We consider first a simplified model for the time-dependent optimal boundary control of the temperature distribution in Ω (cf. Example 3 in Lecture 1):

(44) $$\min \ J(y,u) := \frac{1}{2}\int_{\Omega} \left|y(x,T) - y_\Omega(x)\right|^2 dx + \frac{\lambda}{2}\int_0^T\int_\Gamma |u(x,t)|^2 \, ds\, dt,$$

subject to

(45) $$\boxed{\begin{array}{rcll} y_t - \Delta y &=& 0 & \text{in } Q := \Omega \times (0,T) \\ \partial_\nu y + \alpha\, y &=& \beta\, u & \text{on } \Sigma := \Gamma \times (0,T) \\ y(x,0) &=& y_0(x) & \text{in } \Omega \end{array}}$$

and

(46) $$u_a(x,t) \le u(x,t) \le u_b(x,t) \quad \text{for a.e. } (x,t) \in \Sigma.$$

General strategy

- Show that the IBVP (45) has for every $u \in U_{ad}$ a unique solution in a suitable function space.
- Show that the OCP has an optimal pair $(\bar{y}, \bar{u})$.
- Derivation of first-order optimality conditions (which, due to the convexity of J, are also sufficient).

Before doing this, we again apply the formal Lagrange method in order to get an idea of what sort of optimality conditions can be expected. To this end put

$$U_{ad} := \{u \in L^2(\Sigma) : u_a(x,t) \leq u(x,t) \leq u_b(x,t) \text{ for a.e. } (x,t) \in \Sigma\}.$$

Guessing the first-order necessary conditions

Let $p := (p_1, p_2)$. We consider the Lagrangian

$$\mathscr{L}(y,u,p) = J(y,u) - \iint_Q (y_t - \Delta y)\, p_1 \, dx\, dt - \iint_\Sigma (\partial_\nu y + \alpha\, y - \beta\, u)\, p_2 \, ds\, dt.$$

We expect the necessary optimality conditions:

$D_y \mathscr{L}(\bar y, \bar u, p)\, y$	$=$	0	for all y with $y(0) = 0$
$D_u \mathscr{L}(\bar y, \bar u, p)\,(u - \bar u)$	$\geq$	0	for all $u \in U_{ad}$.

Observing that the derivative of the linear and continuous (?) mapping $y \mapsto y(\cdot, T)$ coincides with the mapping itself, we find that

Guessing the first-order necessary conditions

$$D_y\mathscr{L}(\bar{y},\bar{u},p)\,y \quad = \quad \int_\Omega \big(\bar{y}(T)-y_\Omega\big)\,y(T)\,dx - \iint_Q \big(y_t-\Delta y\big)\,p_1\,dx\,dt$$
$$- \iint_\Sigma \big(\partial_\nu y+\alpha y\big)\,p_2\,ds\,dt.$$

Then, $\forall$ smooth y with $y(0)=0$:

$$\begin{aligned}
0 \quad &= \quad \int_\Omega \big(\bar{y}(T)-y_\Omega\big)\,y(T)\,dx - \int_\Omega y(T)\,p_1(T)\,dx + \iint_Q y\,p_{1,t}\,dx\,dt \\
&\quad + \iint_\Sigma p_1\,\partial_\nu y\,ds\,dt - \iint_\Sigma y\,\partial_\nu p_1\,ds\,dt + \iint_Q y\,\Delta p_1\,dx\,dt \\
&\quad - \iint_\Sigma p_2\,\partial_\nu y\,ds\,dt - \iint_\Sigma \alpha\,y\,p_2\,ds\,dt \\
&= \quad \int_\Omega \big(\bar{y}(T)-y_\Omega-p_1(T)\big)\,y(T)\,dx + \iint_Q \big(p_{1,t}+\Delta p_1\big)\,y\,dx\,dt \\
&\quad - \iint_\Sigma \big(\partial_\nu p_1+\alpha\,p_2\big)\,y\,ds\,dt + \iint_\Sigma \big(p_1-p_2\big)\,\partial_\nu y\,ds\,dt.
\end{aligned}$$

Guessing the first-order necessary conditions

$\forall\, y \in C_0^\infty(Q)$: $\quad y(T), y(0), y, \partial_\nu y$ vanish on Ω, resp. Σ.

$\Longrightarrow$

$$\iint\limits_Q \big(p_{1,t} + \Delta p_1\big)\, y\, dx\, dt = 0 \qquad \forall y \in C_0^\infty(Q).$$

$\Longrightarrow$

$$p_{1,t} + \Delta p_1 = 0 \quad \text{in } Q$$

Next, $\forall\, y \in C^1(\bar{\Omega})$ such that $y_{|_\Sigma} = 0$:

$$\int\limits_\Omega \big(\bar{y}(T) - y_\Omega - p_1(T)\big)\, y(T)\, dx = 0$$

$\Longrightarrow p_1(T) = \bar{y}(T) - y_\Omega$ in Ω.

Guessing the first-order necessary conditions

Now, put $p_{1|_\Sigma} = p_2$. Then we have

$$\iint_\Sigma (\partial_\nu p_1 + \alpha p_1) y \, ds \, dt = 0 \quad \forall y \in C^1(\bar{Q}) .$$

$\Longrightarrow \partial_\nu p_1 + \alpha p_1 = 0 \quad$ in Σ .

Putting $p := p_1$, we have the **adjoint equation**

(47)

$$\begin{array}{rcll} -p_t & = & \Delta p & \text{in } Q \\ \partial_\nu p + \alpha p & = & 0 & \text{on } \Sigma \\ p(T) & = & \bar{y}(T) - y_\Omega & \text{in } \Omega. \end{array}$$

Guessing the first-order necessary conditions

Moreover, we have the variational inequality

(48) $$D_u\mathscr{L}(\bar{y},\bar{u},p)(u-\bar{u}) = \iint_{\Sigma} (\lambda\,\bar{u}+\beta\,p)(u-\bar{u})\,ds\,dt \geq 0 \quad \forall\, u\in U_{ad}\,.$$

Note: This was just formal!

We test (45) by $v \in H^1(\Omega) =: V$. Formally, we obtain

$$\int_\Omega y_t(t)\, v\, dx = -\int_\Omega \nabla y(t) \cdot \nabla v\, dx + \int_\Gamma (\beta\, u(t) - \alpha(t)\, y(t))\, v\, ds \quad \forall\, t \in [0,T], \tag{49}$$

where we write $y(t)(x) := y(x,t)$. Obviously, the right-hand side defines an element $F(t) \in V^*$.

$\Longrightarrow$ We should have $y_t \in V^*$, with a notion of "$\frac{d}{dt}$" yet to be defined

$\Longrightarrow$ spaces of **vector-valued distributions**

Let $(X, \|\cdot\|_X)$ be a Banach space.

Def.:

(i) We denote by $L^p(a,b;X)$, $1 \le p < \infty$, the linear space of all (equivalence classes of) measurable vector-valued functions $y : [a,b] \to X$ having the property that

$$\int_a^b \|y(t)\|_X^p \, dt < \infty.$$

The space $L^p(a,b;X)$ is a Banach space with respect to the norm

$$\|y\|_{L^p(a,b;X)} := \Big(\int_a^b \|y(t)\|_X^p \, dt \Big)^{1/p}.$$

(ii) We denote by $L^\infty(a,b;X)$ the Banach space of all (equivalence classes of) measurable vector-valued functions $y : [a,b] \to X$ having the property that

$$\|y\|_{L^\infty(a,b;X)} := \operatorname{ess} \sup_{t \in [a,b]} \|y(t)\|_X < \infty.$$

Theorem 14: Let $(X,(\cdot,\cdot)_X)$ be a Hilbert space. Then $L^2(a,b;X)$ is a Hilbert space with the scalar product

(50) $$(u,v)_{L^2(a,b;X)} := \int_0^T (u(t),v(t))_X \, dt$$

Def.: Let $(X,\|\cdot\|_X)$ be a Banach space. We say that a vector-valued function $y:[a,b]\to X$ is **continuous at the point** $t\in[a,b]$ if we have $\lim_{\tau\to t}\|y(\tau)-y(t)\|_X = 0$. We denote the space of all vector-valued functions that are continuous at every $t\in[a,b]$ by $C\big([a,b],X\big)$. The space $C\big([a,b],X\big)$ is a Banach space with respect to the norm

$$\|y\|_{C([a,b],X)} = \max_{t\in[a,b]} \|y(t)\|_X .$$

Remark: If $1<p<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$, and $f\in L^q(Q)$, then $f\in L^q(0,T;L^q(\Omega))\subset L^q(0,T;H^1(\Omega)^*)$.

Def.: Let $(H,(\cdot,\cdot)_H)$ be a separable Hilbert space, $(V,\|\cdot\|_V)$ a reflexive separable Banach space. If V is continuously and densely embedded in H, we speak of a **Gelfand triple** $V \subset H \subset V^*$.

Remark: "$H \subset V$" is understood in the following sense: $\forall\, f \in H$ the mapping $u \mapsto (f,u)_H$ belongs to V^*. By Riesz's theorem, we may identify f with this mapping. In this sense,

$$V \subset H \simeq H^* \subset V^* .$$

Standard examples: $H^1(\Omega) \subset L^2(\Omega) \subset H^1(\Omega)^*, \quad H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$.

Remark: Also the embebdding $H \subset V^*$ is dense and continuous!

Def.: Let $V \subset H \subset V^*$ be a Gelfand triple, $1 < p < +\infty$, $y \in L^p(0,T;V)$.

$w \in L^q(0,T;V^*)$ is called **generalized derivative of *y*** (denoted: $w = y_t$) iff

$\frac{1}{p} + \frac{1}{q} = 1$ and

(51) $$\int_0^T y(t)\,\varphi'(t)\,dt = -\int_0^T w(t)\,\varphi(t)\,dt \quad \forall\,\varphi \in C_0^\infty(0,T).$$

Lemma: Let $y \in L^p(0,T;V)$. Then $w = y_t$ if and only if

(52) $$\int_0^T (y(t),v)_H\,\varphi'(t)\,dt = -\int_0^T (w(t),v)_{V^*\times V}\,\varphi(t)\,dt \quad \forall\,v \in V \quad \forall\,\varphi \in C_0^\infty(0,T).$$

Lemma: Let $V \subset H \subset V^*$ be a Gelfand triple, $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

$W_p(0,T) := \{y \in L^p(0,T;V) : \exists\, y_t \in L^q(0,T;V^*)\}$ is a Banach space with the norm

$\|y\|_{W_p(0,T)} := \|y\|_{L^p(0,T;V)} + \|y_t\|_{L^q(0,T;V^*)}$.

Properties:

(i) Every $y \in W_p(0,T)$ coincides—possibly after a suitable modification on a set of zero measure—with an element of $C\big([0,T],H\big)$. In this sense, we have the continuous embedding $W_p(0,T) \hookrightarrow C\big([0,T],H\big)$.

(ii) $y(0), y(T)$ are well-defined elements of H!

(iii) For all $y,\, p \in W_p(0,T)$ the **formula of integration by parts** holds:

$$\begin{aligned}\int_0^T \big(y'(t), p(t)\big)_{V^* \times V}\, dt \quad &= \quad \big(y(T), p(T)\big)_H - \big(y(0), p(0)\big)_H \\ &\quad - \int_0^T \big(p'(t), y(t)\big)_{V^* \times V}\, dt.\end{aligned}$$

(iv) $\forall\, y \in W_p(0,T)$ we have

$$\int_0^T \big(y'(t), y(t)\big)_{V^* \times V}\, dt = \frac{1}{2}\|y(T)\|_H^2 - \frac{1}{2}\|y(0)\|_H^2.$$

(v) The set of "polynomials" $p(t) := \sum_{i=0}^{k} t^i x_i$, with $x_i \in V$, is dense in $W_p(0,T)$

We generally assume: $\Omega \subset \mathbb{R}^N$ open, bounded, with $\Gamma \in C^{0,1}$; $u_a, u_b \in L^2(\Sigma)$; $\beta \in L^\infty(\Sigma)$; $\alpha \in L^\infty(\Sigma)$ with $\alpha \geq 0$ a.e. and $\|\alpha\|_{L^\infty(\Sigma)} > 0$; $y_0 \in L^2(\Omega)$.

Theorem 15: Under the above assumptions, the IBVP (45) has for any $u \in U_{ad}$ a unique weak solution $y \in W_2(0,T)$, where $V = H^1(\Omega)$, $H = L^2(\Omega)$. We have $y(0) = y_0$, and

(53)
$$(y_t(t), v)_{V^* \times V} + \int_\Omega \nabla y(t) \cdot \nabla v \, dx + \int_\Gamma \alpha(t)\, y(t)\, v \, ds = \int_\Gamma \beta(t)\, u(t)\, v \, ds$$
$$\forall\, v \in V, \quad \text{for a.e. } t \in (0,T).$$

Moreover, the mapping $u \mapsto (y, y(0), y(T))$ is continuous from $L^2(\Sigma)$ into $W_2(0,T) \times L^2(\Omega) \times L^2(\Omega)$.

Remark: A corresponding result holds for the state problem with distributed nonstationary control!

We return to the OCP (44)–(46). The mapping $u \mapsto Su := y(T)$ is linear and continuous from $L^2(\Sigma)$ into $L^2(\Omega)$, and hence the reduced cost functional

$$f(u) = J(Su,u) = \frac{1}{2}\int_\Omega |Su - y_\Omega|^2\,dx + \frac{\lambda}{2}\iint_\Sigma |u|^2\,ds\,dt$$

is proper, convex, l.s.c. $\overset{\text{Theorem 3}}{\Longrightarrow}$ OCP has a solution $\bar{u} \in U_{ad}$. If $\lambda > 0$, it is unique. The first-order necessary (and sufficient) optimality condition reads, owing to Theorem 8:

$$\text{(54)}\qquad \begin{aligned} f'(\bar{u})(u-\bar{u}) &= (S^*(S\bar{u} - y_\Omega), u(T) - \bar{u}(T))_{L^2(\Omega)} + (\lambda\,\bar{u}, u-\bar{u})_{L^2(\Sigma)} \\ &= (\bar{y}(T) - y_\Omega, y(T) - \bar{y}(T))_{L^2(\Omega)} + (\lambda\,\bar{u}, u-\bar{u})_{L^2(\Sigma)} \geq 0 \\ &\qquad \forall\, u \in U_{ad}\,. \end{aligned}$$

As always, we have to identify the adjoint operator S^* !

First-order necessary conditions

Put $z := y - \bar{y} = Su - S\bar{u}$. Then z solves

$$z_t - \Delta z = 0 \text{ in } \Omega, \quad \partial_\nu z + \alpha z = \beta(u - \bar{u}) \text{ on } \Gamma, \quad z(0) = 0.$$

Now consider the **adjoint equation** with the **adjoint state** p,

$$-p_t - \Delta p = 0 \text{ in } \Omega, \quad \partial_\nu p + \alpha p = 0 \text{ on } \Gamma, \quad p(T) = \bar{y}(T) - y_\Omega.$$

Clearly, $p \in W_2(0,T)$, and we have:

$$\begin{aligned}
0 &= -\int_0^T (p_t(t) z(t))_{V^* \times V}\, dt + \iint_Q \nabla z \cdot \nabla p \, dx\, dt + \iint_\Sigma \alpha\, p\, z\, ds\, dt \\
&= \int_0^T (z_t(t), p(t))_{V^* \times V}\, dt - \int_\Omega p(T) z(T)\, dx + \iint_Q \nabla z \cdot \nabla p \, dx\, dt + \iint_\Sigma \alpha\, p\, z\, ds\, dt \\
&= \int_0^T (z_t(t), p(t))_{V^* \times V}\, dt - \int_\Omega (\bar{y}(T) - y_\Omega)(y(T) - \bar{y}(T))\, dx + \iint_Q \nabla z \cdot \nabla p \, dx\, dt + \iint_\Sigma \alpha\, p\, z\, ds\, dt.
\end{aligned}$$

Similarly,

$$0 = \int_0^T (z_t(t), p(t))_{V^* \times V}\, dt + \iint_Q \nabla z \cdot \nabla p \, dx\, dt + \iint_\Sigma \alpha\, p\, z\, ds\, dt - \iint_\Sigma p\, \beta\, (u - \bar{u})\, ds\, dt$$

$$\Longrightarrow \int_\Omega (\bar{y}(T) - y_\Omega)(y(T) - \bar{y}(T))\, dx = \iint_\Sigma \beta\, p(u - \bar{u})\, ds\, dt\,.$$

We have thus shown:

Theorem 16: Under the given assumptions, $\bar{u} \in U_{ad}$ is optimal with associated state $\bar{y} \in W_2(0,T)$ if and only if the unique solution $p \in W_2(0,T)$ to the adjoint state equation

$$\begin{array}{rcll} -p_t - \Delta p &=& 0 & \text{in } Q \\ \partial_\nu p + \alpha\, p &=& 0 & \text{on } \Sigma \\ p(T) &=& \bar{y}(T) - y_\Omega & \text{in } \Omega \end{array}$$

satisfies the variational inequality

$$\iint_\Sigma \big(\beta(x,t)\, p(x,t) + \lambda\, \bar{u}(x,t)\big)\, \big(u(x,t) - \bar{u}(x,t)\big)\, ds(x) dt \geq 0 \quad \forall\, u \in U_{ad}\,.$$

Remarks: 1. If $\lambda > 0$, we again obtain the projection formula

$$\bar{u}(x,t) = \mathbb{P}_{[u_a(x,t),u_b(x,t)]}\left\{-\frac{1}{\lambda}\beta(x,t)\,p(x,t)\right\}.$$

2. Consider the optimal nonstationary heat source problem

(55) $$\min J(y,u) := \frac{1}{2}\iint_{\Sigma} \left|y(x,t) - y_\Sigma(x,t)\right|^2 ds(x)\,dt + \frac{\lambda}{2}\iint_{Q} |u(x,t)|^2\,dx\,dt,$$

subject to

(56) $$\boxed{\begin{array}{rclr} y_t - \Delta y & = & \beta\, u & \text{in } Q \\ \partial_\nu y & = & 0 & \text{on } \Sigma \\ y(0) & = & 0 & \text{in } \Omega \end{array}}$$

and

(57) $$u_a(x,t) \le u(x,t) \le u_b(x,t) \qquad \text{for a.e. } (x,t) \in Q.$$

Again, we obtain the existence of an optimal pair $(\bar{y},\bar{u})$ with $\bar{y}\in W_2(0,T)$, and the first-order necessary optimality conditions read:

Adjoint equation:

$$\begin{array}{rcll} -p_t-\Delta y &=& 0 & \text{in } \Omega \\ \partial_\nu p &=& \bar{y}-y_\Sigma & \text{on } \Sigma \\ p(T) &=& 0 & \text{in } \Omega \end{array}$$

Variational inequality:

$$\iint_Q (\beta\, p+\lambda\, \bar{u})(u-\bar{u})\, dx\, dt \geq 0 \quad \forall u\in U_{ad}.$$

If $\lambda>0$, again a projection formula can be derived.