1. Existence, uniqueness and regularity of weak solutions to elliptic BVPs

In this section, we study linear elliptic operators of the form

(6) $$\mathscr{A}\, y(x) = -\sum_{i,j=1}^{N} D_i \big(a_{ij}(x)\, D_j\, y(x)\big), \quad x \in \Omega \subset \mathbb{R}^N.$$

General assumptions:

(H1) $a_{ij} \in L^\infty(\Omega)\,, \quad a_{ij} = a_{ji}\,, \quad \forall\, i, j.$

(H2) $\exists\, \gamma_0 > 0$ such that $\sum_{i,j=1}^{N} a_{ij}(x)\, \xi_i\, \xi_j \geq \gamma_0\, |\xi|^2$ a.e. in $\Omega\,, \quad \forall\, \xi \in \mathbb{R}^N$

We denote

(7) $\partial_{\nu_{\mathscr{A}}} y =$ directional derivative of y in the direction of the **conormal** $\nu_{\mathscr{A}}$, where

(8) $$(\nu_{\mathscr{A}})_i(x) = \sum_{j=1}^{N} a_{ij}(x)\, \nu_j(x)\,, \quad 1 \leq i \leq N$$

Consider the OCP

(9)
$$\min J(y,u,v) \quad := \quad \frac{\lambda_\Omega}{2}\|y-y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda_\Gamma}{2}\|y-y_\Gamma\|^2_{L^2(\Gamma)} + \frac{\lambda_v}{2}\|v\|^2_{L^2(\Omega)} + \frac{\lambda_u}{2}\|u\|^2_{L^2(\Gamma_1)}$$

subject to the constraints

(10)
$$\boxed{\begin{aligned} \mathscr{A}\,y + c_0\,y &= \beta_\Omega\,v && \text{in } \Omega \\ \partial_{\nu_{\mathscr{A}}} y + \alpha\,y &= \beta_\Gamma\,u && \text{on } \Gamma_1 \\ y &= 0 && \text{on } \Gamma_0 \end{aligned}}$$

and

(11)
$$\begin{aligned} v_a(x) \le \quad & v(x) \quad \le v_b(x) && \text{for a.e. } x \in \Omega \\ u_a(x) \le \quad & u(x) \quad \le u_b(x) && \text{for a.e. } x \in \Gamma_1 \end{aligned}$$

General assumptions:

(H3) $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_0 and Γ_1 measurable.

(H4) $c_0 \in L^\infty(\Omega)$, $c_0 \geq 0$ a.e., $\alpha \in L^\infty(\Gamma_1)$, $\alpha \geq 0$ a.e.

(H5) Either $|\Gamma_0| > 0$ or $\Gamma = \Gamma_1$ and $\|c_0\|_{L^2(\Omega)} + \|\alpha\|_{L^2(\Gamma)} > 0$.

(H6) $\beta_\Omega \in L^\infty(\Omega)$, $\beta_\Gamma \in L^\infty(\Gamma_1)$.

(H7) $\lambda_\Omega, \lambda_\Gamma, \lambda_v, \lambda_u$ are given nonnegative constants.

Moreover, we put

$$
\begin{aligned}
V_{ad} &= \left\{ v \in L^2(\Omega) : v_a(x) \leq v(x) \leq v_b(x) \quad \text{for a.e. } x \in \Omega \right\}, \\
U_{ad} &= \left\{ u \in L^2(\Gamma_1) : u_a(x) \leq u(x) \leq u_b(x) \quad \text{for a.e. } x \in \Gamma_1 \right\}
\end{aligned}
$$

with $v_a, v_b \in L^\infty(\Omega)$, $u_a, u_b \in L^\infty(\Gamma_1)$.

The state equation

We consider first the BVP

(12)
$$\begin{array}{rcll} \mathscr{A}\, y + c_0\, y & = & f & \text{in } \Omega \\ \partial_{\nu_{\mathscr{A}}} y + \alpha\, y & = & g & \text{on } \Gamma_1 \\ y & = & 0 & \text{on } \Gamma_0 \end{array}$$

We associate with (12) the following **weak formulation**:

Let $V := \{y \in H^1(\Omega) \, : \, y|_{\Gamma_0} = 0\}$.

Define on V the bilinear form:

(13)
$$a[y,v] := \int_\Omega \sum_{i,j=1}^N a_{ij} D_i y D_j v \, dx + \int_\Omega c_0\, y\, v \, dx + \int_{\Gamma_1} \alpha\, y\, v ds$$

Then the weak form of (12) is to find some $y \in V$ such that

$$a[y,v] = (f,v)_{L^2(\Omega)} + (g,v)_{L^2(\Gamma_1)} \quad \forall v \in V$$

Theorem 5: $\Omega \subset \mathbb{R}^N$ open, bounded; $\partial\Omega \in C^{0,1}$; **(H1)–(H5)** fulfilled.
Then:

- $\forall\, (f,g) \in L^2(\Omega) \times L^2(\Gamma_1)\ \exists_1$ weak solution $y \in V$.
- $\exists\, c_{\mathscr{A}} > 0 : \|y\|_{H^1(\Omega)} \le c_{\mathscr{A}} \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)}\right)$

Lax–Milgram lemma: Let $V, (\cdot,\cdot)_V)$ be a real Hilbert space, and let $a : V \times V \longrightarrow \mathbb{R}$ denote a bilinear form. Moreover, suppose that there exist positive constants α_0 and β_0 such that the following conditions are satisfied for all $v, y \in V$:

(14) $$\left|a[y,v]\right| \le \alpha_0\, \|y\|_V\, \|v\|_V \quad \text{(boundedness)}$$

(15) $$a[y,y] \ge \beta_0\, \|y\|_V^2 \quad (V\text{-ellipticity}).$$

Then for every $F \in V^*$ the variational equation $a[y,v] = F(v)\ \forall\, v \in V$ admits a unique solution $y \in V$. Moreover, there is some constant $c_a > 0$, which does not depend on F, such that

(16) $$\|y\|_V \le c_a\, \|F\|_{V^*}.$$

Proof of existence

We apply the Lax–Milgram lemma with $V = \{y \in H^1(\Omega) \,:\, v_{|\Gamma_0} = 0\}$

$(y, v)_V \;:=\; \int\limits_\Omega (\nabla y \cdot \nabla v + y\, v)\, dx$, and

$F(v) \;:=\; (f, v)_{L^2(\Omega)} + (g, v)_{L^2(\Gamma_1)}$.

Obviously $F \in V^*$, since $\forall\; v \in V$, by the trace theorem,

$$\begin{aligned} |F(v)| \quad &\le \quad \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)} \|v\|_{L^2(\Gamma_1)} \\ &\le \quad \Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)} \Big) \|v\|_{H^1(\Omega)} \end{aligned}$$

By a similar calculation, we have

$|a(y, v)| \le \alpha_0 \|y\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall\, y, v \in V$, for some $\alpha_0 > 0$.

Moreover,

Proof of existence

$$a[y,y] \geq \gamma_0 \int_\Omega |\nabla y(x)|^2\, dx + \int_\Omega c_0(x)|y(x)|^2\, dx + \int_{\Gamma_1} \alpha(x)|y(x)|^2\, ds\,.$$

If $|\Gamma_0| > 0$, then (15) follows from Poincaré's inequality. If $\Gamma = \Gamma_1$, we have:

Lemma (Friedrichs) Let $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be a continuous BF such that $B[y,y] \geq 0 \quad \forall\, y \in H^1(\Omega)$. If $B[h,h] > 0$ for $h \equiv 1$, then the norm

$$\|y\| := \left(\int_\Omega |\nabla y|^2\, dx \right)^{\frac{1}{2}} + B[y,y]^{\frac{1}{2}}$$

is equivalent to $\|\cdot\|_{H^1(\Omega)}$ on $H^1(\Omega)$.

The existence result now follows from the lemma and **(H5)** if we put

$$B[y,v] := \int_\Omega c_0(x)\, y\, v\, dx + \int_{\Gamma_1} \alpha(x)\, y\, v\, ds\,.$$

□

Apply first Theorem 3 of Lecture 1 to show existence for OCP. We take:

$V :=$ as above; $\mathscr{U} := L^2(\Omega) \times L^2(\Gamma_1)$; $\mathscr{U}_{ad} := V_{ad} \times U_{ad}$; $H = C = L^2(\Omega)$;

$A : V \to V^*$ the operator $A \in \mathscr{L}(V, V^*)$ defined by the BF (13);

$B \in \mathscr{L}(\mathscr{U}, V^*)$ the mapping assigning to $(v, u) \in \mathscr{U}_{ad}$ the linear functional $F = B(v, u) \in V^*$:

$$F(y) := \int_{\Omega} \beta_{\Omega} \, v \, y \, dx + \int_{\Gamma_1} \beta_{\Gamma} \, u \, y \, ds .$$

Since $\mathscr{U}_{ad}$ is bounded and $J : V \times \mathscr{U} \to \mathbb{R}$ convex and l.s.c., OCP has at least one minimum.

If $\lambda_{\Omega} > 0$ and $\lambda_v > 0$, then the minimum is unique.

Let: $(U, \|\cdot\|_U), (V, \|\cdot\|_V)$ B-spaces, $\mathscr{U} \subset U$ nonempty, open, $F : \mathscr{U} \to V$.

Def.: Let $u \in \mathscr{U}$.

- If $\exists\, \delta F(u,h) := \lim\limits_{t \downarrow 0} \frac{1}{t} (F(u+th) - F(u))$, then $\delta F(u,h)$ is called the **directional derivative of f at u in the direction h**.
- If $\exists\, \delta F(u,h)\ \forall\, h \in U$, then $h \mapsto \delta(u,h)$ is the **first variation of F at u**.
- Let $\exists$ the first variation $\delta F(u,\cdot)$. F is said to be **Gâteaux differentiable at u** $:\Longleftrightarrow \exists\, A \in \mathscr{L}(U,V)$ such that $\delta F(u,h) = A h \quad \forall\, h \in U$. We write $A = F'_G(u)$.
- F is said to be **Fréchet differentiable at u** $:\Longleftrightarrow \exists\, A \in \mathscr{L}(U,V)$ and a mapping $r(u,\cdot) : U \to V$ such that: for all $h \in U$ with $u+h \in \mathscr{U}$, we have

$$F(u+h) = F(u) + A h + r(u,h)$$

with

$$\frac{\|r(u,h)\|_V}{\|h\|_U} \to 0 \quad \text{as } \|h\|_U \to 0 .$$

We write $F'(u) := A$.

- If F is Fréchet differentiable at every $u \in \mathscr{U}$, then F is said to be Fréchet differentiable on $\mathscr{U}$.
- If $\exists\, F'(u) \quad \forall\, u \in \mathscr{U}$ and the mapping $u \mapsto F'(u)$ is continuous, we speak of **continuous Fréchet differentiability** on $\mathscr{U}$.

Remarks:

- If $\exists\, F'(u)$, then $\exists\, F'_G(u)$, and $F'(u) = F'_G(u)$ (but not vice versa!)
- If $\exists\, F'(u)$, then $F'(u)h = \delta F(u,h) \quad \forall\, h \in U$.
- $F \in \mathscr{L}(U,V) \implies F'(u) = F \quad \forall\, u \in U$.
- If $V = \mathbb{R}$, then $F'(u) \in \mathscr{L}(U,\mathbb{R}) = U^*$.

Example: $(H,(\cdot,\cdot)_H)$ Hilbert space, $F(u) := \|u\|_H^2 = (u,u)_H$.

$\forall\, u,h : F(u+h) - F(u) = 2(u,h)_H + \|h\|_H^2$

$\Longrightarrow F'(u) \in H^*$ given by $F'(u)\,h = 2(u,h)_H \quad \forall\, h \in H$.

Riesz $\Longrightarrow F'(u) \in H^* \cong 2u \in H$. We call $2u$ the **gradient of F at u**.

Theorem 6 (Chain rule)

Let: U,V,Z B-spaces, $\mathscr{U} \subset U, \mathscr{V} \subset V$ open, $F : \mathscr{U} \to \mathscr{V}$ and $G : \mathscr{V} \to Z$ F-differentiable at $u \in \mathscr{U}$ and $F(u) \in \mathscr{V}$, respectively. Then $E := G \circ F$ is F-differentiable at u, and we have

$$E'(u) = G'(F(u))\,F'(u).$$

Example: $(U,(\cdot,\cdot)_U),(H,(\cdot,\cdot)_H)$ Hilbert spaces, $z\in H$ fixed.

Let $S\in\mathscr{L}(U,H)$. Consider the functional $E:U\to\mathbb{R}$,

$$E(u)=\|Su-z\|_H^2$$

Then $E(u)=G\big(F(u)\big)$, where $G(v)=\|v\|_H^2$ and $F(u)=Su-z$.

We know:

$$G'(v)h=(2v,h)_H,\quad F'(u)h=Sh.$$

$$\begin{aligned}\Longrightarrow E'(u)h &= G'(F(u))F'(u)h=\big(2v,F'(u)h\big)_H\\ &= 2\left(Su-z,Sh\right)_H\\ &= 2\left(S^*(Su-z),h\right)_U.\end{aligned}$$

Here, $S^*\in\mathscr{L}(H^*,U^*)$ is the **adjoint of** S.

Def.: Let $(U,\|\cdot\|_U),(V,\|\cdot\|_V)$ be Banach spaces, $A\in\mathscr{L}(U,V)$. Then the mapping

$$A^*\in\mathscr{L}(V^*,U^*),\ (A^*g)(f):=(g\circ A)(f),\ g\in V^*,f\in U^*,$$

is called the **dual operator** of A.

Def.: Let $(U,(\cdot,\cdot)_U),(V,(\cdot,\cdot)_V)$ be Hilbert spaces, $A\in\mathscr{L}(U,V)$. Then an operator $A^\star$ is called the **Hilbert space adjoint of** A if

(17) $$\big(v,Au\big)_V=\big(A^\star v,u\big)_U\quad\forall u\in U,\quad\forall v\in V$$

Using the Riesz representation theorem, Hilbert space adjoint and dual of an operator $A\in\mathscr{L}(U,V)$ can be identified in the case of Hilbert spaces. We do that and always speak of **adjoints**.

The whole theory is based on the following simple results:

Theorem 7: Let $(U, \|\cdot\|_U)$ be a normed space, $J : U \to (-\infty, +\infty]$ a mapping with $J \not\equiv +\infty$. Then: $\bar{u} \in U$ minimizer of $J \iff 0 \in \partial J(\bar{u})$.

Proof: $0 \in \partial J(\bar{u})$ means by definition of $\partial J(\bar{u})$: $\quad J(\bar{u}) - J(u) \leq 0 \quad \forall\, u \in U$. $\square$

Theorem 8: Let $(U, \|\cdot\|_U)$ be a normed space; $C \subset U$ nonempty, convex, closed; $f : \mathscr{U} \to \mathbb{R}$ Gâteaux differentiable, where $C \subset \mathscr{U} \subset U$, $\mathscr{U}$ open. If $\bar{u} \in C$ is a solution to

(18) $$\min_{u \in C} f(u),$$

then $\bar{u}$ solves

(19) $$f'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall\, u \in C.$$

Proof: Since C is convex, $\bar{u}+t(u-\bar{u}) \in C \quad \forall\, t \in [0,1], \quad \forall\, u \in C$. Hence,
$\frac{1}{t}(f(\bar{u}+t(u-\bar{u}))-f(\bar{u})) \geq 0$ for $0<t\leq 1 \implies f'(\bar{u})(u-\bar{u}) \geq 0$. $\square$

We return to the OCP (9)–(11). Obviously, the **control-to-state mapping** $G:(u,v)\mapsto y$ is linear, continuous from $L^2(\Gamma_1)\times L^2(\Omega)$ in V.

Since $H^1(\Omega) \hookrightarrow L^2(\Omega)$, also ($E_Y :=$ identity from $H^1(\Omega)$ into $L^2(\Omega)$)

$$S := E_Y \circ G : L^2(\Gamma_1) \times L^2(\Omega) \to L^2(\Omega)$$

is linear, continuous.

Also, by the trace theorem,

$$S_\Gamma := \tau \circ G, \quad (u,v) \mapsto (\tau \circ G)(u,v) := y_{|\Gamma}$$

is linear, continuous.

We thus may introduce the **reduced cost functional**

$$(20)\quad J(y,u,v) = f(u,v) \;=\; \frac{\lambda_\Omega}{2}\,\|S(u,v)-y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda_\Gamma}{2}\,\|S_\Gamma(u,v)-y_\Gamma\|^2_{L^2(\Gamma)} + \frac{\lambda_v}{2}\,\|v\|^2_{L^2(\Omega)} + \frac{\lambda_u}{2}\,\|u\|^2_{L^2(\Gamma_1)}.$$

To simplify the exposition, we now consider the special case

$$\mathscr{A} = -\Delta, \quad c_0 \equiv 0, \quad \Gamma_0 = \Gamma, \quad \lambda_v = \lambda, \quad \lambda_\Gamma = \lambda_u = 0, \quad \beta_\Gamma \equiv 0, \quad \beta_\Omega \equiv \beta.$$

We thus consider the optimal control problem (where we replace v, v_a, v_b by u, u_a, u_b):

A simplified case

(OCP)*

$$\min J(y,u) := \frac{1}{2}\,\|y - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\,\|u\|^2_{L^2(\Omega)},$$

subject to

$$\boxed{\begin{array}{rcll} -\Delta y &=& \beta\, u & \text{in } \Omega \\ y &=& 0 & \text{on } \Gamma \end{array}}$$

and

$$u_a(x) \le u(x) \le u_b(x) \quad \text{for a.e. } x \in \Omega.$$

We postulate: $\lambda \ge 0$. We have $V = H^1_0(\Omega)$ and

$$U_{ad} = \{u \in L^2(\Omega) : u_a \le u \le u_b \ \text{ a.e. }\}.$$

A simplified case

Reduced functional:

$$f(u) = J(y,u) = \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}\,,$$

where $S = E_Y \circ G$ with $G : L^2(\Omega) \to H^1(\Omega)\,, \quad u \mapsto y$. Clearly

$$f'(u)\,h = (S^*(Su - y_\Omega) + \lambda\, u, h)_{L^2(\Omega)} \quad \forall\, h \in L^2(\Omega)\,,$$

and the variational inequality (19) becomes

$$\left(S^*(S\bar{u} - y_\Omega) + \lambda\,\bar{u}, u - \bar{u}\right)_{L^2(\Omega)} \geq 0 \quad \forall u \in U_{ad}.$$

We need to determine S^*, i.e.,

$$(z, Su)_{L^2(\Omega)} = (S^*z, u)_{L^2(\Omega)} \quad \forall\, z \in L^2(\Omega), \quad \forall\, u \in L^2(\Omega)\,.$$

We use the lemma:

Lemma: Let functions $z, u \in L^2(\Omega)$ and $c_0, \beta \in L^\infty(\Omega)$ with $c_0 \geq 0$ a.e. in Ω be given, and let y and p denote, respectively, the weak solutions to the elliptic boundary value problems

$$\begin{aligned} -\Delta y + c_0\, y &= \beta\, u & \qquad -\Delta p + c_0\, p &= z & \quad \text{in } \Omega \\ y &= 0 & p &= 0 & \quad \text{on } \Gamma. \end{aligned}$$

Then

$$\int_\Omega z\, y\, dx = \int_\Omega \beta\, p\, u\, dx. \tag{21}$$

Proof: We invoke the variational formulations of the above boundary value problems. For y, insertion of the test function $p \in H_0^1(\Omega)$ yields

$$\int_\Omega \Big(\nabla y \cdot \nabla p + c_0\, y\, p\Big)\, dx = \int_\Omega \beta\, p\, u\, dx,$$

while for p we obtain with the test function $y \in H_0^1(\Omega)$ that

$$\int_\Omega \Big(\nabla p \cdot \nabla y + c_0\, p\, y\Big)\, dx = \int_\Omega z\, y\, dx.$$

Since the left-hand sides are equal, the assertion immediately follows. $\square$

Theorem 9: The adjoint operator $S^*: L^2(\Omega) \to L^2(\Omega)$ is given by $S^* z := \beta\, p$, where $p \in H_0^1(\Omega)$ is the weak solution to

$$-\Delta p = z \quad \text{in } \Omega\,, \qquad p = 0 \quad \text{on } \Gamma\,.$$

Proof: By the above lemma, $\forall\, z, u \in L^2(\Omega)$,

$$(z, S u)_{L^2(\Omega)} = (z, y)_{L^2(\Omega)} = (\beta\, p, u)_{L^2(\Omega)}\,.$$

Moreover, the mapping $z \mapsto \beta\, p$ belongs to $\mathscr{L}(L^2(\Omega), L^2(\Omega))$. $\square$

Def.: The weak solution $p \in H_0^1(\Omega)$ to the **adjoint state equation**

(22)
$$\begin{aligned} -\Delta p &= \bar{y} - y_\Omega \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \Gamma \end{aligned}$$

is called the **adjoint state associated** with $\bar{y}$.

We now find:

$$S^*(S\bar{u} - y_\Omega) = S^*(\bar{y} - y_\Omega) = \beta\, p$$

$\Longrightarrow (\beta\, p + \lambda\, \bar{u}, u - \bar{u}) \geq 0 \quad \forall\, u \in U_{ad}$

$\Longrightarrow$ **Optimality system:** a control u, together with the optimal state y and the adjoint state p, is a solution of (OCP)* if and only if

(23)
$$\boxed{\begin{array}{rclcrcl} -\Delta y & = & \beta\, u & \qquad & -\Delta p & = & y - y_\Omega \\ y|_\Gamma & = & 0 & & p|_\Gamma & = & 0 \\ & & & u \in U_{ad} & & & \\ & & \multicolumn{3}{c}{\big(\beta\, p + \lambda\, u, v - u\big)_{L^2(\Omega)} \geq 0 \quad \forall v \in U_{ad}.} & & \end{array}}$$

Pointwise interpretation

The inequality

$$\int_\Omega (\beta\, p + \lambda\, \bar{u})\, \bar{u}\, dx \le \int_\Omega (\beta\, p + \lambda\, \bar{u})\, u\, dx \quad \forall u \in U_{ad}$$

means:

$$\int_\Omega (\beta\, p + \lambda\, \bar{u})\, \bar{u}\, dx = \min_{u \in U_{ad}} \int_\Omega (\beta\, p + \lambda\, \bar{u})\, u\, dx.$$

We easily obtain:

Lemma: The variational inequality is satisfied if and only if, for a.e. $x \in \Omega$,

(24)
$$\bar{u}(x) = \begin{cases} u_a(x) & \text{if} \quad \beta(x)\, p(x) + \lambda\, \bar{u}(x) > 0 \\ \in [u_a(x), u_b(x)] & \text{if} \quad \beta(x)\, p(x) + \lambda\, \bar{u}(x) = 0 \\ u_b(x) & \text{if} \quad \beta(x)\, p(x) + \lambda\, \bar{u}(x) < 0. \end{cases}$$

We obtain as consequences:

Case $\lambda = 0$: Then, a.e. in Ω,

$$\bar{u}(x) = \begin{cases} u_a(x) & \text{if} \quad \beta(x)\,p(x) > 0 \\ u_b(x) & \text{if} \quad \beta(x)\,p(x) < 0. \end{cases}$$

Hence: If $\beta(x)\,p(x) \neq 0$ a.e. in $\Omega \implies \bar{u}$ is a **bang-bang control**.

Case $\lambda > 0$: Then, a.e. in Ω,

(25) $$\bar{u}(x) = \mathbb{P}_{[u_a(x),u_b(x)]}\left\{-\frac{1}{\lambda}\beta(x)\,p(x)\right\} \quad \text{for almost every } x \in \Omega.$$

Notice: Let $\beta \in C^{0,1}(\bar{\Omega})$, $u_a, u_b \in H^1(\Omega)$. Since $\mathbb{P}_{[a,b]}(u) = \min\{b, \max\{a,u\}\}$, and since the adjoint state p belongs to $H^1(\Omega)$, we have $\bar{u} \in H^1(\Omega)$ for $\lambda > 0$!
Hence: The regularizing term $\|u\|^2_{L^2(\Omega)}$ in the cost functional has a regularizing effect on the optimal control.

A convenient method to “guess” the necessary optimality conditions is the formal Lagrange method. We explain it for the OCP (9)–(11):

(9)
$$\min J(y,u,v) := \frac{\lambda_\Omega}{2}\|y-y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda_\Gamma}{2}\|y-y_\Gamma\|^2_{L^2(\Gamma)} + \frac{\lambda_v}{2}\|v\|^2_{L^2(\Omega)} + \frac{\lambda_u}{2}\|u\|^2_{L^2(\Gamma_1)}$$

subject to the constraints

(10)
$$\begin{aligned} \mathscr{A}\,y + c_0\,y &= \beta_\Omega\,v && \text{in } \Omega \\ \partial_{\nu_\mathscr{A}} y + \alpha\,y &= \beta_\Gamma\,u && \text{on } \Gamma_1 \\ y &= 0 && \text{on } \Gamma_0 \end{aligned}$$

and

(11)
$$\begin{aligned} v_a(x) \le\ & v(x) \le v_b(x) && \text{for a.e. } x \in \Omega \\ u_a(x) \le\ & u(x) \le u_b(x) && \text{for a.e. } x \in \Gamma_1 \end{aligned}$$

The formal Lagrange method

The state space was $V = \{y \in H^1(\Omega) : y_{|\Gamma_0} = 0\}$. The general idea is to include the "difficult" equation constraints (10) into the Lagrangian and thus to minimize

(26)
$$\mathscr{L}(y,u,v,p) := J(y,u,v) - \int_\Omega \big(\mathscr{A}\, y + c_0\, y - \beta_\Omega\, v\big)\, p\, dx - \int_{\Gamma_1} \big(\partial_{\nu_{\mathscr{A}}} y + \alpha\, y - \beta_\Gamma\, u\big)\, p\, ds$$

over $V_{ad} \times U_{ad}$. We do not care whether this expression makes sense and simply integrate by parts to find, with the BF (13):

$$\mathscr{L}(y,u,v,p) = J(y,u,v) - a[y,p] + \int_\Omega \beta_\Omega\, v\, p\, dx + \int_{\Gamma_1} \beta_\Gamma\, u\, p\, ds\,.$$

Lagrange's method tells us that we should have $D_y \mathscr{L} = 0$, i.e.:

$$D_y \mathscr{L}(y,u,v,p)\, h = \int_\Omega \lambda_\Omega\, (\bar{y} - y_\Omega)\, h\, dx + \int_\Gamma \lambda_\Gamma\, (\bar{y} - y_\Gamma)\, h\, ds - a[h,p] = 0, \quad \forall\, h \in V\,.$$

$\Longrightarrow$

$p \in H^1(\Omega)$ is the unique weak solution to the adjoint state equation:

$$
(27) \qquad \boxed{\begin{array}{rcll}
\mathscr{A}\, p + c_0\, p & = & \lambda_\Omega\,(\bar{y} - y_\Omega) & \text{in } \Omega \\
\partial_{\nu_\mathscr{A}} p + \alpha\, p & = & \lambda_\Gamma\,(\bar{y} - y_\Gamma) & \text{on } \Gamma_1 \\
p & = & 0 & \text{on } \Gamma_0.
\end{array}}
$$

Also, again by Lagrange's method, we ought to have the variational inequalities:

$$
(28) \qquad D_v \mathscr{L}(\bar{y}, \bar{u}, \bar{v}, p)(v - \bar{v}) = \int_\Omega (\lambda_v\, \bar{v} + \beta_\Omega\, p)(v - \bar{v})\, dx \geq 0 \quad \forall\, v \in V_{ad},
$$

$$
(29) \qquad D_u \mathscr{L}(\bar{y}, \bar{u}, \bar{v}, p)(u - \bar{u}) = \int_{\Gamma_1} (\lambda_u\, \bar{u} + \beta_\Gamma\, p)(u - \bar{u})\, ds \geq 0 \quad \forall\, u \in U_{ad}.
$$

More general elliptic problems

The techniques introduced above can be generalized to semilinear elliptic control problems. In the book of Tröltzsch, elliptic state equations are considered of the type:

(30)
$$\begin{array}{rcll} \mathscr{A}\, y + c_0(x)\, y + d(x,y) & = & f & \text{in } \Omega \\ \partial_{\nu_{\mathscr{A}}} y + \alpha(x)\, y + b(x,y) & = & g & \text{on } \Gamma. \end{array}$$

Problems:

- State space? Existence, uniqueness? Control-to-state operator?
- Differentiability of nonlinearities (Nemytskii operators)?
- Problem nonconvex $\Longrightarrow$ necessary conditions are not sufficient $\Longrightarrow$ need second-order sufficient conditions $\Longrightarrow$ new problems

We consider, as example, the OCP with distributed control

(31) $$\min J(y,u) := \frac{1}{2} \int\limits_{\Omega} \left| y(x) - y_\Omega(x) \right|^2 dx + \frac{\lambda}{2} \int\limits_{\Omega} |u(x)|^2 dx,$$

subject to

(32) $$\boxed{\begin{array}{rcll} -\Delta y + d(x,y) & = & u & \text{in } \Omega \\ y & = & 0 & \text{on } \Gamma \end{array}}$$

and

(33) $u_a(x) \le u(x) \le u_b(x)$ for a.e. $x \in \Omega$, where $u_a, u_b \in L^\infty(\Omega)$.

The BVP (32) is **semilinear**.

- Cannot simply take $V = H_0^1(\Omega)$: if $y \in H_0^1(\Omega)$, do we have $d(\cdot, y(\cdot)) \in L^2(\Omega)$?
- Even if $d(\cdot, y(\cdot)) \in L^2(\Omega)$ for $y \in H_0^1(\Omega)$, is $y \mapsto d(\cdot, y(\cdot))$ Fréchet differentiable?

We avoid this by considering nonlinearities for which we can work in $L^\infty(\Omega)$. We assume

(H1) $\Omega \subset \mathbb{R}^N$ open, bounded, $\Gamma \in C^{0,1}$.

(H2) $d : \Omega \times \mathbb{R} \to \mathbb{R}$ is bounded and measurable with respect to $x \in \Omega$ for every $y \in \mathbb{R}$.

(H3) d is continuous, and increasing and Lipschitz continuous in y for a.e. $x \in \Omega$.

(H4) $d(x, 0) = 0$ for a.e. $x \in \Omega$. **(not really necessary!)**

Existence for the state equation

Theorem 10: Suppose **(H1)–(H4)** are fulfilled. Then the BVP (32) has a unique weak solution $y \in H_0^1(\Omega) \cap C(\bar{\Omega})$, i.e., we have

$$\int_\Omega \nabla y \cdot \nabla v \, dx + \int_\Omega d(x, y(x))\, v(x)\, dx = \int_\Omega u\, v\, dx \quad \forall\, v \in H_0^1(\Omega)\,. \tag{34}$$

Moreover, $\exists\, c_\infty > 0$ such that

$$\|y\|_{H^1(\Omega)} + \|y\|_{C(\bar{\Omega})} \le c_\infty \|u\|_{L^\infty(\Omega)}\,. \tag{35}$$

Remarks:

- In the case $\Gamma \in C^{1,1}$ we also have $y \in W^{2,p}(\Omega)$ for $1 \le p < +\infty$.
- The proof of the theorem uses the Browder–Minty theorem on monotone operators.
- We denote $y = y(u)$.

Guessing the first-order necessary conditions

We first use the formal Lagrange method to “guess” what the necessary conditions should look like. Define

$$\begin{aligned} L(y,u,p) &= J(y,u) - \int_\Omega (-\Delta y + d(x,y(x)) - u)\, p\, dx \\ &= J(y,u) - \int_\Omega (\nabla y \cdot \nabla p + d(x,y(x))\, p - u\, p)\, dx\,. \end{aligned}$$

Then:

$$D_y L(\bar y, \bar u, p)\, h = \int_\Omega (\bar y - \bar y_\Omega)\, h\, dx - \int_\Omega (\nabla h \cdot \nabla p + d_y(x,\bar y(x))\, p\, h)\, dx \overset{!}{=} 0 \quad \forall\, h \in H_0^1(\Omega)$$

(36)
$$\boxed{\begin{aligned} -\Delta p + d_y(x,\bar y(x))\, p &= \bar y - y_\Omega && \text{in } \Omega \\ p &= 0 && \text{on } \Gamma \end{aligned}}$$
(adjoint system)

(37)
$$D_u L(\bar y, \bar u, p)(u - \bar u) = \int_\Omega (\lambda\, \bar u + p)(u - \bar u)\, dx \geq 0 \quad \forall\, u \in U_{ad}\,.$$

The rigorous proof uses:

Theorem 11: Let **(H2)–(H4)** and the following condition be satisfied:

(H5) d is continuously differentiable with respect to y for a.e. $x \in \Omega$, and we have:

- **(i)** $|d_y(x,0)| \leq K$ for a.e. $x \in \Omega$.
- **(ii)** d_y is locally Lipschitz with respect to $y \in \mathbb{R}$.

Then the Nemytskii operator $D : y \mapsto d(\cdot, y(\cdot))$ is continuously Fréchet differentiable from $L^\infty(\Omega)$ into itself, and we have

(38) $$(D'(y)\,h)(x) = d_y(x, y(x))\,h(x) \quad \text{a.e. in } \Omega, \quad \forall\, h \in L^\infty(\Omega).$$

Remark: **(H5)** holds if $d(x,y) = d(y)$ and $d \in C^2(\mathbb{R})$.

By Theorem 10, the **control-to-state mapping** $G : L^\infty(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega})$, $G(u) := y(u)$, is well defined, and it can easily be shown to be globally Lipschitz continuous. A forteriori, we have:

Theorem 12: Let **(H1)–(H4)** be satisfied. Then G is Fréchet differentiable from $L^\infty(\Omega)$ into $H_0^1(\Omega) \cap C(\bar{\Omega})$. The directional derivative at $\bar{u} \in L^\infty(\Omega)$ in the direction h is given by $G'(\bar{u})\,h = y$, where y denotes the weak solution to the linearized BVP

$$
\begin{array}{rl}
-\Delta y + d_y(x,\bar{y})\,y = h & \text{in } \Omega \\
y = 0 & \text{on } \Gamma .
\end{array}
\tag{39}
$$

From this result, using the differentiability of the reduced cost functional

$$f(u) := J(G(u),u) = \frac{1}{2}\int_\Omega |G(u)-y_\Omega|^2\,dx + \frac{\lambda}{2}\int_\Omega |u|^2\,dx\,,$$

we easily derive the first-order optimality conditions:

Theorem 13: Let **(H1)–(H5)** be satisfied, and let $\bar{u} \in U_{ad}$ be locally optimal and $\bar{y} = y(\bar{u})$ be the associated state. Then the adjoint state $p \in H_0^1(\Omega) \cap C(\bar{\Omega})$, which is the unique solution to the adjoint equation

(40)
$$\boxed{\begin{aligned} -\Delta p + d_y(x,\bar{y}(x))\,p &= \bar{y} - y_\Omega \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \Gamma \end{aligned}}$$

satisfies the variational inequality

(41)
$$\int_\Omega (\lambda\,\bar{u} + p)(x)\,(u(x) - \bar{u}(x))\,dx \geq 0 \quad \forall\, u \in U_{ad}\,.$$

Remarks:

1. As in the linear case, we find for $\lambda > 0$:

(42) $$\bar{u}(x) = \mathbb{P}_{[u_a(x),u_b(x)]}\left\{-\frac{1}{\lambda}\,p(x)\right\} \quad \text{for a.e. } x \in \Omega.$$

Consequently:

$$u_a, u_b \in C(\bar{\Omega})\ (H^1(\Omega)) \implies \bar{u} \in C(\bar{\Omega})\ (H^1(\Omega))\,.$$

2. Consider the OCP with boundary control:

$$\min\ J(y,u) := \frac{1}{2}\int_\Omega |y - y_\Omega|^2\,dx + \frac{\lambda}{2}\int_\Omega |u|^2\,dx\,,$$

subject to

(43) $$\boxed{\begin{array}{rcll} -\Delta y + y & = & 0 & \text{in } \Omega \\ \partial_\nu y + d(x,y) & = & u & \text{on } \Gamma \end{array}}$$

and

$$u_a(x) \le u(x) \le u_b(x) \quad \text{for a.e. } x \in \Gamma.$$

One obtains in this case the adjoint equation (with $p \in H^1(\Omega) \cap C(\bar{\Omega})$)

$$\boxed{\begin{array}{rcll} -\Delta p + p & = & y - y_\Omega & \text{in } \Omega \\ \partial_\nu p + d_y(x,\bar{y})\, p & = & 0 & \text{on } \Gamma \end{array}}$$

and the variational inequality

$$\int_\Gamma (\lambda\, \bar{u} + p)(x)\, (u(x) - \bar{u}(x))\, ds \geq 0 \quad \forall\, u \in U_{ad}\,.$$