In the following, we assume:

- U, Z Banach spaces; $G : U \to Z$ is Fréchet differentiable; $C \subset U$ nonempty, convex
- $K \subset Z$ convex cone, i.e., if $\lambda > 0$ and $z \in K$, then $\lambda z \in K$.
  We write for $z_1, z_2 \in Z$: $z_1 \leq_K z_2 \iff z_2 - z_1 \in K$, and
  $z <_K 0 \iff -z \in \text{int}\,(K)$.
- **Def.:** Let $K \subset Z$ be a convex cone. Then the set

$$K^+ = \{z^* \in Z^* : (z^*, z)_{Z^* \times Z} \geq 0 \quad \forall\, z \in K\}$$

  is called the **dual cone of *K***.
- $f : U \to \mathbb{R}$ is Fréchet differentiable.

We consider the general optimization problem:

(58)
$$\boxed{\begin{aligned} &\min f(u), \\ &G(u) \leq_K 0, \quad u \in C. \end{aligned}}$$

We look for **local minimizers** of (58), i.e., for $\bar{u} \in C$ with $G(\bar{u}) \leq_K 0$ such that there is some $\varepsilon > 0$ such that

(59) $f(\bar{u}) \leq f(u)$ whenever $u \in C$, $G(u) \leq_K 0$, and $\|u - \bar{u}\|_U \leq \varepsilon$.

We eliminate the "complicated" constraint $G(u) \leq_K 0$ by means of a Lagrange multiplier and introduce the **Lagrangian**

(60)
$$L : U \times Z^* \to \mathbb{R}, \quad L(u, z^*) = f(u) + (z^*, G(u))_{Z^* \times Z}.$$

Def.: Let $\bar{u}$ be a local minimizer of (58). Any $z^* \in K^+$ such that

(61) $$D_u L(\bar{u}, z^*)(u - \bar{u}) \geq 0 \qquad \forall\, u \in C,$$

(62) $$(z^*, G(\bar{u}))_{Z^* \times Z} = 0 \quad \text{(complementary slackness condition)},$$

is called a **Lagrange multiplier** associated with $\bar{u}$.

Remark: From (60), (61) we obtain:

(63) $$(f'(\bar{u}) + G'(\bar{u})^* z^*, u - \bar{u})_{U^* \times U} \geq 0 \quad \forall\, u \in C,$$

or, equivalently,

(64) $$(f'(\bar{u}), u - \bar{u})_{U^* \times U} + (z^*, G'(\bar{u})(u - \bar{u}))_{Z^* \times Z} \geq 0 \quad \forall\, u \in C.$$

For the existence of such a multiplier we need some so-called **constraint qualification** to be satisfied.

Def.: Suppose that $\bar{u} \in C$ and $G(\bar{u}) \leq_K 0$, and let

$$C(\bar{u}) := \{\alpha\,(u-\bar{u}) \,:\, \alpha \geq 0,\, u \in C\}, \quad K(\bar{z}) := \{\beta\,(z-\bar{z}) \,:\, \beta \geq 0,\, z \in K\}.$$

The condition

(65) $$\boxed{G'(\bar{u})\,C(\bar{u}) + K\big(-G(\bar{u})\big) = Z}$$

is called the **Zowe–Kurcyusz constraint qualification**.

Remarks:

1. (65) $\Longleftrightarrow$ $\forall\, z \in Z\ \exists\, u \in C,\ v \geq_K 0,\ \alpha \geq 0,\ \beta \geq 0$ such that

   (66) $$\alpha\, G'(\bar{u})(u-\bar{u}) + \beta\,\big(v + G(\bar{u})\big) = z$$

2. Let $C = U$, $K = \{0\}$. Then (65) means: $G'(\bar{u})\,U = Z$, i.e., $G'(\bar{u})$ is surjective.

3. If $G(\bar{u}) <_K 0$, i.e., if $-G(\bar{u}) \in \text{int}\,(K)$, then (65) is satisfied.

Problem: for $1 \le p < +\infty$, the positive cones

$$K := \{L^p(\Omega) : u \ge 0 \text{ a.e. in } \Omega\}$$

have empty interior in $L^p(\Omega)$!

4. (65) is satisfied if the **linearized Slater condition** is fulfilled:

(67)
$$\boxed{\exists\, \tilde{u} \in C : G(\bar{u}) + G'(\bar{u})(\tilde{u} - \bar{u}) <_K 0}$$

5. Other, more general constraint qualifications can be found in the book of Neittaanmaki–Tiba–Sprekels.

The following multiplier rule is due to Zowe–Kurcyusz (1979):

Theorem 17: Let $\bar{u} \in C$ be a local minimizer of (58), and let f and G be continuously Fréchet differentiable in an open neighborhood of $\bar{u}$. If the constraint qualification (65) holds, then there exists a Lagrange multiplier $z^* \in Z^*$ associated with $\bar{u}$. Moreover, the set of Lagrange multipliers associated with $\bar{u}$ is bounded.

Example 1: Consider $\min f(u)$, $G(u) = 0$, $u \in C$. Then $K = \{0\}$, and (65) becomes: $G'(\bar{u})C(\bar{u}) = Z$. It holds the variational inequality (61), while the complementary slackness condition (62) is meaningless.

Example 2: Consider the minimization problem

(68) $$\min f(u) := \int_\Omega \psi\big(x,u(x)\big)\,dx, \qquad u(x) \le u_b(x) \quad \text{for a.e. } x \in \Omega$$

Assume: $u_b \in L^\infty(\Omega)$, ψ smooth and such that $f : L^2(\Omega) \to \mathbb{R}$ is continuously Fréchet differentiable.

We put: $Z := U = L^2(\Omega)$, $\quad G(u)(x) := u(x) - u_b(x)$, $\quad K := \{z \in U \,:\, z(x) \ge 0$ for a.e. $x \in \Omega\}$.

Then: $C = L^2(\Omega) \implies C(\bar u) = L^2(\Omega)$, $\quad G'(u) =$ identity mapping. The problem becomes:

$$\min f(u), \quad G(u) \le_K 0.$$

(66) is satisfied with $\beta = 0$, $\alpha = 1$, $u = z + \bar u$ for $z \in Z$.

$\Longrightarrow$ (65) is fulfilled.

$\overset{\text{Theorem 17}}{\Longrightarrow}$ $\exists$ Lagrange multiplier $z^* \in Z^* = L^2(\Omega)^*$. Identify $z^* \in Z^*$ with $\mu \in L^2(\Omega)$, $\mu \geq 0$ a.e. in Ω.

(63) takes the form

$$(f'(\bar{u}) + G'(\bar{u})^* z^*, u - \bar{u})_{L^2(\Omega)^* \times L^2(\Omega)} \geq 0 \quad \forall\, u \in L^2(\Omega).$$

Since $f'(\bar{u})$ can be identified with $\psi_u(\cdot, \bar{u}(\cdot)) \in L^2(\Omega)$, we find:

$\exists$ Lagrange multiplier $\mu \in L^2(\Omega)$, $\mu \geq 0$ a.e. in Ω, such that $\psi_u(\cdot, \bar{u}(\cdot)) + \mu = 0$.

The complementary slackness condition (62) reads:

$$\int_\Omega \mu(x)(\bar{u}(x) - u_b(x))\, dx = 0.$$

Example 3: The distributed control problem

(69) $$\min J(y,u) := \frac{1}{2}\int_{\Omega} \left|y(x) - y_{\Omega}(x)\right|^2 dx + \frac{\lambda}{2}\int_{\Omega} |u(x)|^2 dx$$

subject to

(70) $$\boxed{\begin{array}{rcll} -\Delta y + y & = & u & \text{in } \Omega \\ y & = & 0 & \text{on } \Gamma \end{array}}$$

and

(71) $$u_a(x) \le u(x) \le u_b(x) \quad \text{for a.e. } x \in \Omega$$

is a special case of (31)–(33) with $d(x,y) = y$. The necessary conditions are (36), (37) with $d_y(x,\bar{y}(x)) \equiv 1$.

A distributed control problem

Put: $Y = H_0^1(\Omega)$, $Y^* = H^{-1}(\Omega)$, $E_Y =$ embedding operator $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, $B =$ embedding operator $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$; define $A : Y \to Y^*$ by the BF associated with the BVP

$$a(y, v) := \int_\Omega (\nabla y \cdot \nabla v + y\, v)\, dx, \quad y, v \in H_0^1(\Omega).$$

$\Longrightarrow$

$$\min J(y, v) = \frac{1}{2} \|E_Y\, y - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2} \|v\|^2_{L^2(\Omega)}$$

subject to

$$A\, y = B\, v, \quad v \in V_{ad} := \{v \in L^2(\Omega) : u_a \le v \le u_b \text{ a.e. in } \Omega\}.$$

Now, put: $U := Y \times L^2(\Omega)$; $u := (y, v) \in U$; $G : U \to Y^* =: Z$, $G(u) := A\, y - B\, v$; $C := Y \times V_{ad} \subset U$. Clearly, G is continuously differentiable, with

$G'(\bar{u})(y, v) = A\,y - B\,v$.

$\Longrightarrow \min J(u)\,, \quad G(u) = 0\,, \quad u \in C$

By the Lax–Milgram theorem, $G'(\bar{u})$ is surjective. It even can be shown that $G'(\bar{u})C(\bar{u}) = Z = Y^*$. Hence (65) is satisfied.

$\overset{\text{Theorem 17}}{\Longrightarrow}$ $\exists$ Lagrange multiplier $z^* \in Z^* = (Y^*)^* \cong Y = H_0^1(\Omega)$.

Put: $\quad p := -z^*$. Then:

$$L(u, p) = L(y, v, p) = J(y, v) - (A\,y - B\,v, p)_{Y^* \times Y}\,.$$

$\overset{(61)}{\Longrightarrow} D_{(y,v)} L(\bar{y}, \bar{v}, p)(y - \bar{y}, v - \bar{v}) \geq 0 \quad \forall (y, v) \in C = Y \times V_{ad}$

A distributed control problem

$$\Longrightarrow\ D_y L(\bar{y}, \bar{v}, p)\, y = 0 \quad \forall\, y \in Y$$

$$\Longrightarrow\ (\bar{y} - y_\Omega, y) - (A^* p, y) = 0 \quad \forall\, y \in H_0^1(\Omega)$$

$\Longrightarrow$ p is the weak solution to the adjoint equation (36),

$$\begin{aligned} -\Delta p + p &= \bar{y} - y_\Omega && \text{in } \Omega \\ p &= 0 && \text{on } \Gamma. \end{aligned}$$

We also obtain

$$D_v L(\bar{y}, \bar{v}, p)(v - \bar{v}) \geq 0 \quad \forall\, v \in V_{ad}$$

$\Longrightarrow \displaystyle\int_\Omega (p + \lambda\, \bar{v})(v - \bar{v})\, dx \geq 0 \quad \forall\, v \in V_{ad}$, which is (37).

We consider the general situation

- U, V, Z reflexive Banach spaces; $(H, (\cdot,\cdot)_H)$ Hilbert space, where $V \subset H \subset V^*$ with dense and continuous embeddings.
- $A \in L(V,Z)$, $\quad B \in L(U,Z)$.
- $C \subset H$, $U_{ad} \subset U$ nonempty, convex, closed.
- $\tilde{J} : V \to \mathbb{R}$ convex, l.s.c.

We include the constraints in the cost functional and consider:

(72) $$\min J(y,u) := \tilde{J}(y,u) + I_{K \times U_{ad}}(y,u)$$

subject to

(73) $$Ay = Bu$$

Assume: There is some $(\tilde{y},\tilde{u}) \in K \times U_{ad}$ such that $A\tilde{y} = B\tilde{u}$ and $\tilde{y} \in \text{int}\,(K)$.

Theorem 18: Suppose the above condition is satisfied, and let $\exists A^{-1} \in L(Z,V)$. Then $\bar{u} \in U_{ad}$ is a solution to (72), (73) with optimal state $\bar{y}$ if and only if there is some $p \in Z^*$ such that

$$\boxed{\begin{array}{rcl} A\bar{y} & = & B\bar{u} \\ A^* p & \in & -(\partial J)_1(\bar{y},\bar{u}) \\ B^* p & \in & (\partial J)_2(\bar{y},\bar{u}) \end{array}} \tag{74}$$

(Here: $\partial J = ((\partial J)_1, (\partial J)_2)$ is the subdifferential of J)

"Proof": (Details: book of Neittaanmäki–Sprekels–Tiba)

With $S := A^{-1}B$ we minimize the reduced functional

$$f(u) := \tilde{J}(Su,u) + I_{K\times U_{ad}}(Su,u) \quad \text{on } U\,.$$

$\overset{\text{Theorem 7}}{\iff} \; 0 \in \partial f(\bar{u})$

Tiba's (1977) chain rule for subdifferentials shows

$\iff$

(75) $$0 \in S^*[(\partial J)_1(S\bar{u},\bar{u})] + (\partial J)_2(S\bar{u},\bar{u})\,.$$

Step 1: Let $(\bar{y},\bar{u})$ be optimal. Then $A\bar{y} = B\bar{u}$.

By (75), $\exists\, q \in -(\partial J)_1(S\bar{u},\bar{u})$ such that $w := B^* A^{*-1} q \in (\partial J)_2(S\bar{u},\bar{u})$.

Now put $p := (A^*)^{-1} q$.

Step 2: If (74) holds, then $\bar{y} = A^{-1}B\bar{u} = S\bar{u}$, and

$$S^*[(\partial J)_1(S\bar{u},\bar{u})] + (\partial J)_2(S\bar{u},\bar{u}) \ni -B^*(A^*)^{-1}A^* p + B^* p = 0\,.$$ □

We assume

- $(V,(\cdot,\cdot)_V)$, $(U,(\cdot,\cdot)_U)$ Hilbert spaces.
- $A \in L(V,V^*)$, $\exists\, A^{-1} \in L(V^*,V)$, $B \in L(U,V^*)$.
- $J : V \times U \to \mathbb{R}$ is convex, continuous; $h : V \times U \to \mathbb{R}$ convex, continuous.

We want to solve the OCP

(76) $$\min J(y,u)$$

(77) subject to $A\,y = B\,u$ and $h(y,u) \le 0$.

Let $K := \{(y,u) \in V \times U \;:\; A\,y = B\,u\}$.
We replace J by including the equality constraint:

$$\tilde{J}(y,u) := J(y,u) + I_K(y,u)\,.$$

Theorem 19: Let the above conditions hold, and assume the following **Slater condition** is satisfied:

(78) $$\exists\, (\tilde{y},\tilde{u}) \in K \quad \text{such that} \quad (\tilde{y},\tilde{u}) \in \ \text{int}\,(D)\,, \quad \text{where}$$

$$D := \{(y,u) \in V \times U \,:\, h(y,u) \le 0\}\,.$$

If $(\bar{y},\bar{u})$ is an optimal pair for OCP, then $\exists\, \lambda \ge 0$ such that

(79) $$(0,0) \in \partial J(\bar{y},\bar{u}) + \partial I_K(\bar{y},\bar{u}) + \lambda\, \partial h(\bar{y},\bar{u})$$

(80) $$\lambda\, h(\bar{y},\bar{u}) = 0 \quad \text{(complementary slackness)}$$

Proof: See the book of Neittaanmäki–Sprekels–Tiba.

Corollary: Under the assumptions of Theorem 19, there are $\lambda \geq 0$ and $p \in V$ such that

$$\boxed{\begin{aligned} &-A^* p \in (\partial L)_1(\bar{y},\bar{u}) + \lambda(\partial h)_1(\bar{y},\bar{u}) \\ &B^* p \in (\partial L)_2(\bar{y},\bar{u}) + \lambda(\partial h)_2(\bar{y},\bar{u}) \\ &\lambda\, h(\bar{y},\bar{u}) = 0 \end{aligned}} \tag{81}$$

Proof: Since K is a subspace, $\partial I_K(\bar{y},\bar{u}) = K^\perp$. Now one easily checks that $K^\perp = \{(A^* p, -B^* p) : p \in V\}. \overset{\text{Theorem 19}}{\Longrightarrow}$ assertion. $\square$