conductor

$\rightarrow$ conductivity at $x \in \Omega$: $A(x) = \chi(x)\beta + (1 - \chi(x))\alpha$

- For a given layout χ, the temperature u is solution of

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,(A(x)\nabla u(x)) & = & 1 & \text{in } \Omega \\ u(x) & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

Model problem: optimal radiator (Tartar)

- The most conducting material β is supposed to be more expensive
- Two interesting problems:

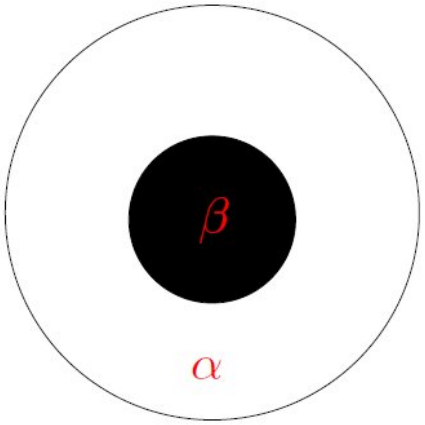
  $1/$ Find the cheapest configuration *maximizing* the mean temperature over Ω:

$$\max_{\chi} \Big(\int_\Omega u(x)dx - \ell \int_\Omega \chi(x)dx \Big), \quad \text{for a given } \ell > 0$$

  $2/$ Find the cheapest configuration *minimizing* the mean temperature over Ω:

$$\min_{\chi} \Big(\int_\Omega u(x)dx + \ell \int_\Omega \chi(x)dx \Big), \quad \text{for a given } \ell > 0$$

Model problem: optimal radiator (Tartar)



• Problem 1: easy computation in polar coordinates. explicit classical solution:

$0 < r < R_0$: material β,

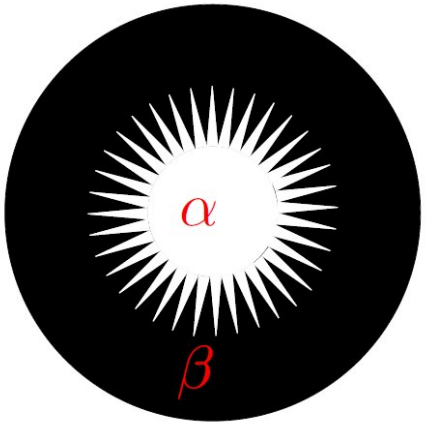
$R_0 < r < 1$: material α, with $R_0 = 2\alpha\sqrt{\dfrac{l}{\beta - \alpha}}$

• Problem 2: minimizing sequence converging to the minimum of the functional :

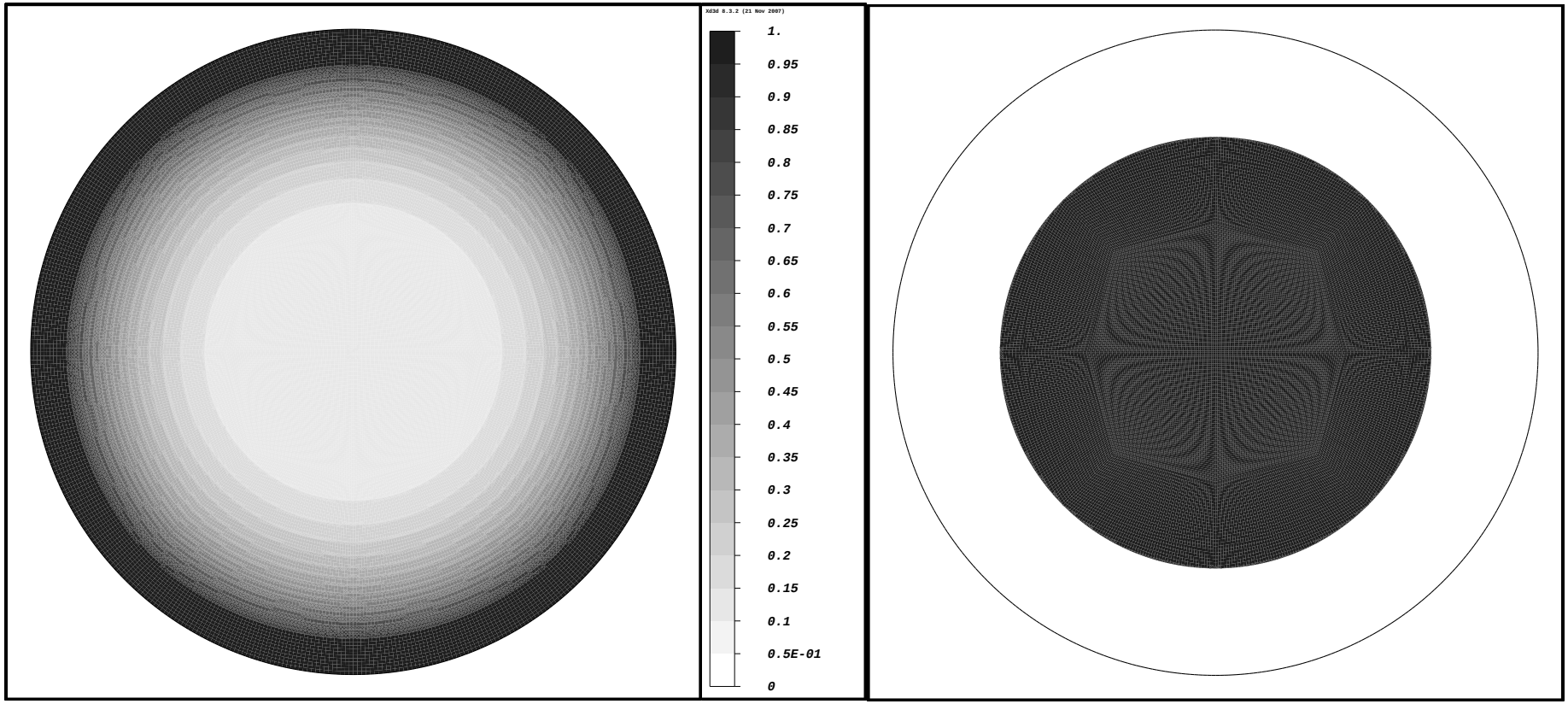
$0 < r < R_0$: material α,

$R_1 < r < 1$: material β, with $R_1 = 2\beta\sqrt{\dfrac{l}{\beta - \alpha}}$

$R_0 < r < R_1$: fine mixture of α and β



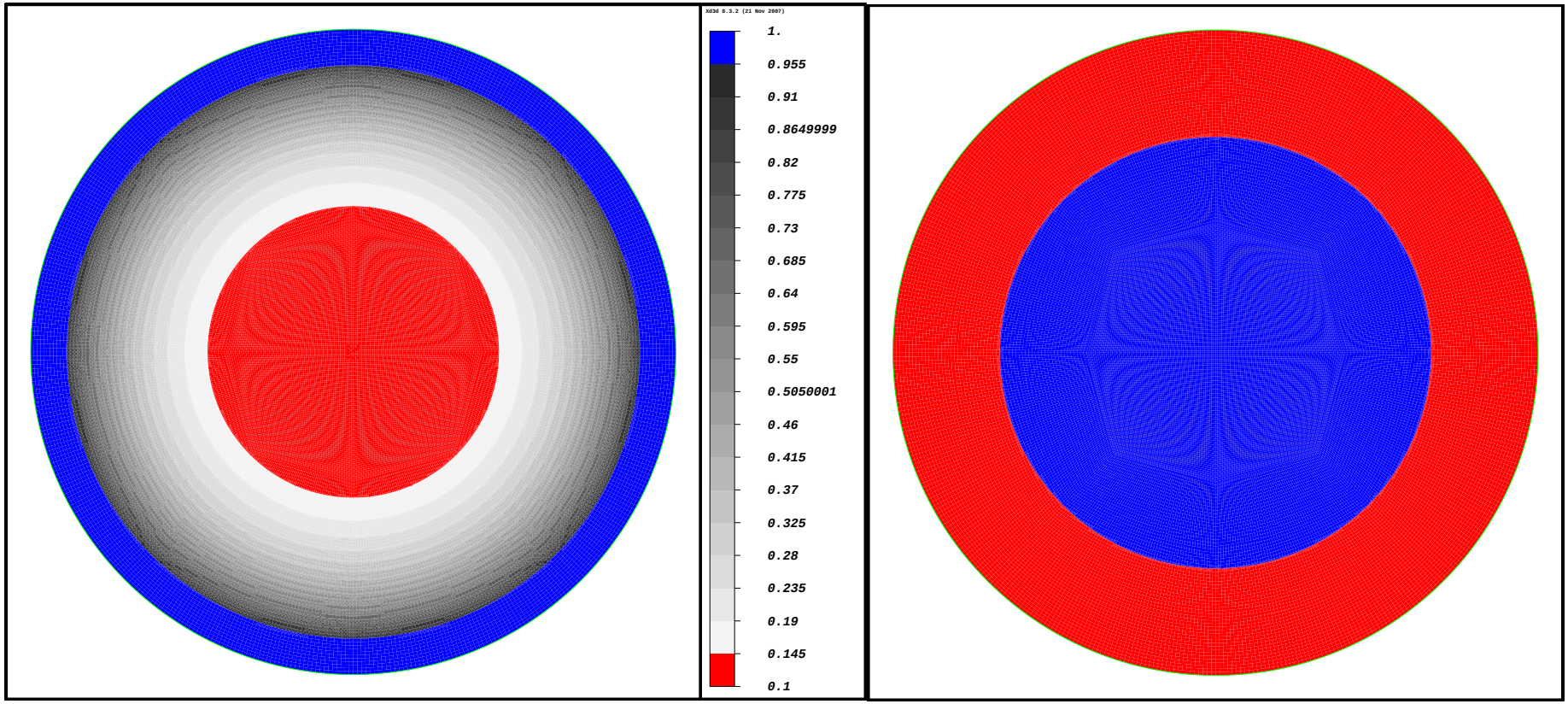
Optimal radiator: numerical solution using homogenization



Problem 2

Problem 1

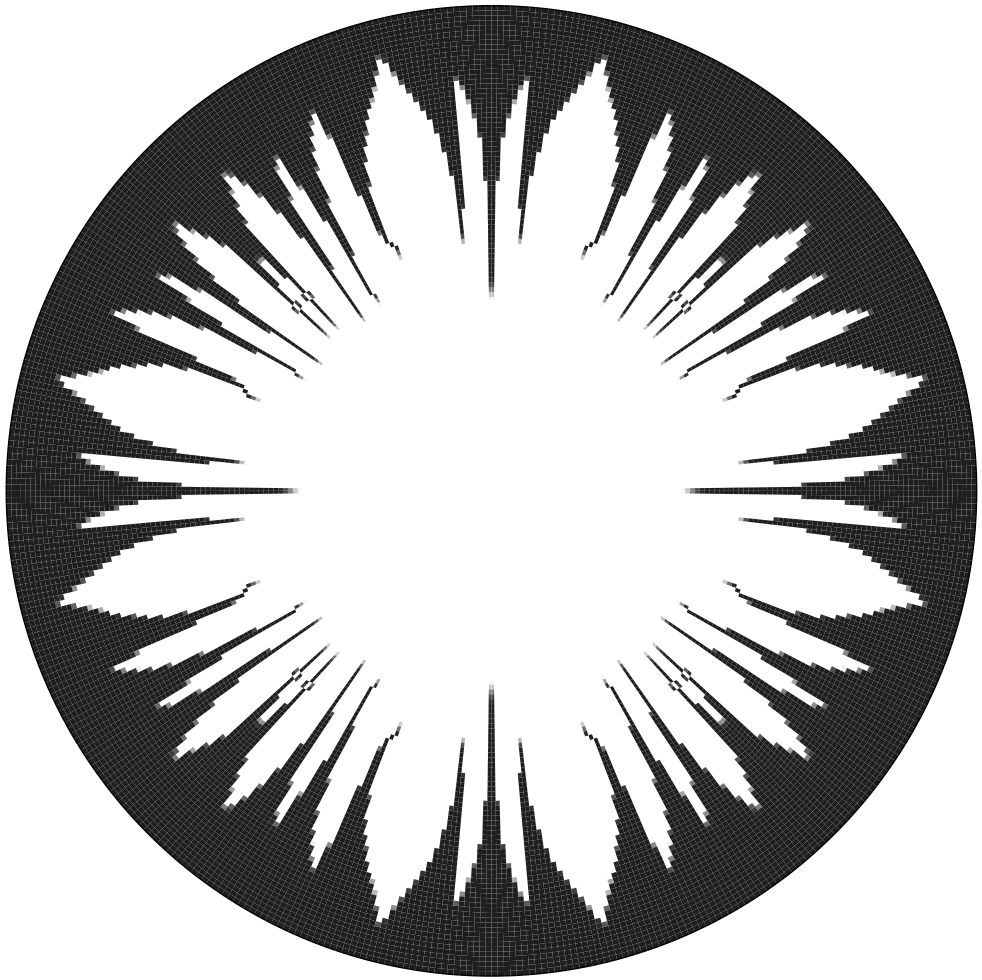
Optimal radiator: numerical solution using homogenization



Problem 2

Problem 1

Optimal radiator: numerical solution using homogenization (penalized design)



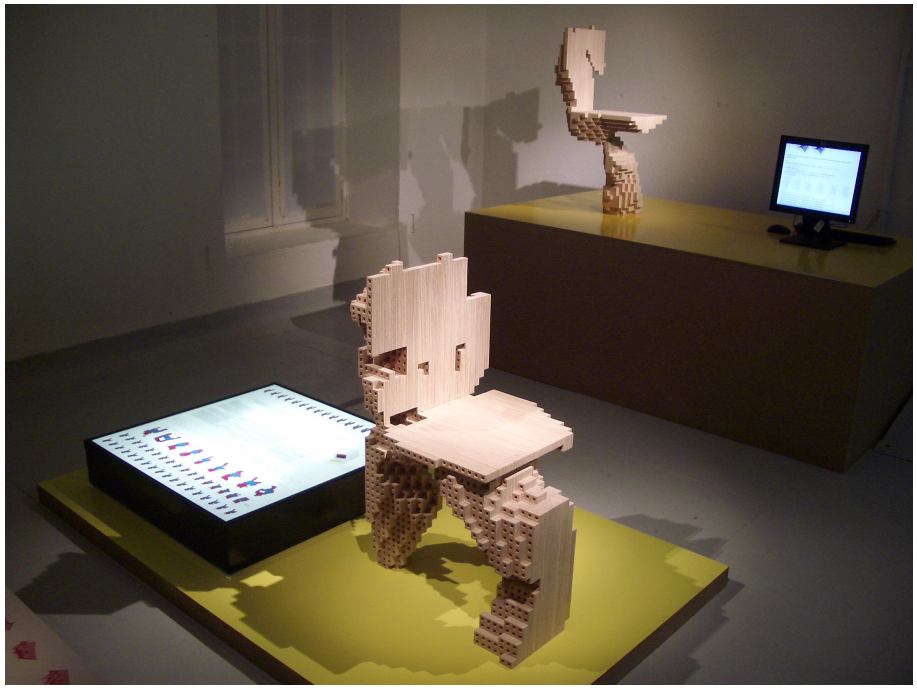
Ill posed problem

3 ways to change it into a well posed problem:

• Change the problem and enlarge the set of admissible solutions (shapes)
$\longrightarrow$ *relaxation* (homogenization method)

• Add constraints (e.g. smoothness of the boundaries, topology) or regularizing terms (perimeter)
$\longrightarrow$ *narrow the set of admissible solutions (e.g. level set method)*

• Work on finite dimension sets and do numerical computations
$\longrightarrow$ *evolutionary algorithms*



Part 1: homogenization method

Historical facts about the homogenization method

- First theoretical intuition: Tartar (early 80's)

- First "computer" simulations: Cherkaev (80's, published in russian)

- First real computer simulations: Bendsoe-Kikuchi (1988) (rectangular holes, then approximated homogenized material $\longrightarrow$ SIMP method).

Basic equations of linearized elasticity

Linear elastic and isotropic material. Hooke's law denoted A and characterized by Lamé coefficients λ, μ.

If u is the displacement field, $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the strain tensor, and the stress tensor

$$\sigma = Ae(u) = \lambda \mathrm{tr}(e(u))I + 2\mu e(u)$$

On ω, the elasticity system is:

$$\begin{cases} -\mathrm{div}\sigma &= 0 & \text{in } \omega \\ \sigma \cdot n &= f & \text{on } \partial\omega_N \\ u &= u_0 & \text{on } \partial\omega_D \end{cases}$$

It admits a unique solution if $meas(\omega_D) > 0$ or if the external forces are well balanced, i.e.

$$\int_{\partial\omega_N} f ds = 0$$

Basic equations of linearized elasticity - fictitious material

ω is characterized by a *characteristic function* $\chi : \Omega \longrightarrow \{0, 1\}$ such that

$$\omega = \{x \in \Omega \, ; \chi(x) = 1\}$$

The system of elasticity can be written on the whole Ω:

$$\left\{ \begin{array}{rcll} -\mathrm{div}\chi(x) Ae(u) & = & 0 & \text{in } \Omega \\ \sigma \cdot n & = & f & \text{on } \partial\Omega \\ \sigma \cdot n & = & 0 & \text{on } \partial\omega \setminus \partial\Omega \\ \sigma & = & 0 & \text{in } \Omega \setminus \omega \end{array} \right.$$

with a natural (zero traction) on the boundaries of holes of ω

Remark: in numerical computations of topology optimization, "void" is often replaced by a "soft" material in order to get a well posed problem over Ω. (fictitious material)

Various objective functions

- **Compliance :**

$$W(\omega) = \int_\Omega \chi(x) Ae(u) : e(u) dx = \int_\Omega A^{-1}\sigma : \sigma dx = \int_{\partial\Omega} f \cdot u \, ds = c(\omega)$$

Global measurement of the rigidity.
The most popular in topology optimization of structures (for practical and mathematical reasons).
Achieved theory and powerful numerical methods.

- **Cost function depending on the displacement:** example

$$W(\omega) = \left\{ \int_\omega C(x)|u(x) - u_0(x)|^\alpha dx \right\}^{1/\alpha}$$

where u_0 is a given *target displacement* and $\alpha \geq 2$ a constant
$\longrightarrow$ *compliant micromechanisms optimization (MEMS)*

- **Cost function depending on the stresses:** example $W(\omega) = \int_\Omega |\sigma|^2$

Other objective functions

- **Weighted sum of compliances :**

$$W(\omega) = \sum_i C_i \int_\Omega \chi(x) A e(u_i) : e(u_i) dx = \sum_i C_i \int_\Omega A^{-1}\sigma_i : \sigma_i dx$$

$$= \sum_i C_i \int_{\partial\Omega} f_i \cdot u_i ds$$

Find the best compromise for a structure *successively* submitted to several loadings f_i. Displacement fields u_i are solutions of the elasticity system for the loading case f_i.

- **Eigenfrequencies :** The relevant problem is to maximize the smallest eigenvalues of the structure. Using Rayleigh quotient, the problem is similar to the compliance case.

Compliance case

To get rid of the volume constraint, we introduce a positive Lagrange multiplier ℓ:

$$\inf_{\omega \subset \Omega} \{c(\omega) + \ell|\omega|\}$$

Sum of two contradictory terms:
Find the best compromise between rigidity and weight

Complementary energy principle:

$$c(\omega) = \int_{\partial\Omega} f \cdot u \, ds = \min_{\begin{cases} -\mathrm{div}\tau = 0 & \text{in } \omega \\ \tau \cdot n = f & \text{on } \partial\omega \cap \partial\Omega \\ \tau \cdot n = 0 & \text{on } \partial\omega \setminus \partial\Omega \end{cases}} \int_{\omega} A^{-1}\tau \cdot \tau$$

Compliance case

If τ is extended by 0 inside holes:

$$c(\omega) = \int_{\partial\Omega} f \cdot u ds = \min_{\left\{ \begin{array}{ll} -\mathrm{div}\tau = 0 & \text{in } \Omega \\ \tau \cdot n = f & \text{on } \partial\Omega \end{array} \right.} \int_\Omega (\chi(x)A)^{-1}\tau \cdot \tau$$

Minimize $c(\omega)$ over all $\omega \subset \Omega$ $\quad \rightarrow$ double minimization. Switch the two minimization:

$$\inf_{\omega \subset \Omega} \{c(\omega) + \ell|\omega|\} = \inf_{\left\{ \begin{array}{ll} -\mathrm{div}\tau = 0 & \text{in } \Omega \\ \tau \cdot n = f & \text{on } \partial\Omega \end{array} \right.} \int_\Omega F(\tau)$$

where $F(\tau) = \left\{ \begin{array}{ll} A^{-1}\tau \cdot \tau + \ell & \text{if } \tau \neq 0 \\ 0 & \text{if } \tau = 0 \end{array} \right.$

Ill posed problem (again)

• Mathematical hints: The minimizing sequences of characteristic functions can converge to something else than a characteristic function.

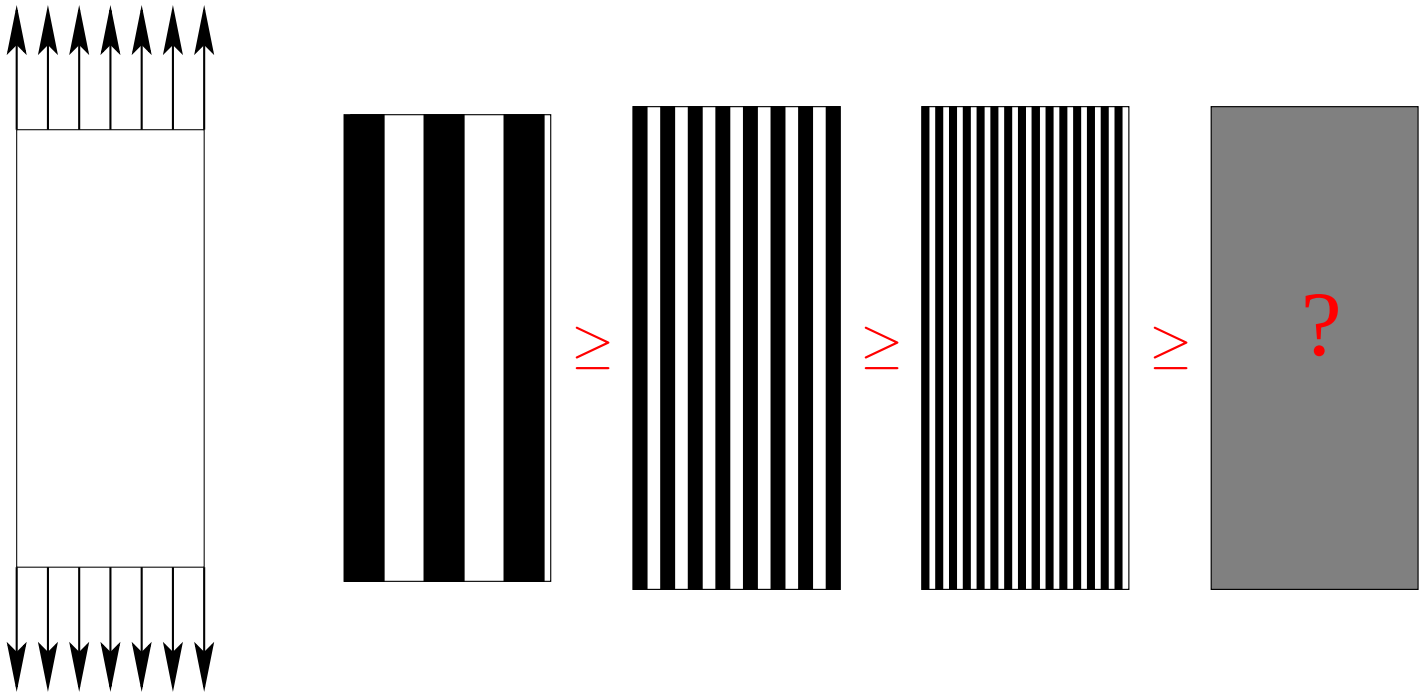
• Mechanically: Many small holes may be more effective than a big one having the same volume. Composite material having micro-holes may be solutions.

• Remedies:

- *Fight:* reduce the space of admissible shape to be sure that the computed solution belong to it: add constraints and look for classical solutions (cf. part 2: level set).

- *Give up:* admit there is a problem and allow the weird things: enlarge the space of admissible solution to allow composite materials with micro-holes $\longrightarrow$ homogenization method.

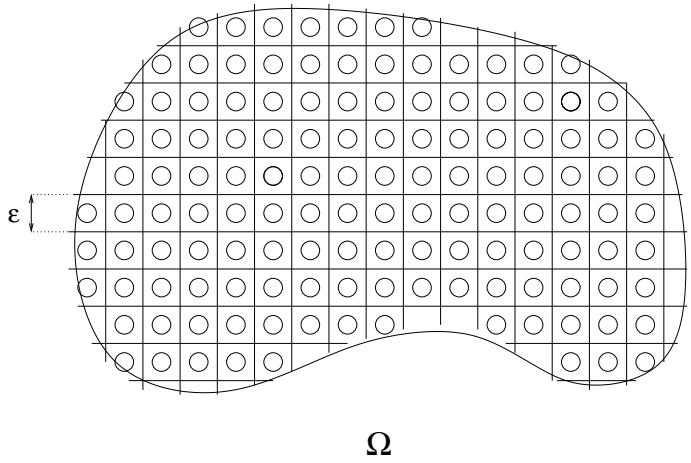
Example



Homogenization

Rigorous way of computing the *effective properties* (macroscopic) of heterogeneous and composite media.

For an heterogeneous medium where (small) ε is the typical size of small details, the direct computation is too expensive. We look for an *equivalent material*, at the macroscopic scale, that will be the limit material when $\varepsilon \longrightarrow 0$.



1d example

Consider $a > 0$ a 1-periodic function and the following problem:

$$\left\{ \begin{array}{rcl} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(a\left(\dfrac{x}{\varepsilon}\right)\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) & = & f \quad \text{in }]\alpha,\beta[\\ u(\alpha) = u(\beta) & = & 0 \end{array} \right.$$

The *limit problem* as $\varepsilon \longrightarrow 0$ writes:

$$\left\{ \begin{array}{rcl} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(a^*\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) & = & f \quad \text{in }]\alpha,\beta[\\ u(\alpha) = u(\beta) & = & 0 \end{array} \right.$$

where

$$a^* = \frac{1}{\dfrac{1}{\beta-\alpha}\displaystyle\int_\alpha^\beta \frac{dy}{a(y)}}$$

Harmonic mean value: *not so intuitive!*

Homogenization - Periodic case

If the microstructure is periodic, the homogenized coefficients can be computed solving PDEs posed on the periodicity cell. In practice this computation is not explicit except in some particular cases.

Two steps :

• Formal asymptotic expansion of the solution leading to the cell problems.

• Proof of the convergence to the homogenized equation (Tartar energy method) as $\varepsilon \longrightarrow 0$ and estimation of the error between the exact solution and the homogenized one for a given value of ε.

Homogenization - Periodic case

Consider a $Y-$periodic Hooke's law $A(y)$ and the following elasticity problem with oscillating coefficients:

$$\begin{cases} -\text{div}\ \left(A\left(\frac{x}{\varepsilon}\right) e(u_\varepsilon)\right) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

Ansazt for the two-scales asymptotic expansion:

$$u_\varepsilon(x) = \sum_{i=0}^{+\infty} u_i\left(x, \frac{x}{\varepsilon}\right),$$

where $u_i(x,y)$ is a function of both variables x and y, periodic in y with period $Y = (0,1)^N$. The following derivation rule is used

$$\nabla\left(u_i\left(x, \frac{x}{\varepsilon}\right)\right) = \left(\varepsilon^{-1}\nabla_y u_i + \nabla_x u_i\right)\left(x, \frac{x}{\varepsilon}\right),$$

where ∇_y and ∇_x denote the partial derivative with respect to the first and second variable of $u_i(x,y)$.

Homogenization - Periodic case

This series is plugged in the equation

$$f(x) = -\varepsilon^{-2}\left[\mathrm{div_y}\ (Ae_y(u_0))\right]\left(x, \frac{x}{\varepsilon}\right)$$

$$-\varepsilon^{-1}\left[\mathrm{div_y}\ (A(e_x(u_0) + e_y(u_1))) + \mathrm{div_x}\ (Ae_y(u_0))\right]\left(x, \frac{x}{\varepsilon}\right)$$

$$-\sum_{i=0}^{+\infty}\varepsilon^i\left[\mathrm{div_x}\ (A(e_x(u_i) + e_y(u_{i+1}))) + \mathrm{div_y}\ (A(e_x(u_{i+1}) + e_y(u_{i+2})))\right]\left(x, \frac{x}{\varepsilon}\right)$$

Identifying each coefficient as an individual equation yields a cascade of equations that lead to the homogenized equation

$$-\mathrm{div_x}\ (A^* e_x(u(x))) = f(x) \text{ in } \Omega,$$

Homogenization - Periodic case

The homogenized Hooke's law A^* is given by

$$A^*_{ijkl} = \int_Y \left(A(y) e_y(w_{ij})_{kl} + A_{ijkl}(y) \right) dy$$

where the fields w_{ij} are solutions of the cell problems

$$\begin{cases} -\mathrm{div_y} \ \left(A(y) \left(e_{ij} + e_y(w_{ij}(y)) \right) \right) = 0 & \text{in } Y \\ y \to w_{ij}(y) & Y\text{-periodic,} \end{cases}$$

where e_{ij} are the elements of the canonical basis of the space of symmetric matrices in dimension d.

Homogenization - Non periodic case

In non periodic settings there is no explicit characterization of the homogenized Hooke's law but the limit problem can be established as well using the H-convergence (or G-convergence) theory.

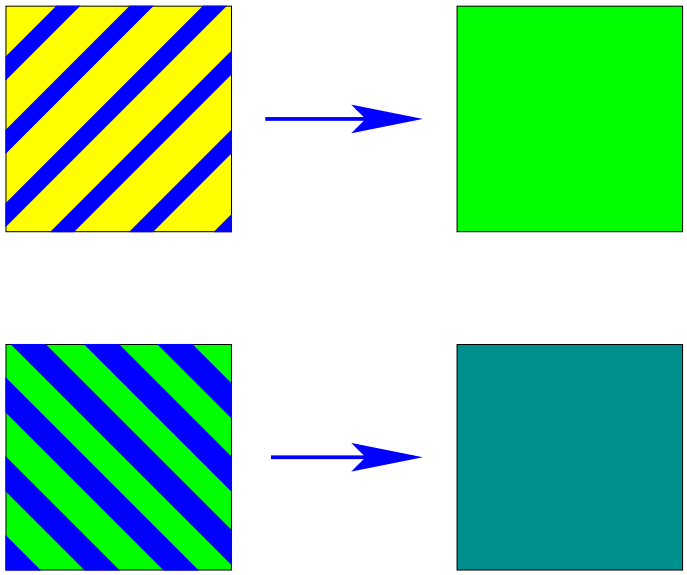
Moreover, if

$$G_\theta = \left\{ \begin{array}{l} \text{all Hooke's laws that can be built by homogenization} \\ \text{mixing 2 materials } A_1 \text{ and } A_2 \text{ in proportions } \theta \text{ and } (1-\theta) \end{array} \right\},$$

some bounds of the energy over G_θ can be computed and are achieved by a particular, explicit, class of periodic materials: the sequential laminates.

Sequential laminates

- Particular composites built sequentially
- Described by a small number of parameters: global proportion of each constitutive material, lamination rank, lamination directions and lamination proportions along each direction.
- *Explicit formulas* for A^* function of these parameters



Optimal bounds of the energy

• If $d = 2$, consider given 2 symmetric matrix τ whose eigenvalues are τ_1 and τ_2 ("principal stresses"). Let A be a given isotropic Hooke's law characterized by its bulk modulus κ and its shear modulus μ, and $\theta \in [0, 1]$ a given constant. If G_θ is the set of all materials that can be built in mixing A and void in proportions θ and $(1 - \theta)$ then

$$\min_{A^* \in G_\theta} A^{*-1}\tau \cdot \tau = A^{-1}\tau \cdot \tau + \frac{(\kappa + \mu)(1 - \theta)}{4\kappa\mu\theta}(|\tau_1| + |\tau_2|)^2$$

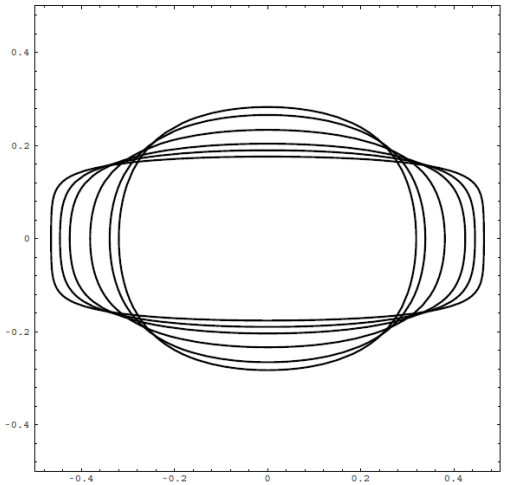
This minimum is attained for a rank 2 laminate whose lamination directions are the eigenvectors of τ. The lamination proportions are $\frac{|\tau_1|}{|\tau_1|+|\tau_2|}$ and $\frac{|\tau_2|}{|\tau_1|+|\tau_2|}$.

• Easy optimization over $\theta \in [0, 1]$: $\theta_{optimal} = \min\left\{1, (|\tau_1| + |\tau_2|)\sqrt{\frac{(\kappa + \mu)}{4\kappa\mu\ell}}\right\}$

• If $d = 3$ there is an analogous result, although involving more complicated formulas. The energy bounds are attained for a rank 3 laminate (or rank 1 or 2 in some particular cases) and the lamination directions are aligned with the principal directions of τ.

Remarks

• Non uniqueness of the optimal microstructures: there exist other, less usable, optimal microstructures: confocal ellipsoids (2d, 3d), Vidgergauz inclusions (2d).



• *Non uniqueness of the optimal laminate:* example if $\tau = pI$ (hydrostatic pressure), the lamination directions can be any directions. For a given τ in 2d, it is also possible to find rank $2n$ optimal laminates, for any n.

Relaxation

Definition: enlarge *just what is necessary* the set of admissible shapes and change the less possible the cost function so that the modified problem has a solution in the new set of admissible solutions.

Initial formulation of the shape optimization problem:

$$\inf_{\chi \in L^\infty(\Omega;\{0,1\})} J(\chi) = \int_\Omega (\chi(x)A)^{-1}\tau \cdot \tau dx + \ell \int_\Omega \chi(x)dx,$$

where τ is solution of

$$\left\{ \begin{array}{rcll} -\mathrm{div}\tau & = & 0 & \text{in } \Omega \\ \tau \cdot n & = & f & \text{on } \partial\Omega, \end{array} \right.$$

relaxed formulation:

$$\min_{(\theta,A^*)\in \mathcal{CD}} J^*(\theta, A^*) = \int_\Omega (A^*)^{-1}\tau \cdot \tau dx + \ell \int_\Omega \theta(x)dx,$$

where τ is solution of

$$\left\{ \begin{array}{rcll} -\mathrm{div}\tau & = & 0 & \text{in } \Omega \\ \tau \cdot n & = & f & \text{on } \partial\Omega, \end{array} \right.$$

and $\mathcal{CD}$ denotes the set of the generalized shapes that include composite materials.

Relaxation

The new set of admissible shapes is

$$\mathcal{CD} = \left\{ \theta \in L^\infty(\Omega; [0,1]),\ A^*(x) \in G_{\theta(x)},\ \forall x \in \Omega \right\},$$

where for each given $\theta(x) \in [0,1]$, G_θ is the set of all *homogenized Hooke's laws* obtained by mixing A and void in respective proportions θ and $(1-\theta)$.

G_θ is unknown for elasticity !

But: it can be proved that G_θ is the set of all the Hooke's law obtained by mixing A and void *in a periodic way* with proportions θ and $(1-\theta)$.

Relaxation does not change (too much) the problem

$$\inf_{\chi \in L^\infty(\Omega;\{0,1\})} J(\chi) = \min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*)$$

• All the homogenized solutions can be attained as limits of minimizing sequences of classical designs.

• If the initial problem admits a classical solution, it is solution of the relaxed problem as well.

Miracle !

The relaxed formulation is expressed as a minimization problem over a set that is partially unknown.

But the 2 minimizations, over the designs (θ, A^*) and the admissible stresses τ can be exchanged. Moreover, thanks to the local character of the $G - closure$, the minimization over the admissible designs can be put under the integral:

$$\min_{\substack{-\mathrm{div}\tau=0 \quad \text{in } \Omega \\ \tau\cdot n=f \quad \text{on } \partial\Omega}} \int_\Omega \min_{\substack{A^*\in G_\theta \\ 0\leq\theta\leq 1}} (A^{*-1}\tau\cdot\tau + \ell\theta)$$

Then, for a given τ, minimizing $A^{*-1}\tau\cdot\tau$ over G_θ amounts to compute the bound of the energy $\longrightarrow$ explicit solution using sequential laminates.

Numerical algorithm

$$\min_{\begin{cases} -\mathrm{div}\tau = 0 & \text{in } \Omega \\ \tau \cdot n = f & \text{on } \partial\Omega \end{cases}} \int_\Omega \min_{\substack{A^* \in G_\theta \\ 0 \le \theta \le 1}} (A^{*-1}\tau \cdot \tau + \ell\theta)$$

• First minimization: *solve an elasticity problem for a given* $A^*(x)$ $\longrightarrow$ classical FEM (or any efficient method).

• Second minimization: *minimization over the shape parameters* (θ, A^*) *for given* $\tau(x)$ $\longrightarrow$ explicit optimality conditions and optimal composites

Algorithm (alternate directions like):

1. Initialization of the design parameters (θ_0, A_0^*) (e.g. $\theta_0 = 1$ and $A_0^* = A$).

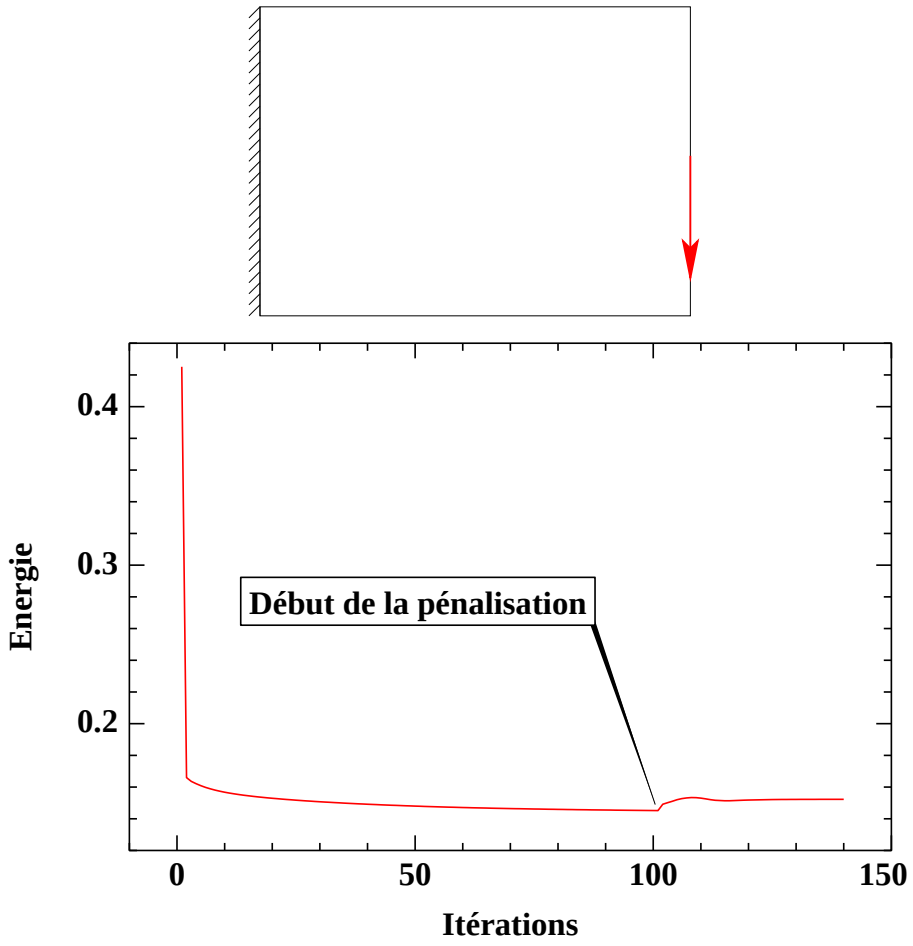
2. Itérations until convergence:

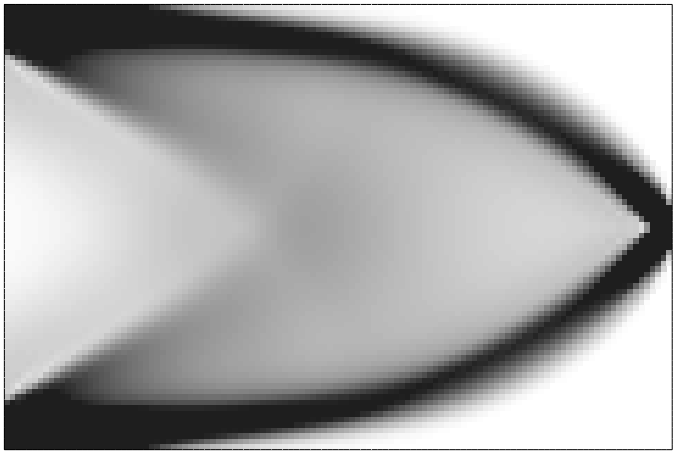
   (a) Compute τ_n solving elasticity for the design variables $(\theta_{n-1}, A_{n-1}^*)$.
   (b) Update the design variable (θ_n, A_n^*) using the explicit optimal composites for the stress tensor τ_n.

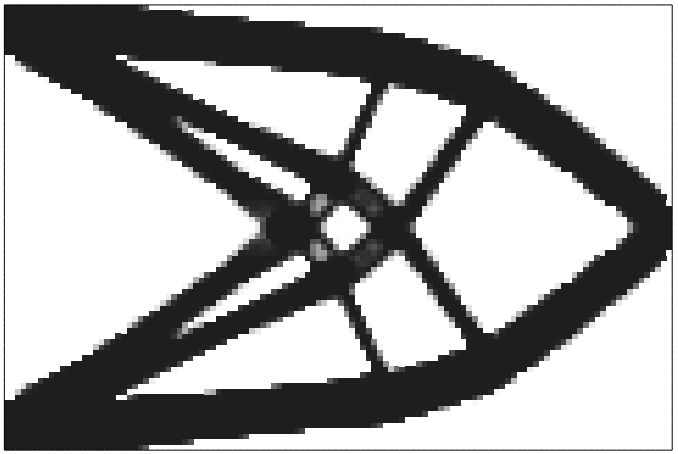
Remarks

- The energy decreases at each iteration
- Not sensitive to initial guess and to mesh refinement
- Convergence $\Leftrightarrow$ energy and shape parameters are stationary
- The weight parameter ℓ can be adjusted numerically to satisfy a volume constraint
- "void" is replaced by a "weak" material in order to have a valid system on all Ω.
- In 2d, rank 2 laminates are degenerate: their Hooke's law has to be regularized
- In 2d, with square elements, some "checkerboard instabilities" can appear (no clear explanation). They are easily filtered...
- After convergence to a generalized solution (with composites) it is possible to penalize the composites to find a classical solution (mesh dependent)
- Displacement u is discretized by the FEM, and the shape parameters (θ, A^*) are taken constant in each cell

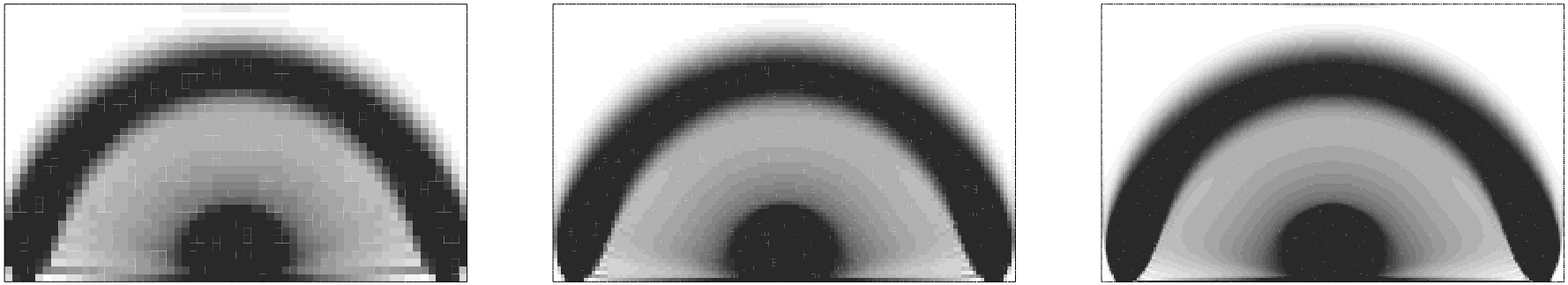
Cantilever


0.4
0.3
0.2
Energie
Début de la pénalisation
0
50
100
150
Itérations

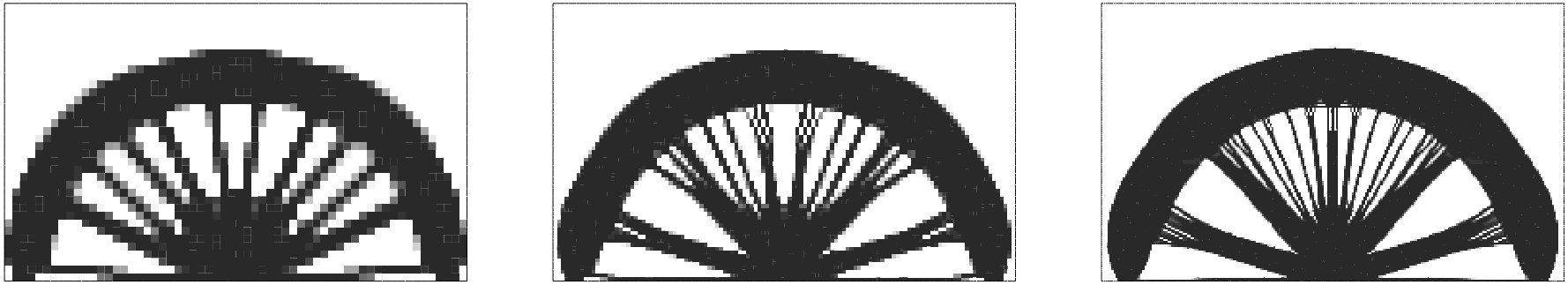




Stable by mesh refinement



Composite designs



Penalized designs

Other methods

• Rectangular holes + rotation in each cell. Homogenization computed numerically and tabulated (Bendsoe-Kikuchi 88).

Drawbacks:

- No explicit properties of the composites
- Suboptimal composites $\longrightarrow$ less efficient

• Isotropic composites. Consider only the isotropic composite in G_θ. The energy bound over this subset is attained by rank 3 laminates in 2d and rank 6 laminates in 3d with given direction:

Advantages: only 1 remaining shape parameter (θ) $\longrightarrow$ simpler

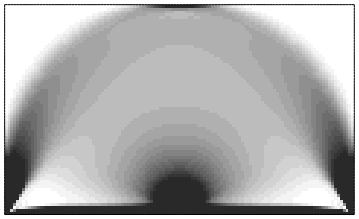
Drawbacks: suboptimal composites

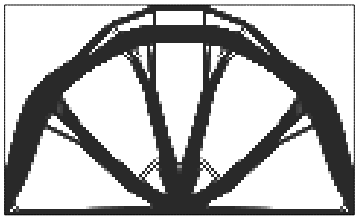
Other methods

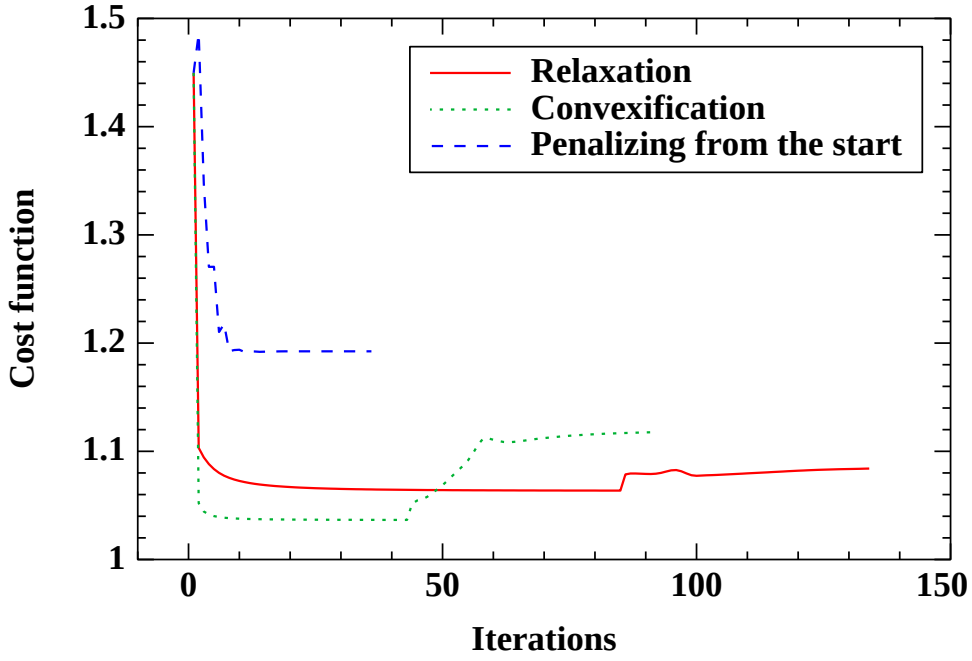
- Convexification:

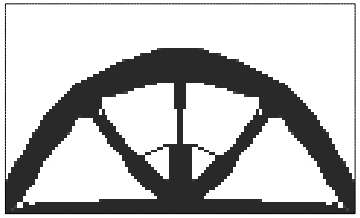
$$\min_{\tau} \int_{\Omega} \min_{0\le\theta\le 1} ((\theta A)^{-1}\tau\cdot\tau + \ell\theta)$$

$\longrightarrow$ explicit and simple optimization of θ (the only shape parameter).
Some admissible shapes lie outside of G_θ. The local information of the lamination direction is lost $\longrightarrow$ *poorer solutions*.









Variant:
$\min_{0\le\theta\le 1} ((\theta^p A)^{-1}\tau\cdot\tau + \ell\theta)$, where p is a (quite) secret parameter $\longrightarrow$ SIMP method

Multiple loads

$$W(\omega) = \sum_i \int_\Omega A^{-1}\sigma_i : \sigma_i dx = \sum_i \int_{\partial\Omega} f_i \cdot u_i ds$$

f_i are different loadings applied successively to the structure.

Relaxed problem can be written in the same way:

$$\min_{(\theta, A^*)\in\mathcal{CD}} J^*(\theta, A^*) = \int_\Omega \sum_i \left((A^*)^{-1}\tau_i \cdot \tau_i\right) dx + \ell \int_\Omega \theta(x) dx,$$

where $\forall i, \tau_i$ is solution of

$$\left\{ \begin{array}{rcll} -\mathrm{div}\tau_i & = & 0 & \text{in } \Omega \\ \tau_i \cdot n & = & f_i & \text{on } \partial\Omega, \end{array} \right.$$

It can be proved (Francfort-Tartar) that there exists a rank 3 laminate in 2d and a rank 6 laminate in 3d that attains the bound on a sum of energies. But these laminate cannot computed explicitly like in the single load case.

Multiple loads

The *explicit* phase of optimization over the shape parameters is replaced by a *numerical* optimization:

• We look for a quasi-optimal composite in the class of the sequential laminates

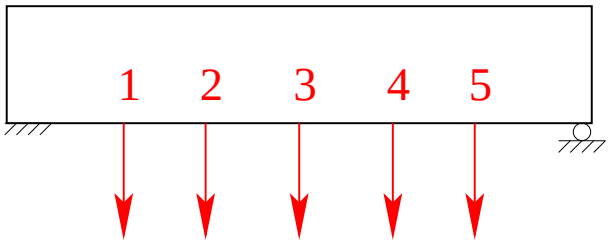
• The number of lamination directions q is fixed as well as the directions: typically $q = 12$ directions in 2d and $q = 100$ in 3d are enough to cover the whole half space

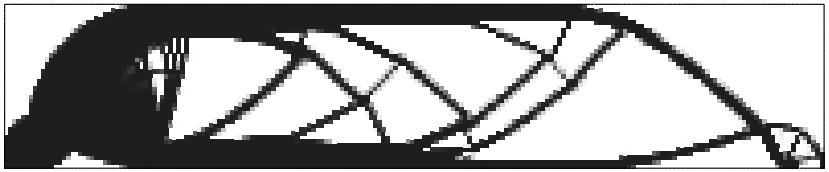
• The remaining parameters θ (global proportion) and $(m_i)_{1 \leq i \leq q}$ are optimized in each cell of the mesh

• The optimization of m_i is done by a gradient method. It is efficient since the problem is convex in m_i for fixed θ and lamination directions

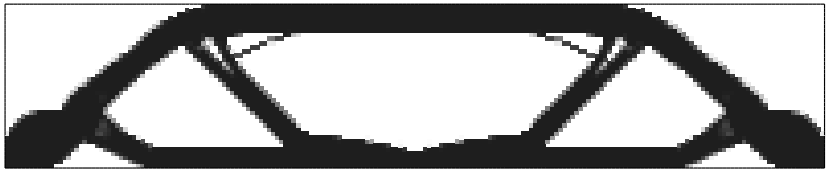
• The optimization of θ remains explicit

Multiple loads example

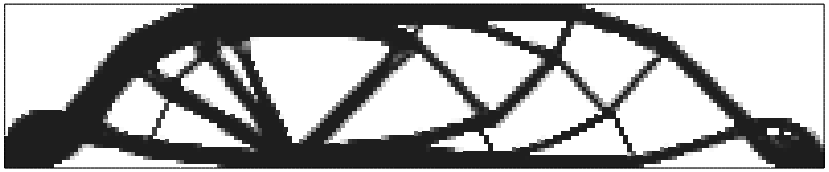




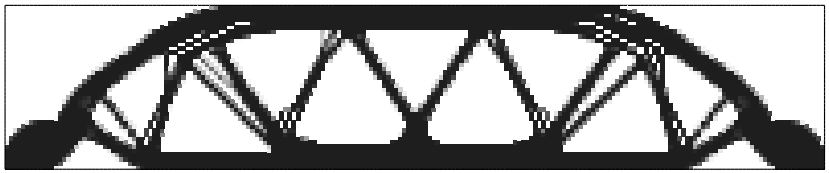
Solution for loading 1



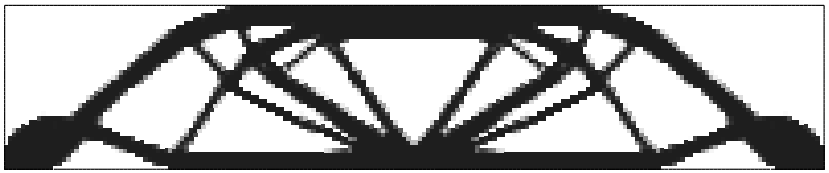
Solution for the 5 loadings together



Solution for loading 2



Multi-loadings solution



Solution for loading 3

Another cost function depending on the displacement

$$W(\omega) = \left\{ \int_\omega C(x)|u(x) - u_0(x)|^\alpha dx \right\}^{1/\alpha}$$

where u_0 is a given target displacement, $\alpha \geq 2$ and $C \in L^\infty(\Omega)$ is a given function used to localize

u solution of

$$\left\{ \begin{array}{rcll} -\mathrm{div}(Ae(u)) & = & 0 & \text{in } \omega \\ Ae(u) \cdot n & = & f & \text{on } \partial\omega \cap \partial\Omega \end{array} \right.$$

2 objectives:

- Minimize the displacement over a given area ($u_0 = 0$) $\rightarrow$ more local criterion than the compliance to optimize the rigidity

- *Minimize the difference to a target displacement*. It can be used as well to maximize the displacement along a given direction $\rightarrow$ compliant mechanisms, MEMS

Relaxed formulation (total relaxation)

Find

$$\min_{(\theta,A^*)\in\mathcal{CD}} J^*(\theta,A^*),$$

$$J^*(\theta,A^*)=\left(\int_\Omega \theta(x)C(x)|u(x)-u_0(x)|^\alpha dx\right)^{\frac{1}{\alpha}}+\ell\int_\Omega\theta(x)\,dx,$$

where u is solution of

$$\left\{\begin{array}{rcll} -\mathrm{div}\,(A^*e(u)) & = & 0 & \text{in } \Omega \\ A^*e(u)\cdot n & = & f & \text{on } \partial\Omega, \end{array}\right.$$

and $\mathcal{CD}$ is the set of admissible generalized shapes

$$\mathcal{CD}=\left\{\theta\in L^\infty(\Omega;[0,1]),\ A^*(x)\in G_{\theta(x)},\ \forall x\in\Omega\right\},$$

Partial relaxation

$$\min_{(\theta, A^*) \in \mathcal{LD}} J^*(\theta, A^*),$$

with

$$\mathcal{LD} = \left\{ \theta \in L^\infty(\Omega; [0,1]),\ A^*(x) \in L_{\theta(x)},\ \forall x \in \Omega \right\},$$

where $L_{\theta(x)} = \{$ set of all the Hooke's laws obtained by sequential lamination of A and void in proportions θ and $1 - \theta$ $\}$.

Remarks

• In the compliance case or Σ compliances, partial relaxation $\Leftrightarrow$ total relaxation.

• For all other objective functions, it is not true (it is maybe true but not proved yet...)

• An ill posed problem is replaced by another a priori ill posed problem. We expect that it is at least "less ill posed"...

• In practical computations, the set of admissible composites is further restricted to rank q laminates (q is fixed) with fixed lamination directions, just like in the multi-loads case

Adjoint state

Since we derive with respect to shape parameters a function depending on the solution of a PDE posed on this domain, the computation of the adjoint state in needed.

Remark: compliance is *self-adjoint*

The adjoint state p is solution of

$$\begin{cases} -\mathrm{div}\left(A^* e(p)\right) = c_\alpha \theta C |u - u_0|^{\alpha-2}(u - u_0) & \text{on } \Omega \\ +\text{boundary conditions} \end{cases}$$

where $c_\alpha = \left(\int_\Omega \theta(x) C(x) |u(x) - u_0(x)|^\alpha dx \right)^{\frac{1-\alpha}{\alpha}}$.

p allows to compute the derivatives of J^* with respect to the shape parameters θ and (m_i).

Numerical algorithm (projected gradient like)

- Initialization of the shape parameters θ_0 and $(m_i)_0$ in each cell.
- Iteration until convergence $k \geq 0$:
  * Computation of u_k (displacement) and p_k (adjoint) for fixed θ_k and $(m_i)_k$.
  * Updating of the shape parameters:

$$\begin{aligned} \theta_{k+1} &= \max(0, \min(1, \theta_k - t_k \nabla_\theta J_k^*)) \\ m_{i,k+1} &= \max(0, m_{i,k} - t'_k \nabla_{m_i} J_k^*) \end{aligned}$$

* The constraint $\sum m_{i,k} = 1$ is adjusted by a Lagrange multiplier.

* Descent steps t_k and t'_k are such that $J^*(\theta_{k+1}, m_{i,k+1}) < J^*(\theta_k, m_{i,k})$

Examples of numerical simulations