# Semi-Lagrangian Methods for Advection-Diffusion of Differential Form

Holger Heumann

Laboratoire de Mathématique J.A. Dieudonné, Université de Nice - Sophia Antipolis
Laboratoire de recherche sur la Fusion Contrôlée

joint work with R. Hiptmair (ETH Zürich)

Recent Advances on Theory and Applications
of Semi-Lagrangian Methods,
Rome, Dec 2011

# Case Study: Magnetic Advection

Magnetoquasistatic electrodynamic equations in moving media:

# Case Study: Magnetic Advection

Magnetoquasistatic electrodynamic equations in moving media:

$$
\begin{array}{lrcl}
\text{Faraday's Law} & \mathbf{curl}\,\mathbf{E} & = & -\partial_t \mathbf{B}\,, \\
\text{Ampere's Law} & \mathbf{curl}\,\mathbf{H} & = & \mathbf{j}\,, \\
\text{Ohm's Law} & \mathbf{j} & = & \sigma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B})\,, \\
 & \mathbf{B} & = & \mu \mathbf{H}\,, \\
\text{Constraint} & \operatorname{div} \mathbf{B} & = & 0\,,
\end{array}
$$

# Case Study: Magnetic Advection

Magnetoquasistatic electrodynamic equations in moving media:

| | | | | |
|---|---|---|---|---|
| Faraday's Law | $\mathbf{curl}\,\mathbf{E}$ | $=$ | $-\partial_t \mathbf{B}$ , | |
| Ampere's Law | $\mathbf{curl}\,\mathbf{H}$ | $=$ | $\mathbf{j}$ , | |
| Ohm's Law | $\mathbf{j}$ | $=$ | $\sigma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B})$ , | |
| | $\mathbf{B}$ | $=$ | $\mu \mathbf{H}$ , | |
| Constraint | $\mathrm{div}\,\mathbf{B}$ | $=$ | $0$ , | |

unknown:

Electric Field $\mathbf{E}$
Magnetic Field $\mathbf{H}$
Magnetic Induction $\mathbf{B}$
Current Density $\mathbf{J}$

# Case Study: Magnetic Advection

Magnetoquasistatic electrodynamic equations in moving media:

| | | |
|---|---|---|
| Faraday's Law | $\mathbf{curl}\,\mathbf{E} = -\partial_t \mathbf{B}\,,$ | unknown: |
| Ampere's Law | $\mathbf{curl}\,\mathbf{H} = \mathbf{j}\,,$ | Electric Field $\mathbf{E}$ |
| Ohm's Law | $\mathbf{j} = \sigma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B})\,,$ | Magnetic Field $\mathbf{H}$ |
| | $\mathbf{B} = \mu \mathbf{H}\,,$ | Magnetic Induction $\mathbf{B}$ |
| Constraint | $\operatorname{div} \mathbf{B} = 0\,,$ | Current Density $\mathbf{J}$ |

Given: velocity $\boldsymbol{\beta}$ and material parameter $\mu$, $\sigma$

# Case Study: Magnetic Advection

Magnetoquasistatic electrodynamic equations in moving media:

| | | |
|---|---|---|
| Faraday's Law | $\mathbf{curl}\,\mathbf{E} = -\partial_t \mathbf{B}$ , | |
| Ampere's Law | $\mathbf{curl}\,\mathbf{H} = \mathbf{j}$ , | |
| Ohm's Law | $\mathbf{j} = \sigma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B})$ , | |
| | $\mathbf{B} = \mu \mathbf{H}$ , | |
| Constraint | $\operatorname{div} \mathbf{B} = 0$ , | |

unknown:

Electric Field $\mathbf{E}$
Magnetic Field $\mathbf{H}$
Magnetic Induction $\mathbf{B}$
Current Density $\mathbf{J}$

Given: velocity $\boldsymbol{\beta}$ and material parameter $\mu$, $\sigma$

a) formulation for $\mathbf{H}$:

$$\mathbf{curl}\,\sigma^{-1}\,\mathbf{curl}\,\mathbf{H} = -\partial_t \mu \mathbf{H} + \mathbf{curl}(\boldsymbol{\beta} \times \mu \mathbf{H})$$

# Case Study: Magnetic Advection

Magnetoquasistatic electrodynamic equations in moving media:

| | | | | |
|---|---|---|---|---|
| Faraday's Law | $\mathbf{curl}\,\mathbf{E}$ | $=$ | $-\partial_t \mathbf{B}$ , | |
| Ampere's Law | $\mathbf{curl}\,\mathbf{H}$ | $=$ | $\mathbf{j}$ , | |
| Ohm's Law | $\mathbf{j}$ | $=$ | $\sigma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B})$ , | |
| | $\mathbf{B}$ | $=$ | $\mu \mathbf{H}$ , | |
| Constraint | $\mathrm{div}\,\mathbf{B}$ | $=$ | $0$ , | |

unknown:

Electric Field $\mathbf{E}$
Magnetic Field $\mathbf{H}$
Magnetic Induction $\mathbf{B}$
Current Density $\mathbf{J}$

Given: velocity $\boldsymbol{\beta}$ and material parameter $\mu$, $\sigma$

a) formulation for $\mathbf{H}$:

$$\mathbf{curl}\,\sigma^{-1}\,\mathbf{curl}\,\mathbf{H} = -\partial_t \mu \mathbf{H} + \mathbf{curl}(\boldsymbol{\beta} \times \mu \mathbf{H})$$

b) formulation for vector potential $\mathbf{A}$ ($\mathbf{A} = \mathbf{curl}\,\mathbf{B}$):

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{A} = -\sigma \partial_t \mathbf{A} + \sigma \boldsymbol{\beta} \times \mathbf{curl}\,\mathbf{A}$$

# Case Study: Magnetic Advection

Magnetoquasistatic electrodynamic equations in moving media:

| | | | | |
|---|---|---|---|---|
| Faraday's Law | $\mathbf{curl}\,\mathbf{E}$ | $=$ | $-\partial_t \mathbf{B}$ , | |
| Ampere's Law | $\mathbf{curl}\,\mathbf{H}$ | $=$ | $\mathbf{j}$ , | |
| Ohm's Law | $\mathbf{j}$ | $=$ | $\sigma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B})$ , | |
| | $\mathbf{B}$ | $=$ | $\mu \mathbf{H}$ , | |
| Constraint | $\mathrm{div}\,\mathbf{B}$ | $=$ | $0$ , | |

unknown:

Electric Field $\mathbf{E}$
Magnetic Field $\mathbf{H}$
Magnetic Induction $\mathbf{B}$
Current Density $\mathbf{J}$

Given: velocity $\boldsymbol{\beta}$ and material parameter $\mu$, $\sigma$

a) formulation for $\mathbf{H}$:

$$\mathbf{curl}\,\sigma^{-1}\,\mathbf{curl}\,\mathbf{H} = -\partial_t \mu \mathbf{H} + \mathbf{curl}(\boldsymbol{\beta} \times \mu \mathbf{H})$$

b) formulation for vector potential $\mathbf{A}$ ($\mathbf{A} = \mathbf{curl}\,\mathbf{B}$):

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{A} = -\sigma \partial_t \mathbf{A} + \sigma \boldsymbol{\beta} \times \mathbf{curl}\,\mathbf{A}$$

Second order operator and first order operator that is parametrized by velocity.

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u - \mathrm{div}(\varepsilon\, \mathbf{grad}\, u) \quad = \quad f \quad \text{in } \Omega\,.$$

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u - \operatorname{div}(\varepsilon\, \mathbf{grad}\, u) \quad = \quad f \quad \text{in } \Omega\,.$$

diffusive term

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u - \mathrm{div}(\varepsilon\, \mathbf{grad}\, u) = f \quad \text{in } \Omega\,.$$

advective term

diffusive term

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\underbrace{\boldsymbol{\beta} \cdot \mathbf{grad}\, u}_{\text{advective term}} - \underbrace{\mathrm{div}(\varepsilon\, \mathbf{grad}\, u)}_{\text{diffusive term}} \quad = \quad f \quad \text{in } \Omega\,.$$

Recall: general 2nd-order boundary value problem

$$\delta\,(\varepsilon \mathsf{d}\,\omega) \quad = \quad \varphi \quad \text{in } \Omega\,.$$

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u - \mathrm{div}(\varepsilon\, \mathbf{grad}\, u) \quad = \quad f \quad \text{in } \Omega\,.$$

advective term

diffusive term

Recall: general 2nd-order boundary value problem

$$\delta\,(\varepsilon \mathbf{d}\, \omega) \quad = \quad \varphi \quad \text{in } \Omega\,.$$

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\underbrace{\boldsymbol{\beta} \cdot \mathbf{grad}\, u}_{\text{advective term}} - \underbrace{\mathrm{div}(\varepsilon\, \mathbf{grad}\, u)}_{\text{diffusive term}} = f \quad \text{in } \Omega\,.$$

Recall: general 2nd-order boundary value problem

$$\delta\,(\varepsilon \mathrm{d}\,\omega) = \varphi \quad \text{in } \Omega\,.$$

$\omega \mathrel{\hat{=}} k$-form

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u - \mathrm{div}(\varepsilon\, \mathbf{grad}\, u) = f \quad \text{in } \Omega\,.$$

advective term

diffusive term

Recall: general 2nd-order boundary value problem

$$\delta\,(\varepsilon \mathrm{d}\,\omega) = \varphi \quad \text{in } \Omega\,.$$

$\mathrm{d} \mathrel{\hat{=}}$ exterior derivative

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u - \mathrm{div}(\varepsilon\, \mathbf{grad}\, u) \quad = \quad f \quad \text{in } \Omega\,.$$

advective term

diffusive term

Recall: general 2nd-order boundary value problem

$$\delta\,(\varepsilon \mathsf{d}\,\omega) \quad = \quad \varphi \quad \text{in } \Omega\,.$$

$\delta \mathrel{\hat{=}}$ adjoint exterior derivative

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\underbrace{\boldsymbol{\beta} \cdot \mathbf{grad}\, u}_{\text{advective term}} - \underbrace{\mathrm{div}(\varepsilon\, \mathbf{grad}\, u)}_{\text{diffusive term}} = f \quad \text{in } \Omega\,.$$

Recall: general 2nd-order boundary value problem

$$\delta\,(\varepsilon \mathsf{d}\,\omega) = \varphi \quad \text{in } \Omega\,.$$

$\delta \mathrel{\hat{=}}$ adjoint exterior derivative $\quad$ $\mathsf{d} \mathrel{\hat{=}}$ exterior derivative $\quad$ $\omega \mathrel{\hat{=}} k$-form

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u - \mathrm{div}(\varepsilon\, \mathbf{grad}\, u) \quad = \quad f \quad \text{in } \Omega\,.$$

advective term   diffusive term

Recall: general 2nd-order boundary value problem

$$\delta\,(\varepsilon \mathrm{d}\,\omega) \quad = \quad \varphi \quad \text{in } \Omega\,.$$

$\delta \mathrel{\hat{=}}$ adjoint exterior derivative   $\mathrm{d} \mathrel{\hat{=}}$ exterior derivative   $\omega \mathrel{\hat{=}} k$-form

$k$-form $\omega \in \Lambda^k(\Omega) \quad \mathrel{\hat{=}} \quad$ mapping $\quad \omega : \left\{ \begin{array}{c} \text{Oriented} \\ k\text{-dimensional} \\ \text{sub-manifolds} \subset \Omega \end{array} \right. \mapsto \mathbb{R}$

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\underbrace{\boldsymbol{\beta} \cdot \mathbf{grad}\, u}_{\text{advective term}} - \underbrace{\mathrm{div}(\varepsilon\, \mathbf{grad}\, u)}_{\text{diffusive term}} = f \quad \text{in } \Omega\,.$$

Recall: general 2nd-order boundary value problem

$$\delta\,(\varepsilon \mathrm{d}\,\omega) = \varphi \quad \text{in } \Omega\,.$$

$\delta \triangleq$ adjoint exterior derivative  $\mathrm{d} \triangleq$ exterior derivative  $\omega \triangleq k$-form

$k$-form $\omega \in \Lambda^k(\Omega)$ $\triangleq$ mapping $\omega : \left\{ \begin{array}{c} \text{Oriented} \\ k\text{-dimensional} \\ \text{sub-manifolds} \subset \Omega \end{array} \right. \mapsto \mathbb{R}$

In 3D: equivalent vector proxy formulation

$$\begin{array}{lcll} k=0 & : & -\mathrm{div}(\varepsilon\, \mathbf{grad}\, u) = f & \rightarrow \text{diffusion} \\ k=1 & : & \mathbf{curl}(\varepsilon\, \mathbf{curl}\, \mathbf{u}) = \mathbf{f} & \rightarrow \text{magnetostatics} \\ k=2 & : & -\mathbf{grad}(\varepsilon\, \mathrm{div}\, \mathbf{u}) = \mathbf{f} & \end{array}$$

# Scalar Advection-Diffusion Problem

Given: (continuous) velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^d$ on bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u - \operatorname{div}(\varepsilon\, \mathbf{grad}\, u) \quad = \quad f \quad \text{in } \Omega\,.$$

advective term diffusive term

Sobolev space

Recall: general 2nd-order boundary value problem in $H\Lambda^k(\mathrm{d}, \Omega)$

$$\delta\,(\varepsilon \mathrm{d}\,\omega) \quad = \quad \varphi \quad \text{in } \Omega\,.$$

$\delta \mathrel{\hat{=}}$ adjoint exterior derivative $\quad$ $\mathrm{d} \mathrel{\hat{=}}$ exterior derivative $\quad$ $\omega \mathrel{\hat{=}} k$-form

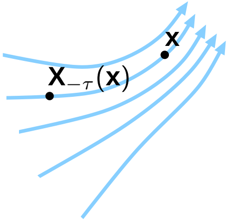
$$k\text{-form } \omega \in \Lambda^k(\Omega) \quad \hat{=} \quad \text{mapping} \quad \omega : \left\{ \begin{array}{c} \text{Oriented} \\ k\text{-dimensional} \\ \text{sub-manifolds} \subset \Omega \end{array} \right. \mapsto \mathbb{R}$$

In 3D: equivalent vector proxy formulation

$$\begin{array}{lcll} k = 0 & : & -\operatorname{div}(\varepsilon\, \mathbf{grad}\, u) = f & \to \text{diffusion} \\ k = 1 & : & \mathbf{curl}(\varepsilon\, \mathbf{curl}\, \mathbf{u}) = \mathbf{f} & \to \text{magnetostatics} \\ k = 2 & : & -\mathbf{grad}(\varepsilon \operatorname{div} \mathbf{u}) = \mathbf{f} & \end{array}$$

# Lie Derivatives

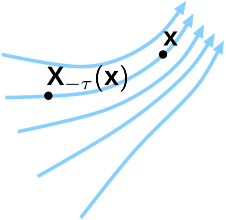
Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:




Special case $d = 3$, Lie derivatives:

# Lie Derivatives

Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:

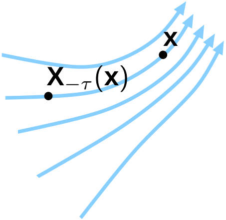



$$(\boldsymbol{\beta} \cdot \mathbf{grad}\, u)(\mathbf{x}) = \frac{d}{dt} u(\mathbf{X}_t(\mathbf{x})) = \lim_{\tau \to 0} \frac{u(\mathbf{x}) - u(\mathbf{X}_{-\tau}(\mathbf{x}))}{\tau}$$

Special case $d = 3$, Lie derivatives:

# Lie Derivatives

Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:




$$(\boldsymbol{\beta} \cdot \mathbf{grad}\, u)(\mathbf{x}) = \frac{d}{dt} u(\mathbf{X}_t(\mathbf{x})) = \lim_{\tau \to 0} \frac{u(\mathbf{x}) - u(\mathbf{X}_{-\tau}(\mathbf{x}))}{\tau}$$

Special case $d = 3$, Lie derivatives:

$k = 0$: $\quad \frac{d}{dt} u(\mathbf{X}_t) = \boldsymbol{\beta} \cdot \mathbf{grad}\, u\,,$

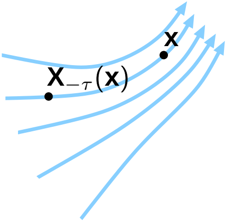
$k = 1$:

$k = 2$:

$k = 3$:

# Lie Derivatives

Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:




$$(\boldsymbol{\beta} \cdot \mathbf{grad}\, u)(\mathbf{x}) = \tfrac{d}{dt} u(\mathbf{X}_t(\mathbf{x})) = \lim_{\tau \to 0} \frac{u(\mathbf{x}) - u(\mathbf{X}_{-\tau}(\mathbf{x}))}{\tau}$$

Special case $d = 3$, Lie derivatives:

$k = 0$: $\quad \tfrac{d}{dt} u(\mathbf{X}_t) \;=\; \boldsymbol{\beta} \cdot \mathbf{grad}\, u$ ,

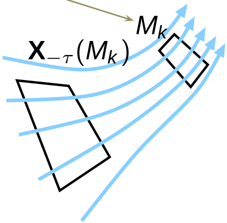
$k = 1$:

$k = 2$:

$k = 3$:

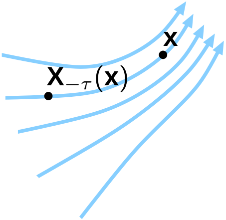
$k$-dim. manifold

$$\begin{aligned}
\int_{M_k} \mathsf{L}_{\boldsymbol{\beta}}\, \omega := &\quad \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{\mathbf{X}_{-\tau}(M_k)} \omega \right) \\
= &\quad \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{M_k} \mathbf{X}^*_{-\tau} \omega \right) \\
= &\quad \int_{M_k} \frac{d}{dt} \mathbf{X}^*_{-\tau} \omega
\end{aligned}$$



# Lie Derivatives

Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:




$$(\boldsymbol{\beta} \cdot \mathbf{grad}\, u)(\mathbf{x}) = \frac{d}{dt} u(\mathbf{X}_t(\mathbf{x})) = \lim_{\tau \to 0} \frac{u(\mathbf{x}) - u(\mathbf{X}_{-\tau}(\mathbf{x}))}{\tau}$$

Special case $d = 3$, Lie derivatives:

$k = 0$: $\quad \frac{d}{dt} u(\mathbf{X}_t) = \boldsymbol{\beta} \cdot \mathbf{grad}\, u$,

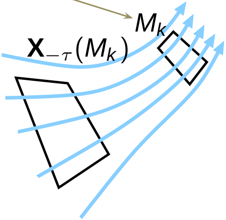
$k = 1$:

$k = 2$:

$k = 3$: $\quad \frac{d}{dt}(\det(D\mathbf{X}_t) u(\mathbf{X}_t)) = \operatorname{div}(\boldsymbol{\beta} u)$,

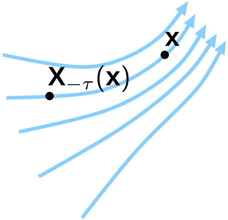
$k$-dim. manifold

$$\begin{aligned} \int_{M_k} \mathsf{L}_{\boldsymbol{\beta}}\, \omega &:= \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{\mathbf{X}_{-\tau}(M_k)} \omega \right) \\ &= \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{M_k} \mathbf{X}^*_{-\tau} \omega \right) \\ &= \int_{M_k} \frac{d}{dt} \mathbf{X}^*_{-\tau} \omega \end{aligned}$$



# Lie Derivatives

Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:



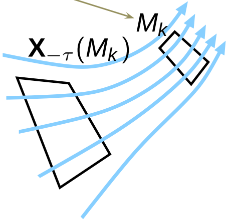

$$(\boldsymbol{\beta} \cdot \mathbf{grad}\, u)(\mathbf{x}) = \frac{d}{dt} u(\mathbf{X}_t(\mathbf{x})) = \lim_{\tau \to 0} \frac{u(\mathbf{x}) - u(\mathbf{X}_{-\tau}(\mathbf{x}))}{\tau}$$

Special case $d = 3$, Lie derivatives:

$$\begin{aligned}
k = 0: \quad & \frac{d}{dt} u(\mathbf{X}_t) &&= \boldsymbol{\beta} \cdot \mathbf{grad}\, u\,, \\
k = 1: \quad & \frac{d}{dt}(\mathrm{D}\mathbf{X}_t^T \mathbf{u}(\mathbf{X}_t)) &&= \\
k = 2: \quad & \frac{d}{dt}(\det(\mathrm{D}\mathbf{X}_t)\mathrm{D}\mathbf{X}_t^{-1} \mathbf{u}(\mathbf{X}_t)) &&= \\
k = 3: \quad & \frac{d}{dt}(\det(\mathrm{D}\mathbf{X}_t) u(\mathbf{X}_t)) &&= \operatorname{div}(\boldsymbol{\beta} u)\,,
\end{aligned}$$

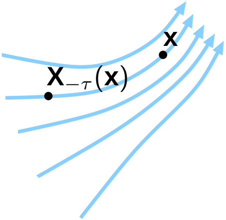
$k$-dim. manifold

$$\begin{aligned}
\int_{M_k} \mathrm{L}_{\boldsymbol{\beta}}\, \omega := \quad & \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{\mathbf{X}_{-\tau}(M_k)} \omega \right) \\
= \quad & \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{M_k} \mathbf{X}^*_{-\tau} \omega \right) \\
= \quad & \int_{M_k} \frac{d}{dt} \mathbf{X}^*_{-\tau} \omega
\end{aligned}$$



# Lie Derivatives

Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:



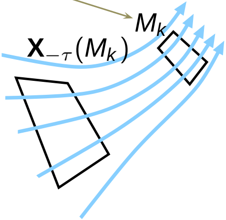

$$(\boldsymbol{\beta} \cdot \mathbf{grad}\, u)(\mathbf{x}) = \frac{d}{dt} u(\mathbf{X}_t(\mathbf{x})) = \lim_{\tau \to 0} \frac{u(\mathbf{x}) - u(\mathbf{X}_{-\tau}(\mathbf{x}))}{\tau}$$

Special case $d = 3$, Lie derivatives:

$$\begin{aligned}
k = 0: &\quad \frac{d}{dt} u(\mathbf{X}_t) &=&\quad \boldsymbol{\beta} \cdot \mathbf{grad}\, u\,, \\
k = 1: &\quad \frac{d}{dt}(D\mathbf{X}_t^T \mathbf{u}(\mathbf{X}_t)) &=&\quad \mathbf{curl}\, \mathbf{u} \times \boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u})\,, \\
k = 2: &\quad \frac{d}{dt}(\det(D\mathbf{X}_t) D\mathbf{X}_t^{-1} \mathbf{u}(\mathbf{X}_t)) &=& \\
k = 3: &\quad \frac{d}{dt}(\det(D\mathbf{X}_t) u(\mathbf{X}_t)) &=&\quad \operatorname{div}(\boldsymbol{\beta} u)\,,
\end{aligned}$$

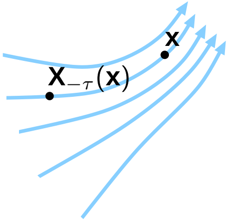
$k$-dim. manifold

$$\begin{aligned}
\int_{M_k} \mathsf{L}_{\boldsymbol{\beta}}\, \omega &:= \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{\mathbf{X}_{-\tau}(M_k)} \omega \right) \\
&= \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{M_k} \mathbf{X}^*_{-\tau} \omega \right) \\
&= \int_{M_k} \frac{d}{dt} \mathbf{X}^*_{-\tau} \omega
\end{aligned}$$



# Lie Derivatives

Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:



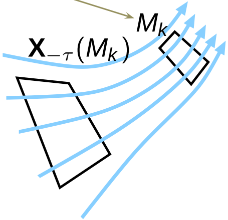

$$(\boldsymbol{\beta} \cdot \mathbf{grad}\, u)(\mathbf{x}) = \tfrac{d}{dt} u(\mathbf{X}_t(\mathbf{x})) = \lim_{\tau \to 0} \frac{u(\mathbf{x}) - u(\mathbf{X}_{-\tau}(\mathbf{x}))}{\tau}$$

Special case $d = 3$, Lie derivatives:

$$\begin{aligned}
k = 0: &\quad \tfrac{d}{dt} u(\mathbf{X}_t) &= \quad& \boldsymbol{\beta} \cdot \mathbf{grad}\, u\,, \\
k = 1: &\quad \tfrac{d}{dt}(D\mathbf{X}_t^T \mathbf{u}(\mathbf{X}_t)) &= \quad& \mathbf{curl}\, \mathbf{u} \times \boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u})\,, \\
k = 2: &\quad \tfrac{d}{dt}(\det(D\mathbf{X}_t) D\mathbf{X}_t^{-1} \mathbf{u}(\mathbf{X}_t)) &= \quad& \boldsymbol{\beta}\, \mathrm{div}\, \mathbf{u} + \mathbf{curl}(\mathbf{u} \times \boldsymbol{\beta})\,, \\
k = 3: &\quad \tfrac{d}{dt}(\det(D\mathbf{X}_t) u(\mathbf{X}_t)) &= \quad& \mathrm{div}(\boldsymbol{\beta} u)\,,
\end{aligned}$$

$k$-dim. manifold

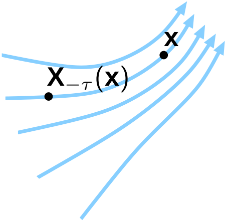
$$\begin{aligned}
\int_{M_k} \mathrm{L}_{\boldsymbol{\beta}}\, \omega := &\quad \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{\mathbf{X}_{-\tau}(M_k)} \omega \right) \\
= &\quad \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{M_k} \mathbf{X}_{-\tau}^* \omega \right) \\
= &\quad \int_{M_k} \tfrac{d}{dt} \mathbf{X}_{-\tau}^* \omega
\end{aligned}$$




$$\boxed{\text{Cartan's formula:} \quad \mathrm{L}_{\boldsymbol{\beta}}\, \omega := \tfrac{d}{dt} \mathbf{X}_{-\tau}^* \omega = \mathrm{d}\, \mathrm{i}_{\boldsymbol{\beta}} \omega + \mathrm{i}_{\boldsymbol{\beta}}\, \mathrm{d}\, \omega.}$$

# Lie Derivatives

Let $\mathbf{X}_t : \Omega \to \Omega \quad \hat{=} \quad$ flow induced by $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x})$:



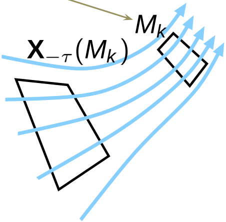

$$(\boldsymbol{\beta} \cdot \mathbf{grad}\, u)(\mathbf{x}) = \tfrac{d}{dt} u(\mathbf{X}_t(\mathbf{x})) = \lim_{\tau \to 0} \frac{u(\mathbf{x}) - u(\mathbf{X}_{-\tau}(\mathbf{x}))}{\tau}$$

Special case $d = 3$, Lie derivatives:

$$\begin{aligned}
k = 0: &\quad \tfrac{d}{dt} u(\mathbf{X}_t) &&= \boldsymbol{\beta} \cdot \mathbf{grad}\, u\,, \\
k = 1: &\quad \tfrac{d}{dt} (\mathrm{D}\mathbf{X}_t^T \mathbf{u}(\mathbf{X}_t)) &&= \mathbf{curl}\, \mathbf{u} \times \boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u})\,, \\
k = 2: &\quad \tfrac{d}{dt} (\det(\mathrm{D}\mathbf{X}_t) \mathrm{D}\mathbf{X}_t^{-1} \mathbf{u}(\mathbf{X}_t)) &&= \boldsymbol{\beta}\, \mathrm{div}\, \mathbf{u} + \mathbf{curl}(\mathbf{u} \times \boldsymbol{\beta})\,, \\
k = 3: &\quad \tfrac{d}{dt} (\det(\mathrm{D}\mathbf{X}_t) u(\mathbf{X}_t)) &&= \mathrm{div}(\boldsymbol{\beta} u)\,,
\end{aligned}$$

$k$-dim. manifold

$$\begin{aligned}
\int_{M_k} \mathrm{L}_{\boldsymbol{\beta}}\, \omega := &\ \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{\mathbf{X}_{-\tau}(M_k)} \omega \right) \\
= &\ \lim_{\tau \to 0} \frac{1}{\tau} \left( \int_{M_k} \omega - \int_{M_k} \mathbf{X}_{-\tau}^* \omega \right) \\
= &\ \int_{M_k} \frac{d}{dt} \mathbf{X}_{-\tau}^* \omega
\end{aligned}$$




contraction

$\boxed{\text{Cartan's formula:} \quad \mathrm{L}_{\boldsymbol{\beta}}\, \omega := \frac{d}{dt} \mathbf{X}_{-\tau}^* \omega = \mathrm{d}\, \mathrm{i}_{\boldsymbol{\beta}} \omega + \mathrm{i}_{\boldsymbol{\beta}}\, \mathrm{d}\, \omega.}$

# Generalized Advection-Diffusion Problem

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{array}{rcl} \partial_t \omega + \delta(\varepsilon \, \mathrm{d}\, \omega) + \mathsf{L}_{\boldsymbol{\beta}}\, \omega &=& \varphi(t) \\ \omega(0) &=& \omega_0 \\ \mathrm{tr}\, \omega &=& \psi(t) \end{array}\right\} \text{ or } \left\{ \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$$

# Generalized Advection-Diffusion Problem

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{array}{rcl} \partial_t \omega + \delta(\varepsilon\, \mathsf{d}\, \omega) + \mathsf{L}_{\boldsymbol{\beta}}\, \omega & = & \varphi(t) \\ \omega(0) & = & \omega_0 \\ \operatorname{tr} \omega & = & \psi(t) \end{array}\right\} \text{ or } \left\{ \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$$

Special case $d = 3$:

$$\begin{array}{ll} k = 0: & \partial_t u - \operatorname{div}(\varepsilon\, \mathbf{grad}\, u) + \boldsymbol{\beta} \cdot \mathbf{grad}\, u = f\,, \\ k = 1: & \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\, \mathbf{curl}\, \mathbf{u}) + \mathbf{curl}\, \mathbf{u} \times \boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u}) = \mathbf{f}\,, \\ k = 2: & \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon \operatorname{div} \mathbf{u}) + \boldsymbol{\beta} \operatorname{div} \mathbf{u} + \mathbf{curl}(\mathbf{u} \times \boldsymbol{\beta}) = \mathbf{f}\,, \\ k = 3: & \partial_t u + \operatorname{div}(\boldsymbol{\beta} u) = f\,. \end{array}$$

# Generalized Advection-Diffusion Problem

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{array}{rcl} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &=& \varphi(t) \\ \omega(0) &=& \omega_0 \\ \operatorname{tr}\omega &=& \psi(t) \end{array}\right\} \text{ or } \left\{\begin{array}{rcll} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega &=& \varphi(t) & \text{in } \Omega\,, \\ \omega(0) &=& \omega_0 & \text{in } \Omega\,, \\ \operatorname{tr}\omega &=& \psi(t) & \text{on } \partial\Omega\,. \end{array}\right.$$

Special case $d = 3$:

$$\begin{aligned}
k = 0&: \partial_t u - \operatorname{div}(\varepsilon\,\mathbf{grad}\,u) + \boldsymbol{\beta}\cdot\mathbf{grad}\,u = f\,,\\
k = 1&: \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\,\mathbf{curl}\,\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) = \mathbf{f}\,,\\
k = 2&: \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon\operatorname{div}\mathbf{u}) + \boldsymbol{\beta}\operatorname{div}\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = \mathbf{f}\,,\\
k = 3&: \partial_t u + \operatorname{div}(\boldsymbol{\beta} u) = f\,.
\end{aligned}$$

# Generalized Advection-Diffusion Problem

adjoint Lie Derivative

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{array}{rcl} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &=& \varphi(t) \\ \omega(0) &=& \omega_0 \\ \operatorname{tr}\omega &=& \psi(t) \end{array}\right\} \text{ or } \left\{\begin{array}{rcll} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega &=& \varphi(t) & \text{in } \Omega\,, \\ \omega(0) &=& \omega_0 & \text{in } \Omega\,, \\ \operatorname{tr}\omega &=& \psi(t) & \text{on } \partial\Omega\,. \end{array}\right.$$

Special case $d = 3$:

$$\begin{array}{ll} k = 0: & \partial_t u - \operatorname{div}(\varepsilon\,\mathbf{grad}\,u) + \boldsymbol{\beta}\cdot\mathbf{grad}\,u = f\,, \\ k = 1: & \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\,\mathbf{curl}\,\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) = \mathbf{f}\,, \\ k = 2: & \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon\,\operatorname{div}\mathbf{u}) + \boldsymbol{\beta}\operatorname{div}\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = \mathbf{f}\,, \\ k = 3: & \partial_t u + \operatorname{div}(\boldsymbol{\beta}u) = f\,. \end{array}$$

# Generalized Advection-Diffusion Problem

adjoint Lie Derivative

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{aligned} \partial_t\omega + \delta(\varepsilon\,\mathrm{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &= \varphi(t) \\ \omega(0) &= \omega_0 \\ \mathrm{tr}\,\omega &= \psi(t) \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \partial_t\omega + \delta(\varepsilon\,\mathrm{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega &= \varphi(t) && \text{in } \Omega\,, \\ \omega(0) &= \omega_0 && \text{in } \Omega\,, \\ \mathrm{tr}\,\omega &= \psi(t) && \text{on } \partial\Omega\,. \end{aligned}\right.$$

Special case $d = 3$:

$$\begin{aligned} k = 0&: \partial_t u - \mathrm{div}(\varepsilon\,\mathbf{grad}\,u) + \boldsymbol{\beta}\cdot\mathbf{grad}\,u = f\,, \\ k = 1&: \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\,\mathbf{curl}\,\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) = \mathbf{f}\,, \\ k = 2&: \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon\,\mathrm{div}\,\mathbf{u}) + \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = \mathbf{f}\,, \\ k = 3&: \partial_t u + \mathrm{div}(\boldsymbol{\beta} u) = f\,. \end{aligned}$$

Special case $d = 3$, adjoint:

$$\begin{aligned} k = 0&: \partial_t u - \mathrm{div}(\varepsilon\,\mathbf{grad}\,u) + \mathrm{div}(\boldsymbol{\beta} u) = f\,, \\ k = 1&: \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\,\mathbf{curl}\,\mathbf{u}) + \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = \mathbf{f}\,, \\ k = 2&: \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon\,\mathrm{div}\,\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) = \mathbf{f}\,, \\ k = 3&: \partial_t u + \boldsymbol{\beta}\cdot\mathbf{grad}\,u = f\,. \end{aligned}$$

# Generalized Advection-Diffusion Problem

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

adjoint Lie Derivative

$$\left.\begin{array}{rcl} \partial_t\omega + \delta(\varepsilon\, \mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &=& \varphi(t) \\ \omega(0) &=& \omega_0 \\ \operatorname{tr}\omega &=& \psi(t) \end{array}\right\} \text{ or } \left\{\begin{array}{rclr} \partial_t\omega + \delta(\varepsilon\, \mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega &=& \varphi(t) & \text{in } \Omega\,, \\ \omega(0) &=& \omega_0 & \text{in } \Omega\,, \\ \operatorname{tr}\omega &=& \psi(t) & \text{on } \partial\Omega\,. \end{array}\right.$$

Special case $d = 3$:

$$\begin{array}{lll} k=0: & \partial_t u - \operatorname{div}(\varepsilon\,\mathbf{grad}\,u) + \boldsymbol{\beta}\cdot\mathbf{grad}\,u = f\,, & \text{well studied} \\ k=1: & \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\,\mathbf{curl}\,\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) = \mathbf{f}\,, & \\ k=2: & \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon\operatorname{div}\mathbf{u}) + \boldsymbol{\beta}\operatorname{div}\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = \mathbf{f}\,, & \\ k=3: & \partial_t u + \operatorname{div}(\boldsymbol{\beta}u) = f\,. & \text{well studied} \end{array}$$

Special case $d = 3$, adjoint:

$$\begin{array}{lll} k=0: & \partial_t u - \operatorname{div}(\varepsilon\,\mathbf{grad}\,u) + \operatorname{div}(\boldsymbol{\beta}u) = f\,, & \text{well studied} \\ k=1: & \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\,\mathbf{curl}\,\mathbf{u}) + \boldsymbol{\beta}\operatorname{div}\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = \mathbf{f}\,, & \\ k=2: & \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon\operatorname{div}\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) = \mathbf{f}\,, & \\ k=3: & \partial_t u + \boldsymbol{\beta}\cdot\mathbf{grad}\,u = f\,. & \text{well studied} \end{array}$$

# Generalized Advection-Diffusion Problem

adjoint Lie Derivative

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{aligned} \partial_t\omega + \delta(\varepsilon\, \mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &= \varphi(t) \\ \omega(0) &= \omega_0 \\ \operatorname{tr}\omega &= \psi(t) \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \partial_t\omega + \delta(\varepsilon\, \mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega &= \varphi(t) && \text{in } \Omega\,, \\ \omega(0) &= \omega_0 && \text{in } \Omega\,, \\ \operatorname{tr}\omega &= \psi(t) && \text{on } \partial\Omega\,. \end{aligned}\right.$$

Special case $d = 3$:

$$\begin{aligned} k = 0&: \partial_t u - \operatorname{div}(\varepsilon\, \mathbf{grad}\, u) + \boldsymbol{\beta}\cdot\mathbf{grad}\, u = f\,, \\ k = 1&: \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\, \mathbf{curl}\,\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) = \mathbf{f}\,, \quad \text{important in MHD} \\ k = 2&: \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon \operatorname{div}\mathbf{u}) + \boldsymbol{\beta}\operatorname{div}\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = \mathbf{f}\,, \\ k = 3&: \partial_t u + \operatorname{div}(\boldsymbol{\beta} u) = f\,. \end{aligned}$$

MagnetoHydroDynamics with constraint $\operatorname{div}\mathbf{u} = 0$

Special case $d = 3$, adjoint:

$$\begin{aligned} k = 0&: \partial_t u - \operatorname{div}(\varepsilon\, \mathbf{grad}\, u) + \operatorname{div}(\boldsymbol{\beta} u) = f\,, \\ k = 1&: \partial_t \mathbf{u} + \mathbf{curl}(\varepsilon\, \mathbf{curl}\,\mathbf{u}) + \boldsymbol{\beta}\operatorname{div}\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = \mathbf{f}\,, \quad \text{important in MHD} \\ k = 2&: \partial_t \mathbf{u} - \mathbf{grad}(\varepsilon \operatorname{div}\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) = \mathbf{f}\,, \\ k = 3&: \partial_t u + \boldsymbol{\beta}\cdot\mathbf{grad}\, u = f\,. \end{aligned}$$

# Discrete Differential Forms

# Discrete Differential Forms

= finite elements for differential forms: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d}, \Omega)$

# Discrete Differential Forms

simplicial mesh

= finite elements for differential forms: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d}, \Omega)$

- piecewise polynomials, locally supported basis functions $\omega_j$

# Discrete Differential Forms

simplicial mesh

= finite elements for differential forms: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d}, \Omega)$

- piecewise polynomials, locally supported basis functions $\omega_j$
- d.o.f. $\leftrightarrow$ integrals over $\ell$-facets ($\ell \geq k$) of $\mathcal{T}$

# Discrete Differential Forms

simplicial mesh

= finite elements for differential forms: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d}, \Omega)$

- piecewise polynomials, locally supported basis functions $\omega_j$
- d.o.f. $\leftrightarrow$ integrals over $\ell$-facets ($\ell \geq k$) of $\mathcal{T}$

▶ nodal interpolation operators $I^k : \Lambda^k(\Omega) \mapsto \Lambda_h^k(\Omega)$: $\boxed{\mathrm{d} \circ I^k = I^{k+1} \circ \mathrm{d}}$

# Discrete Differential Forms

simplicial mesh

= finite elements for differential forms: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d}, \Omega)$

- piecewise polynomials, locally supported basis functions $\omega_j$
- d.o.f. $\leftrightarrow$ integrals over $\ell$-facets ($\ell \geq k$) of $\mathcal{T}$

► nodal interpolation operators $I^k : \Lambda^k(\Omega) \mapsto \Lambda_h^k(\Omega)$: $\boxed{\mathrm{d} \circ I^k = I^{k+1} \circ \mathrm{d}}$

Examples: edge elements (Nédélec FE), face elements (Raviart-Thomas FE)

# Discrete Differential Forms

simplicial mesh

= finite elements for differential forms: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$

- piecewise polynomials, locally supported basis functions $\omega_j$
- d.o.f. $\leftrightarrow$ integrals over $\ell$-facets ($\ell \geq k$) of $\mathcal{T}$

nodal interpolation operators $I^k : \Lambda^k(\Omega) \mapsto \Lambda_h^k(\Omega)$: $\boxed{\mathrm{d} \circ I^k = I^{k+1} \circ \mathrm{d}}$

Examples: edge elements (Nédélec FE), face elements (Raviart-Thomas FE)

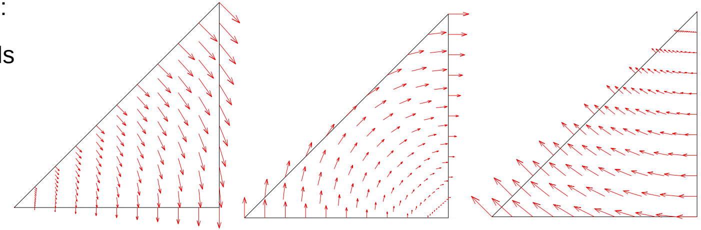
discrete 1-forms, 2D:

d.o.f. **=** edge integrals

vector proxies

$\updownarrow$

tangentially continuous, p.w. linear vectorfields



# Discrete Differential Forms

simplicial mesh

**=** finite elements for differential forms: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$

- piecewise polynomials, locally supported basis functions $\omega_j$
- d.o.f. $\leftrightarrow$ integrals over $\ell$-facets ($\ell \geq k$) of $\mathcal{T}$

nodal interpolation operators $I^k : \Lambda^k(\Omega) \mapsto \Lambda_h^k(\Omega)$: $\boxed{\mathrm{d} \circ I^k = I^{k+1} \circ \mathrm{d}}$

Examples: edge elements (Nédélec FE), face elements (Raviart-Thomas FE)

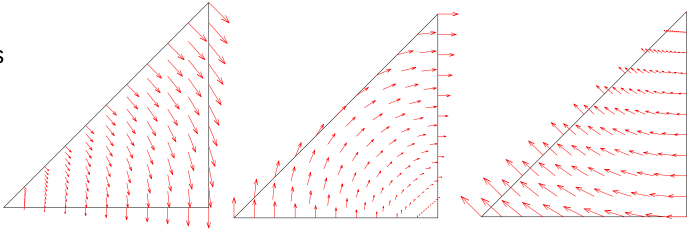
discrete 1-forms, 2D:

d.o.f. **=** edge integrals

vector proxies

$\updownarrow$

tangentially continuous, p.w. linear vectorfields



**Theorem:**($d > k > 0$) For $r \geq 0$ there are $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ that contain only piecewise polynomials of degree $s \leq r$ and:

$$\|\omega - I^k\omega\|_{L^2\Lambda^k} = O(h^{r+1}) \quad \text{and} \quad \|\,\mathrm{d}\,\omega - I^{k+1}\,\mathrm{d}\,\omega\|_{L^2\Lambda^{k+1}} = O(h^{r+1})\,.$$

# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r = 0$

# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi\,.$$

# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi \,.$$

canonical interpolation operator

# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi\,.$$

canonical interpolation operator

Canonical degrees of freedom (d.o.f.) of $\Lambda_h^k(\mathcal{T})$:

$$\text{set of dofs} = \left\{ \int_s \cdot : \Lambda^k(\Omega) \mapsto \mathbb{R}, s \text{ is subsimplex of } \mathcal{T} \right\}.$$

# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

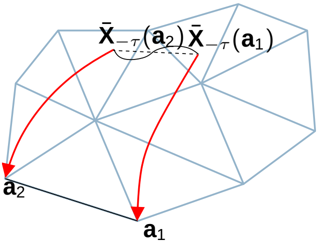
$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi \,.$$

canonical interpolation operator

Canonical degrees of freedom (d.o.f.) of $\Lambda_h^k(\mathcal{T})$:

$$\text{set of dofs} = \left\{ \int_s \cdot : \Lambda^k(\Omega) \mapsto \mathbb{R}, s \text{ is subsimplex of } \mathcal{T} \right\} .$$

$$\int_s \mathbf{X}_{-\tau}^* \omega = \int_{\mathbf{X}_{-\tau}(s)} \omega$$



# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi\,.$$

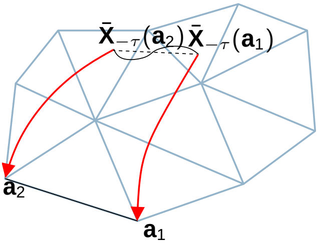
canonical interpolation operator

Canonical degrees of freedom (d.o.f.) of $\Lambda_h^k(\mathcal{T})$:

$$\text{set of dofs} = \left\{ \int_s \cdot : \Lambda^k(\Omega) \mapsto \mathbb{R}, s \text{ is subsimplex of } \mathcal{T} \right\}.$$

$$\int_s \mathbf{X}_{-\tau}^* \omega = \int_{\mathbf{X}_{-\tau}(s)} \omega$$

Advantages:



# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi\,.$$

canonical interpolation operator

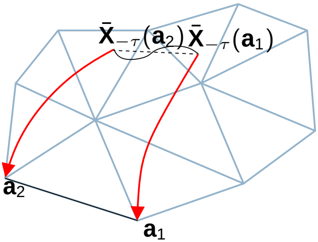
Canonical degrees of freedom (d.o.f.) of $\Lambda_h^k(\mathcal{T})$:

$$\text{set of dofs} = \left\{ \int_s \cdot : \Lambda^k(\Omega) \mapsto \mathbb{R}, s \text{ is subsimplex of } \mathcal{T} \right\}.$$

$$\int_s \mathbf{X}_{-\tau}^* \omega = \int_{\mathbf{X}_{-\tau}(s)} \omega$$

Advantages:

- preserves $\mathrm{d}\,\omega_h^0$, if $\phi = 0$:
  $\mathrm{d}\,\omega_h^n = \mathrm{d}\, I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} = I^k \mathbf{X}_{-\tau}^* \,\mathrm{d}\,\omega_h^{n-1}$;



# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi \,.$$

canonical interpolation operator

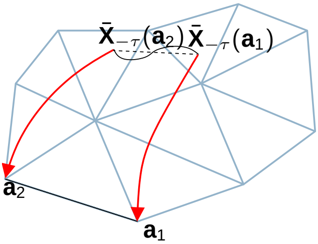
Canonical degrees of freedom (d.o.f.) of $\Lambda_h^k(\mathcal{T})$:

$$\text{set of dofs} = \left\{ \int_s \cdot : \Lambda^k(\Omega) \mapsto \mathbb{R}, s \text{ is subsimplex of } \mathcal{T} \right\} .$$

$$\int_s \mathbf{X}_{-\tau}^* \omega = \int_{\mathbf{X}_{-\tau}(s)} \omega$$

Advantages:

- preserves $\mathrm{d}\,\omega_h^0$, if $\phi = 0$:
  $\mathrm{d}\,\omega_h^n = \mathrm{d}\, I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} = I^k \mathbf{X}_{-\tau}^*\, \mathrm{d}\,\omega_h^{n-1}$;
- experiments show converges;

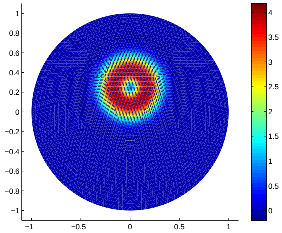


# Numerical Experiment

$$\partial_t \mathbf{u} + \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u} + \mathbf{curl}(\mathbf{u} \times \boldsymbol{\beta}) = 0 \quad \text{in unitcircle}\,,$$

$$\mathbf{u}(0) = (\text{localized bump})^2\,,$$

with $\quad \boldsymbol{\beta} = \begin{pmatrix} y \\ -x \end{pmatrix}.$

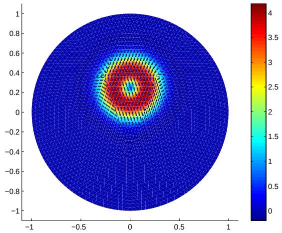


# Numerical Experiment

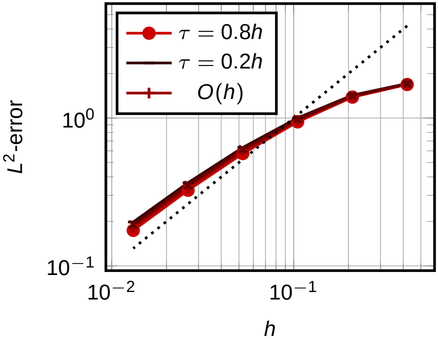
$$\partial_t \mathbf{u} + \boldsymbol{\beta} \operatorname{div} \mathbf{u} + \mathbf{curl}(\mathbf{u} \times \boldsymbol{\beta}) = 0 \quad \text{in unitcircle},$$

$$\mathbf{u}(0) = (\text{localized bump})^2,$$

$$\text{with} \quad \boldsymbol{\beta} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$




Convergence

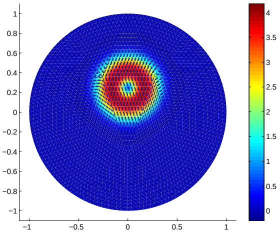


# Numerical Experiment

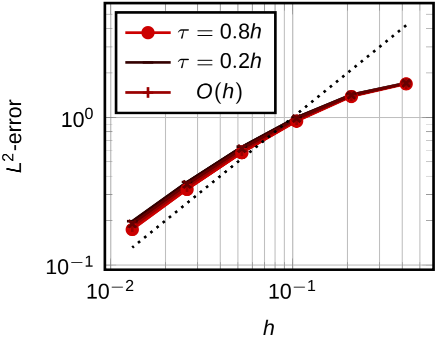
$$\partial_t \mathbf{u} + \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u} + \mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta}) = 0 \quad \text{in unitcircle}\,,$$

$$\mathbf{u}(0) = (\text{localized bump})^2\,,$$

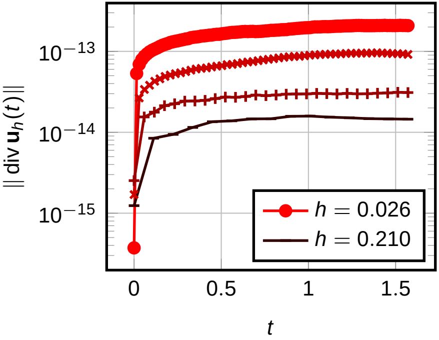
$$\text{with} \quad \boldsymbol{\beta} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$




Convergence




Preservation of closedness



# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d}, \Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi \,.$$

canonical interpolation operator

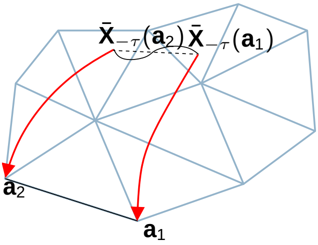
Canonical degrees of freedom (d.o.f.) of $\Lambda_h^k(\mathcal{T})$:

$$\text{set of dofs} = \left\{ \int_s \cdot : \Lambda^k(\Omega) \mapsto \mathbb{R}, s \text{ is subsimplex of } \mathcal{T} \right\} .$$

$$\int_s \mathbf{X}_{-\tau}^* \omega = \int_{\mathbf{X}_{-\tau}(s)} \omega$$

Advantages:

- preserves $\mathrm{d}\,\omega_h^0$, i.e. if $\phi = 0$:
  $\mathrm{d}\,\omega_h^n = \mathrm{d}\, I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} = I^k \mathbf{X}_{-\tau}^*\, \mathrm{d}\,\omega_h^{n-1}$;
- converges;



# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d},\Omega)$ lowest order $r=0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega = \varphi \,.$$

canonical interpolation operator

Canonical degrees of freedom (d.o.f.) of $\Lambda_h^k(\mathcal{T})$:

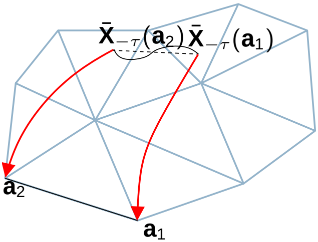
$$\text{set of dofs} = \left\{ \int_s \cdot : \Lambda^k(\Omega) \mapsto \mathbb{R}, s \text{ is subsimplex of } \mathcal{T} \right\} .$$

$$\textstyle\int_s \mathbf{X}_{-\tau}^* \omega = \int_{\mathbf{X}_{-\tau}(s)} \omega$$

Advantages:

- preserves $\mathrm{d}\,\omega_h^0$, i.e. if $\phi = 0$: $\mathrm{d}\,\omega_h^n = \mathrm{d}\, I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} = I^k \mathbf{X}_{-\tau}^*\, \mathrm{d}\,\omega_h^{n-1}$;
- converges;

Disadvantage:



# Our Favourite SL-Scheme

Focus: $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\mathrm{d}, \Omega)$ lowest order $r = 0$

Interpolation-based semi-Lagrangian scheme:

$$\omega_h^n = I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} + \tau I^k \varphi(t^n) \qquad \text{for} \qquad \partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}} \omega = \varphi\,.$$

canonical interpolation operator

Canonical degrees of freedom (d.o.f.) of $\Lambda_h^k(\mathcal{T})$:

$$\text{set of dofs} = \left\{ \int_s \cdot : \Lambda^k(\Omega) \mapsto \mathbb{R}, s \text{ is subsimplex of } \mathcal{T} \right\}\,.$$

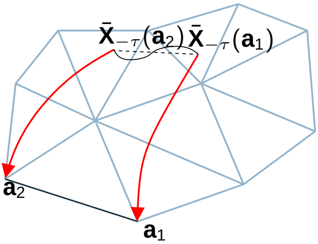
$$\int_s \mathbf{X}_{-\tau}^* \omega = \int_{\mathbf{X}_{-\tau}(s)} \omega$$

Advantages:

- preserves $\mathrm{d}\,\omega_h^0$, i.e. if $\phi = 0$: $\mathrm{d}\,\omega_h^n = \mathrm{d}\, I^k \mathbf{X}_{-\tau}^* \omega_h^{n-1} = I^k \mathbf{X}_{-\tau}^*\, \mathrm{d}\,\omega_h^{n-1}$;
- converges;

Disadvantage:

- lack of $L^2/L^\infty$-stability property, $\|I^k \mathbf{X}_{-\tau}^* \omega_h\| \leq (1 + C\tau)\|\omega_h\| \quad k > 0$, hence no proof of convergence!



# Advection-Diffusion: Convergence Theory

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{array}{rcl} \partial_t\omega + \delta(\varepsilon\, \mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega & = & \varphi(t) \\ \omega(0) & = & \omega_0 \\ \operatorname{tr}\omega & = & \psi(t) \end{array}\right\} \text{ or } \left\{\begin{array}{rcll} \partial_t\omega + \delta(\varepsilon\, \mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega & = & \varphi(t) & \text{in } \Omega\,, \\ \omega(0) & = & \omega_0 & \text{in } \Omega\,, \\ \operatorname{tr}\omega & = & \psi(t) & \text{on } \partial\Omega\,. \end{array}\right.$$

# Advection-Diffusion: Convergence Theory

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{array}{rcl} \partial_t \omega + \delta(\varepsilon\, \mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega & = & \varphi(t) \\ \omega(0) & = & \omega_0 \\ \operatorname{tr}\omega & = & \psi(t) \end{array}\right\} \text{ or } \left\{\begin{array}{rcll} \partial_t \omega + \delta(\varepsilon\, \mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega & = & \varphi(t) & \text{in } \Omega\,, \\ \omega(0) & = & \omega_0 & \text{in } \Omega\,, \\ \operatorname{tr}\omega & = & \psi(t) & \text{on } \partial\Omega\,. \end{array}\right.$$

Focus: $\varepsilon \ll 1$, small diffusion!

# Advection-Diffusion: Convergence Theory

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{aligned} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &= \varphi(t) \\ \omega(0) &= \omega_0 \\ \operatorname{tr}\omega &= \psi(t) \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega &= \varphi(t) && \text{in } \Omega\,, \\ \omega(0) &= \omega_0 && \text{in } \Omega\,, \\ \operatorname{tr}\omega &= \psi(t) && \text{on } \partial\Omega\,. \end{aligned}\right.$$

Focus: $\varepsilon \ll 1$, small diffusion!

Known $\varepsilon$-uniform error estimates ($k = 0$, scalar advection-diffusion):

# Advection-Diffusion: Convergence Theory

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{array}{rcl} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega & = & \varphi(t) \\ \omega(0) & = & \omega_0 \\ \mathrm{tr}\,\omega & = & \psi(t) \end{array}\right\} \text{ or } \left\{\begin{array}{rcll} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega & = & \varphi(t) & \text{in } \Omega\,, \\ \omega(0) & = & \omega_0 & \text{in } \Omega\,, \\ \mathrm{tr}\,\omega & = & \psi(t) & \text{on } \partial\Omega\,. \end{array}\right.$$

Focus: $\varepsilon \ll 1$, small diffusion!

Known $\varepsilon$-uniform error estimates ($k = 0$, scalar advection-diffusion):

Semi-Lagrangian Methods:

$$\text{error} = O(\tau + \tau^{-1}h^{r+1})$$

(Bause/Knabner, Wang/Wang)

# Advection-Diffusion: Convergence Theory

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{aligned} \partial_t\omega + \delta(\varepsilon\,\mathrm{d}\,\omega) + \mathrm{L}_{\boldsymbol{\beta}}\,\omega &= \varphi(t) \\ \omega(0) &= \omega_0 \\ \mathrm{tr}\,\omega &= \psi(t) \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \partial_t\omega + \delta(\varepsilon\,\mathrm{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega &= \varphi(t) && \text{in } \Omega\,, \\ \omega(0) &= \omega_0 && \text{in } \Omega\,, \\ \mathrm{tr}\,\omega &= \psi(t) && \text{on } \partial\Omega\,. \end{aligned}\right.$$

Focus: $\varepsilon \ll 1$, small diffusion!

Known $\varepsilon$-uniform error estimates ($k = 0$, scalar advection-diffusion):
Semi-Lagrangian Methods:

$$\text{error} = O(\tau + \tau^{-1}h^{r+1})$$

(Bause/Knabner, Wang/Wang)

unsatisfactory especially
if $r = 0$ and $\tau = O(h)$

# Advection-Diffusion: Convergence Theory

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{array}{rcl} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega & = & \varphi(t) \\ \omega(0) & = & \omega_0 \\ \operatorname{tr}\omega & = & \psi(t) \end{array}\right\} \text{ or } \left\{\begin{array}{rcll} \partial_t\omega + \delta(\varepsilon\,\mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega & = & \varphi(t) & \text{in } \Omega\,, \\ \omega(0) & = & \omega_0 & \text{in } \Omega\,, \\ \operatorname{tr}\omega & = & \psi(t) & \text{on } \partial\Omega\,. \end{array}\right.$$

Focus: $\varepsilon \ll 1$, small diffusion!

Known $\varepsilon$-uniform error estimates ($k = 0$, scalar advection-diffusion):

Semi-Lagrangian Methods:

$$\text{error} = O(\tau + \tau^{-1}h^{r+1})$$

(Bause/Knabner, Wang/Wang)

Eulerian Methods:

$$\text{error} = O(\tau + h^{r+\frac{1}{2}})$$

(upwind DG, SUPG/SDFEM)

unsatisfactory especially
if $r = 0$ and $\tau = O(h)$

# Advection-Diffusion: Convergence Theory

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{aligned} \partial_t \omega + \delta(\varepsilon\, \mathsf{d}\, \omega) + \mathsf{L}_{\boldsymbol{\beta}}\, \omega &= \varphi(t) \\ \omega(0) &= \omega_0 \\ \operatorname{tr} \omega &= \psi(t) \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \partial_t \omega + \delta(\varepsilon\, \mathsf{d}\, \omega) - \mathcal{L}_{\boldsymbol{\beta}} \omega &= \varphi(t) && \text{in } \Omega\,, \\ \omega(0) &= \omega_0 && \text{in } \Omega\,, \\ \operatorname{tr} \omega &= \psi(t) && \text{on } \partial\Omega\,. \end{aligned}\right.$$

Focus: $\varepsilon \ll 1$, small diffusion!

Known $\varepsilon$-uniform error estimates ($k = 0$, scalar advection-diffusion):

**Semi-Lagrangian Methods:**

$$\text{error} = O(\tau + \tau^{-1} h^{r+1})$$

(Bause/Knabner, Wang/Wang)

**Eulerian Methods:**

$$\text{error} = O(\tau + h^{r+\frac{1}{2}})$$

(upwind DG, SUPG/SDFEM)

unsatisfactory especially
if $r = 0$ and $\tau = O(h)$

Outline Proof(Eulerian methods)

- stabilized discretization of stat. problem;
- Ritz-Galerkin projector $R_h$;
- error recursion for $R_h \omega(t^n) - \omega_h^n$.

# Advection-Diffusion: Convergence Theory

Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\left.\begin{aligned} \partial_t\omega + \delta(\varepsilon\, \mathsf{d}\,\omega) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &= \varphi(t) \\ \omega(0) &= \omega_0 \\ \operatorname{tr}\omega &= \psi(t) \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \partial_t\omega + \delta(\varepsilon\, \mathsf{d}\,\omega) - \mathcal{L}_{\boldsymbol{\beta}}\omega &= \varphi(t) && \text{in } \Omega\,, \\ \omega(0) &= \omega_0 && \text{in } \Omega\,, \\ \operatorname{tr}\omega &= \psi(t) && \text{on } \partial\Omega\,. \end{aligned}\right.$$

Focus: $\varepsilon \ll 1$, small diffusion!

Known $\varepsilon$-uniform error estimates ($k = 0$, scalar advection-diffusion):

Semi-Lagrangian Methods:

$$\text{error} = O(\tau + \tau^{-1}h^{r+1})$$

(Bause/Knabner, Wang/Wang)

Eulerian Methods:

$$\text{error} = O(\tau + h^{r+\frac{1}{2}})$$

(upwind DG, SUPG/SDFEM)

unsatisfactory especially if $r = 0$ and $\tau = O(h)$

Outline Proof(Eulerian methods)

- stabilized discretization of stat. problem;
- Ritz-Galerkin projector $R_h$;
- error recursion for $R_h\omega(t^n) - \omega_h^n$.

Can we use a similar argument for SL-methods?

- yes, at least for SL-Methods based on $L^2$-projection.

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau-t^n}\varphi(\tau)\mathrm{d}\tau$$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\,\omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

discretization ( $\omega_h^n \in \Lambda_h^k(\mathcal{T})$):

$$\omega_h^n = P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1} + P_h \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

discretization ( $\omega_h^n \in \Lambda_h^k(\mathcal{T})$):

$$\omega_h^n = P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1} + P_h \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

discretization ( $\omega_h^n \in \Lambda_h^k(\mathcal{T})$):

$$\omega_h^n = P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1} + P_h \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

Theorem:(Johnson) Standard analysis ($P_h$ is $L^2$-projection):

$$\max_{0 \le n \le N} \|\omega(t^n) - \omega_h^n\| \le C\tau^{-\frac{1}{2}} \max_{0 \le n \le N} \|\omega(t^n) - P_h\omega(t^n)\| = O(\tau^{-\frac{1}{2}} h^{r+1})\,.$$

Assumptions: $\left\{ \begin{array}{l} P_h \mathbf{X}^*_{-\tau}\omega_h \text{ can be computed exactly,} \\ \\ \end{array} \right.$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

discretization ( $\omega_h^n \in \Lambda_h^k(\mathcal{T})$):

$$\omega_h^n = P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1} + P_h \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

**Theorem:**(Johnson) Standard analysis ($P_h$ is $L^2$-projection):

$$\|\eta(t^n)\|^2 := \|\omega(t^n) - \omega_h^n\|^2 = \|\omega(t^n) - P_h\omega(t^n)\|^2 + \|P_h\omega(t^n) - \omega_h^n\|^2$$

$$\max_{0 \le n \le N} \|\omega(t^n) - \omega_h^n\| \le C\tau^{-\frac{1}{2}} \max_{0 \le n \le N} \|\omega(t^n) - P_h\omega(t^n)\| = O(\tau^{-\frac{1}{2}} h^{r+1})\,.$$

Assumptions: $\left\{ \begin{array}{l} P_h \mathbf{X}^*_{-\tau}\omega_h \text{ can be computed exactly,} \\ \end{array} \right.$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

discretization ( $\omega_h^n \in \Lambda_h^k(\mathcal{T})$):

$$\omega_h^n = P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1} + P_h \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

**Theorem:** (Johnson) Standard analysis ($P_h$ is $L^2$-projection):

$$\begin{aligned}
\|\eta(t^n)\|^2 := \|\omega(t^n) - \omega_h^n\|^2 &= \|\omega(t^n) - P_h\omega(t^n)\|^2 + \|P_h\omega(t^n) - \omega_h^n\|^2 \\
&\leq \|\omega(t^n) - P_h\omega(t^n)\|^2 + \left\|\mathbf{X}^*_{-\tau}\omega(t^{n-1}) - \mathbf{X}^*_{-\tau}\omega_h^{n-1}\right\|^2
\end{aligned}$$

$$\max_{0\leq n\leq N} \|\omega(t^n) - \omega_h^n\| \leq C\tau^{-\frac{1}{2}} \max_{0\leq n\leq N} \|\omega(t^n) - P_h\omega(t^n)\| = O(\tau^{-\frac{1}{2}} h^{r+1})\,.$$

Assumptions: $\left\{ \begin{array}{l} P_h\mathbf{X}^*_{-\tau}\omega_h \text{ can be computed exactly,} \\ \\ \end{array} \right.$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

discretization ( $\omega_h^n \in \Lambda_h^k(\mathcal{T})$):

$$\omega_h^n = P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1} + P_h \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

**Theorem:**(Johnson) Standard analysis ($P_h$ is $L^2$-projection):

$$\begin{aligned}
\|\eta(t^n)\|^2 := \|\omega(t^n) - \omega_h^n\|^2 &= \|\omega(t^n) - P_h\omega(t^n)\|^2 + \|P_h\omega(t^n) - \omega_h^n\|^2 \\
&\leq \|\omega(t^n) - P_h\omega(t^n)\|^2 + \left\|\mathbf{X}^*_{-\tau}\omega(t^{n-1}) - \mathbf{X}^*_{-\tau}\omega_h^{n-1}\right\|^2 \\
&\leq \|\omega(t^n) - P_h\omega(t^n)\|^2 + (1 + C\tau)\left\|\eta^{n-1}\right\|^2
\end{aligned}$$

$$\max_{0\leq n\leq N} \|\omega(t^n) - \omega_h^n\| \leq C\tau^{-\frac{1}{2}} \max_{0\leq n\leq N} \|\omega(t^n) - P_h\omega(t^n)\| = O(\tau^{-\frac{1}{2}}h^{r+1})\,.$$

Assumptions: $\begin{cases} P_h\mathbf{X}^*_{-\tau}\omega_h \text{ can be computed exactly,} \\ \left\|\mathbf{X}^*_{-\tau}\omega\right\|^2 \leq (1 + C\tau)\left\|\omega\right\|^2. \end{cases}$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

discretization ( $\omega_h^n \in \Lambda_h^k(\mathcal{T})$):

$$\omega_h^n = P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1} + P_h \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

**Theorem:** (Johnson) Standard analysis ($P_h$ is $L^2$-projection):

$$\begin{aligned}
\|\eta(t^n)\|^2 := \|\omega(t^n) - \omega_h^n\|^2 &= \|\omega(t^n) - P_h\omega(t^n)\|^2 + \|P_h\omega(t^n) - \omega_h^n\|^2 \\
&\le \|\omega(t^n) - P_h\omega(t^n)\|^2 + \left\|\mathbf{X}^*_{-\tau}\omega(t^{n-1}) - \mathbf{X}^*_{-\tau}\omega_h^{n-1}\right\|^2 \\
&\le \|\omega(t^n) - P_h\omega(t^n)\|^2 + (1 + C\tau)\left\|\eta^{n-1}\right\|^2
\end{aligned}$$

$$\stackrel{\text{Gronwall}}{\Rightarrow} \max_{0\le n\le N} \|\omega(t^n) - \omega_h^n\| \le C\tau^{-\frac{1}{2}} \max_{0\le n\le N} \|\omega(t^n) - P_h\omega(t^n)\| = O(\tau^{-\frac{1}{2}} h^{r+1})\,.$$

Assumptions: $\begin{cases} P_h \mathbf{X}^*_{-\tau}\omega_h \text{ can be computed exactly,} \\ \left\|\mathbf{X}^*_{-\tau}\omega\right\|^2 \le (1 + C\tau)\|\omega\|^2\,. \end{cases}$

# Pure Advection I

Non-stationary advection: Find $\omega = \omega(t) \in \Lambda^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\, \omega \quad = \quad \varphi \quad \text{in } \Omega\,,$$

has formal solution:

$$\omega(t^n) = \mathbf{X}^*_{-\tau}\omega(t^{n-1}) + \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

discretization ( $\omega_h^n \in \Lambda_h^k(\mathcal{T})$):

$$\omega_h^n = P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1} + P_h \int_{t^{n-1}}^{t^n} \mathbf{X}^*_{\tau - t^n}\varphi(\tau)\mathrm{d}\tau$$

Theorem:(Johnson) Standard analysis ($P_h$ is $L^2$-projection):

$$\begin{aligned}
\|\eta(t^n)\|^2 := \|\omega(t^n) - \omega_h^n\|^2 &= \|\omega(t^n) - P_h\omega(t^n)\|^2 + \|P_h\omega(t^n) - \omega_h^n\|^2 \\
&\leq \|\omega(t^n) - P_h\omega(t^n)\|^2 + \left\|\mathbf{X}^*_{-\tau}\omega(t^{n-1}) - \mathbf{X}^*_{-\tau}\omega_h^{n-1}\right\|^2 \\
&\leq \|\omega(t^n) - P_h\omega(t^n)\|^2 + (1 + C\tau)\left\|\eta^{n-1}\right\|^2
\end{aligned}$$

$$\overset{\text{Gronwall}}{\Rightarrow} \max_{0\leq n\leq N} \|\omega(t^n) - \omega_h^n\| \leq C\tau^{-\frac{1}{2}} \max_{0\leq n\leq N} \|\omega(t^n) - P_h\omega(t^n)\| = O(\tau^{-\frac{1}{2}} h^{r+1})\,.$$

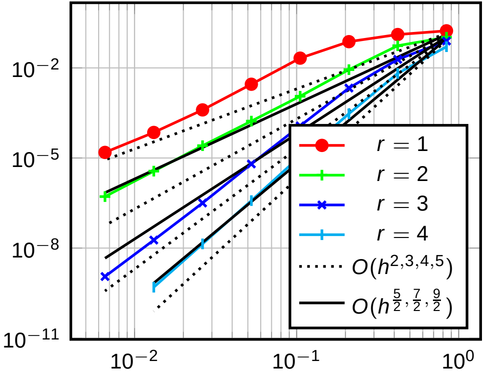
Assumptions: $\begin{cases} P_h \mathbf{X}^*_{-\tau}\omega_h \text{ can be computed exactly,} \\ \left\|\mathbf{X}^*_{-\tau}\omega\right\|^2 \leq (1 + C\tau)\,\|\omega\|^2\,. \end{cases}$ $\leftarrow \omega$ has compact sup.

# Pure Advection II

Numerical Experiment: $k = 0$, scalar advection, monitor $L^2$-error
rotating bump on unit-circle, $\beta = (-y, x)$, smooth initial data, $\tau = \frac{0.8}{\sqrt{2}}h$

# Pure Advection II

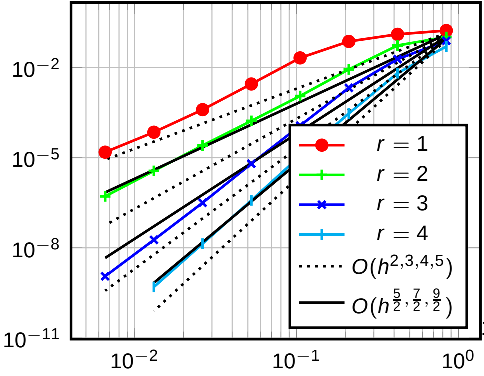
Numerical Experiment: $k = 0$, scalar advection, monitor $L^2$-error
rotating bump on unit-circle, $\beta = (-y, x)$, smooth initial data, $\tau = \frac{0.8}{\sqrt{2}} h$



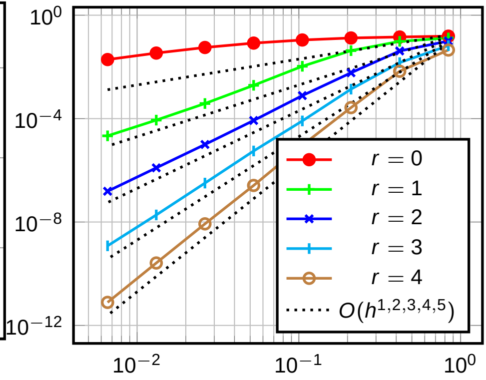

continuous finite elements

# Pure Advection II

Numerical Experiment: $k = 0$, scalar advection, monitor $L^2$-error
rotating bump on unit-circle, $\beta = (-y, x)$, smooth initial data, $\tau = \frac{0.8}{\sqrt{2}} h$



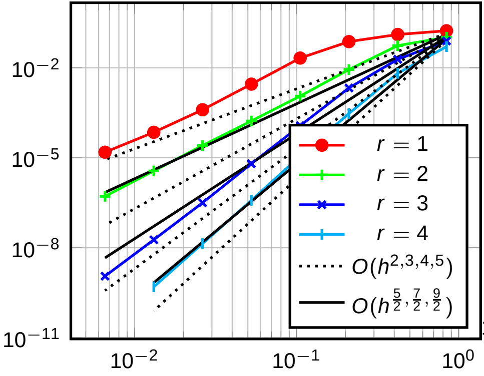
continuous finite elements



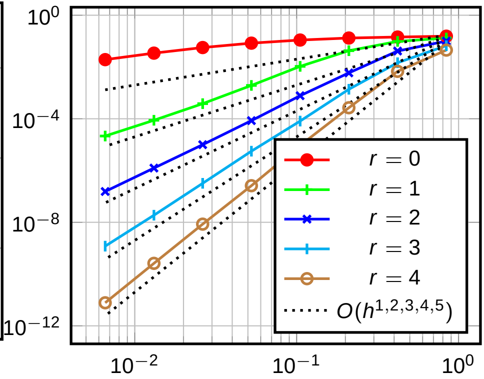
discontinuous finite elements

# Pure Advection II

Numerical Experiment: $k = 0$, scalar advection, monitor $L^2$-error
rotating bump on unit-circle, $\beta = (-y, x)$, smooth initial data, $\tau = \frac{0.8}{\sqrt{2}} h$




continuous finite elements

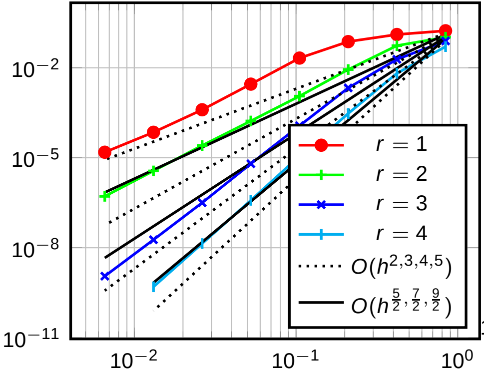



discontinuous finite elements

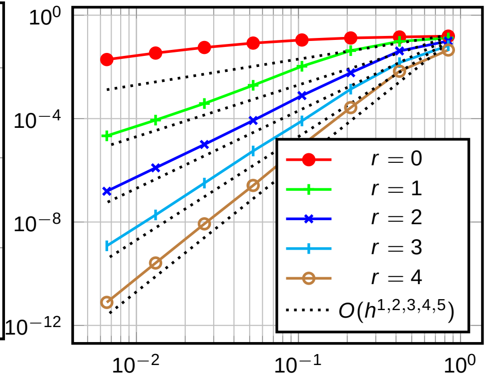
Super-convergence $O(h^{r+1})$(vs. $O(h^{-\frac{1}{2}} h^{r+1})$) except for cont. elements, $r$ even.

# Pure Advection II

Numerical Experiment: $k = 0$, scalar advection, monitor $L^2$-error
rotating bump on unit-circle, $\beta = (-y, x)$, smooth initial data, $\tau = \frac{0.8}{\sqrt{2}} h$




continuous finite elements




discontinuous finite elements

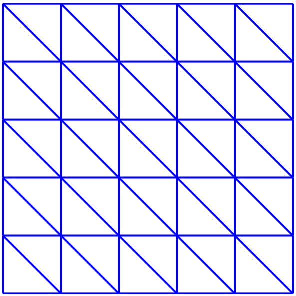
Super-convergence $O(h^{r+1})$(vs. $O(h^{-\frac{1}{2}}h^{r+1})$) except for cont. elements, $r$ even.

Computation (approximation) of $P_h \mathbf{X}^*_{-\tau} \omega_h^{n-1}$ is the essence of SL-schemes!

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau} \omega_h^{n-1}$:

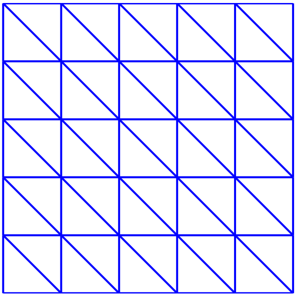
$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau} \omega_h^{n-1}, \eta_h\right)_\Omega + \ldots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$



# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$

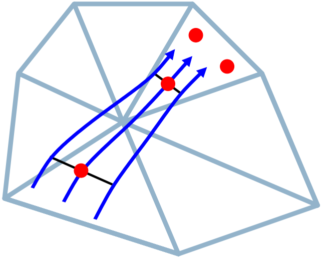


- FEM-Quadrature on $\mathcal{T}$:

# Fully Discrete SL-Methods

Numerical Experiment: Pure advection ($k = 1$, $d = 2$) and quadrature

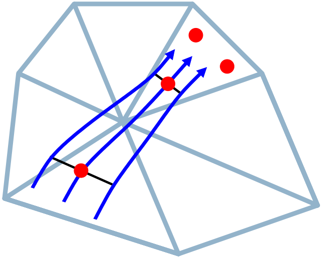
$$\left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_\Omega \approx \sum_{T \in \mathcal{T}} \left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_{T,h} \quad \omega_h \in \Lambda^1_h(\mathcal{T}), \text{ lowest order}\,.$$

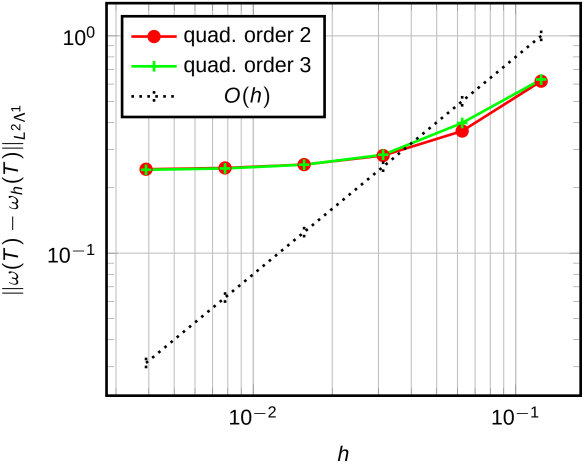


# Fully Discrete SL-Methods

Numerical Experiment: Pure advection ($k = 1$, $d = 2$) and quadrature

$$\left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_\Omega \approx \sum_{T\in\mathcal{T}} \left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_{T,h} \quad \omega_h \in \Lambda^1_h(\mathcal{T}), \text{ lowest order}.$$

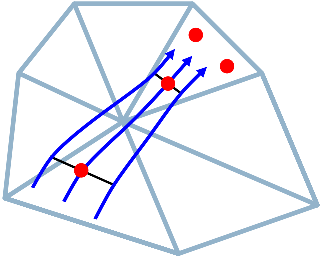




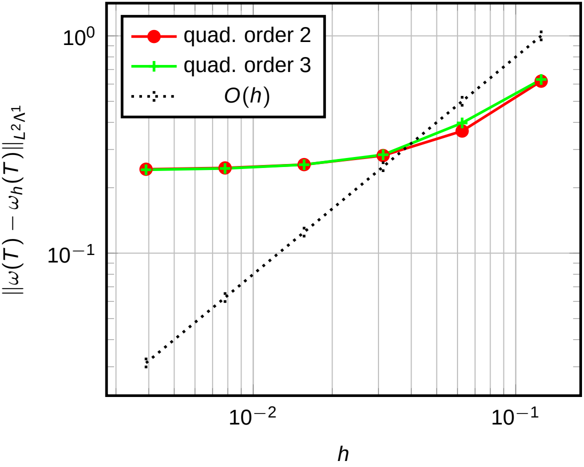
# Fully Discrete SL-Methods

Numerical Experiment: Pure advection ($k = 1$, $d = 2$) and quadrature

$$\left(\mathbf{X}_{-\tau}^{*}\omega_h, \eta_h\right)_{\Omega} \approx \sum_{T\in\mathcal{T}} \left(\mathbf{X}_{-\tau}^{*}\omega_h, \eta_h\right)_{T,h} \quad \omega_h \in \Lambda_h^1(\mathcal{T}), \text{ lowest order .}$$



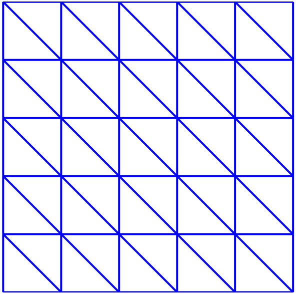
No convergence, why?



# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau} \omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau} \omega_h^{n-1}, \eta_h\right)_\Omega + \dots , \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$

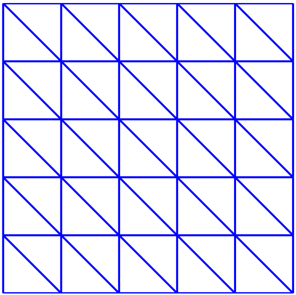


- FEM-Quadrature on $\mathcal{T}$:

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$

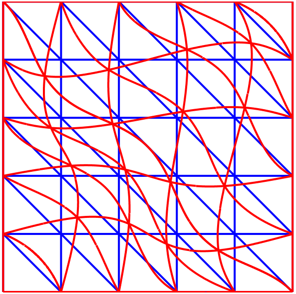


- FEM-Quadrature on $\mathcal{T}$: fails;

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau} \omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau} \omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$

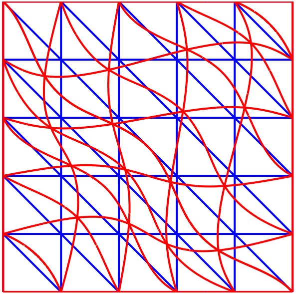


- FEM-Quadrature on $\mathcal{T}$: fails;

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau} \omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau} \omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$

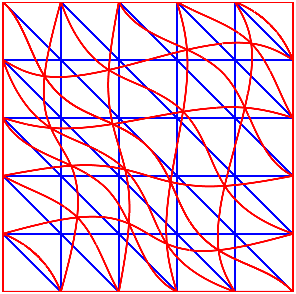


- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$



- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;
- approximate flow $\bar{\mathbf{X}}_\tau(\mathbf{x})$ :

$$\bar{\mathbf{X}}_\tau(\mathbf{x}) := \sum_i \bar{\mathbf{X}}_{\tau,i}\lambda_i(\mathbf{x}) \in (\mathcal{P}_{m_1}(\mathcal{T}))^d,$$

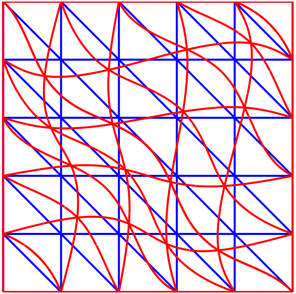
$$\text{with } \left\|\mathbf{X}_\tau(\mathbf{a}_i) - \bar{\mathbf{X}}_{\tau,i}\right\| = O(\tau^{m_2})$$

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$

finite element (FE) function



- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;
- approximate flow $\bar{\mathbf{X}}_\tau(\mathbf{x})$ :

$$\bar{\mathbf{X}}_\tau(\mathbf{x}) := \sum_i \bar{\mathbf{X}}_{\tau,i}\lambda_i(\mathbf{x}) \in (\mathcal{P}_{m_1}(\mathcal{T}))^d,$$

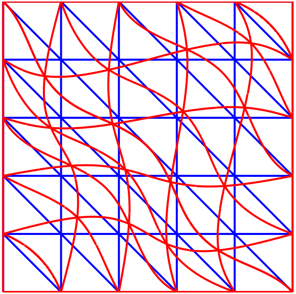
$$\text{with } \left\|\mathbf{X}_\tau(\mathbf{a}_i) - \bar{\mathbf{X}}_{\tau,i}\right\| = O(\tau^{m_2})$$

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$

finite element (FE) function



- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;
- approximate flow $\bar{\mathbf{X}}_\tau(\mathbf{x})$ :

approx. flow of FE-d.o.f.s

$$\bar{\mathbf{X}}_\tau(\mathbf{x}) := \textstyle\sum_i \bar{\mathbf{X}}_{\tau,i}\lambda_i(\mathbf{x}) \in (\mathcal{P}_{m_1}(\mathcal{T}))^d,$$

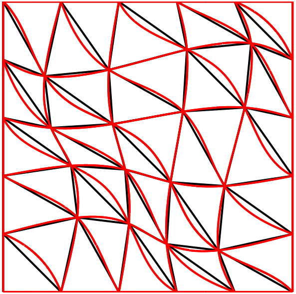
with $\left\|\mathbf{X}_\tau(\mathbf{a}_i) - \bar{\mathbf{X}}_{\tau,i}\right\| = O(\tau^{m_2})$

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1}$:

finite element (FE) function

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$



- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;
- approximate flow $\bar{\mathbf{X}}_\tau(\mathbf{x})$ :

approx. flow of FE-d.o.f.s

$$\bar{\mathbf{X}}_\tau(\mathbf{x}) := \sum_i \bar{\mathbf{X}}_{\tau,i}\lambda_i(\mathbf{x}) \in (\mathcal{P}_{m_1}(\mathcal{T}))^d,$$

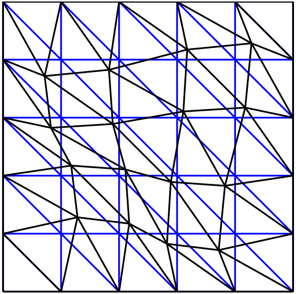
with $\left\|\mathbf{X}_\tau(\mathbf{a}_i) - \bar{\mathbf{X}}_{\tau,i}\right\| = O(\tau^{m_2})$

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1}$:

finite element (FE) function

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$



- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;
- approximate flow $\bar{\mathbf{X}}_\tau(\mathbf{x})$ :

approx. flow of FE-d.o.f.s

$$\bar{\mathbf{X}}_\tau(\mathbf{x}) := \sum_i \bar{\mathbf{X}}_{\tau,i}\lambda_i(\mathbf{x}) \in (\mathcal{P}_{m_1}(\mathcal{T}))^d,$$

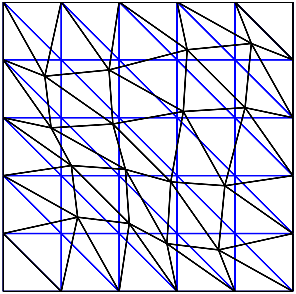
with $\left\|\mathbf{X}_\tau(\mathbf{a}_i) - \bar{\mathbf{X}}_{\tau,i}\right\| = O(\tau^{m_2})$

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau}\omega_h^{n-1}$:

finite element (FE) function

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots\,, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$



- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;
- approximate flow $\bar{\mathbf{X}}_\tau(\mathbf{x})$ :

approx. flow of FE-d.o.f.s

$$\bar{\mathbf{X}}_\tau(\mathbf{x}) := \textstyle\sum_i \bar{\mathbf{X}}_{\tau,i}\lambda_i(\mathbf{x}) \in \left(\mathcal{P}_{m_1}(\mathcal{T})\right)^d,$$

$$\text{with } \left\|\mathbf{X}_\tau(\mathbf{a}_i) - \bar{\mathbf{X}}_{\tau,i}\right\| = O(\tau^{m_2})$$

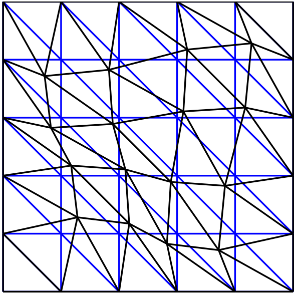
$$\Rightarrow \left\|\mathbf{X}^*_{-\tau}\omega - \bar{\mathbf{X}}^*_{-\tau}\omega\right\|_{L^2\Lambda^k} \leq C(h^{m_1}\tau + \tau^{m_2})\,\|\omega\|_{H^1\Lambda^k}$$

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau} \omega_h^{n-1}$:

finite element (FE) function

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \ldots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$



- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;
- approximate flow $\bar{\mathbf{X}}_\tau(\mathbf{x})$ :

approx. flow of FE-d.o.f.s

$$\bar{\mathbf{X}}_\tau(\mathbf{x}) := \textstyle\sum_i \bar{\mathbf{X}}_{\tau,i}\lambda_i(\mathbf{x}) \in (\mathcal{P}_{m_1}(\mathcal{T}))^d,$$

$$\text{with } \left\|\mathbf{X}_\tau(\mathbf{a}_i) - \bar{\mathbf{X}}_{\tau,i}\right\| = O(\tau^{m_2})$$

$$\Rightarrow \left\|\mathbf{X}^*_{-\tau}\omega - \bar{\mathbf{X}}^*_{-\tau}\omega\right\|_{L^2\Lambda^k} \leq C(h^{m_1}\tau + \tau^{m_2})\,\|\omega\|_{H^1\Lambda^k}$$

Pure Advection: Standard analysis & perturbations:

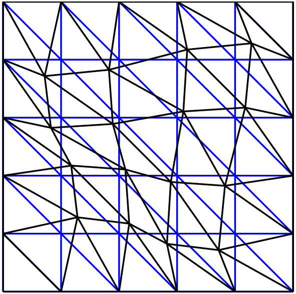
$$\Rightarrow \max_n \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k} = O(\tau^{-\frac{1}{2}}h^{r+1} + h^{m_1} + \tau^{m_2-1})$$

# Fully Discrete SL-Methods

Approximation of $P_h \mathbf{X}^*_{-\tau} \omega_h^{n-1}$:

$$(\omega_h^n, \eta_h)_\Omega = \left(\mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta_h\right)_\Omega + \dots, \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T})$$

finite element (FE) function



- FEM-Quadrature on $\mathcal{T}$: fails;
- exact projection: unfeasible;
- approximate flow $\bar{\mathbf{X}}_\tau(\mathbf{x})$ :

approx. flow of FE-d.o.f.s

$$\bar{\mathbf{X}}_\tau(\mathbf{x}) := \sum_i \bar{\mathbf{X}}_{\tau,i} \lambda_i(\mathbf{x}) \in (\mathcal{P}_{m_1}(\mathcal{T}))^d,$$

$$\text{with } \left\| \mathbf{X}_\tau(\mathbf{a}_i) - \bar{\mathbf{X}}_{\tau,i} \right\| = O(\tau^{m_2})$$

$$\Rightarrow \left\| \mathbf{X}^*_{-\tau}\omega - \bar{\mathbf{X}}^*_{-\tau}\omega \right\|_{L^2\Lambda^k} \leq C(h^{m_1}\tau + \tau^{m_2}) \left\| \omega \right\|_{H^1\Lambda^k}$$

Pure Advection: Standard analysis & perturbations:

$$\Rightarrow \max_n \left\| \omega(t^n) - \omega_h^n \right\|_{L^2\Lambda^k} = O(\tau^{-\frac{1}{2}} h^{r+1} + h^{m_1} + \tau^{m_2 - 1})$$

# Advection-Diffusion I

Goal: error $= O(\tau + \tau^{-\frac{1}{2}} h^{r+1})$

Johnson argument does not apply to the case of advection-diffusion!

# Advection-Diffusion I

Goal: error $= O(\tau + \tau^{-\frac{1}{2}} h^{r+1})$

Johnson argument does not apply to the case of advection-diffusion!

Compare:

Semi-Lagrange:

$$\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta\right)_\Omega,$$

# Advection-Diffusion I

Goal: error $= O(\tau + \tau^{-\frac{1}{2}} h^{r+1})$

Johnson argument does not apply to the case of advection-diffusion!

Compare:

| Semi-Lagrange: | Explicit Euler: |
|---|---|
| $\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}_{-\tau}^{*}\omega_h^{n-1}, \eta\right)_\Omega,$ | $\frac{1}{\tau}\left(\omega_h^n - \omega_h^{n-1}, \eta\right)_\Omega + \mathsf{a}\left(\omega_h^{n-1}, \eta\right).$ |

# Advection-Diffusion I

Goal: error $= O(\tau + \tau^{-\frac{1}{2}} h^{r+1})$

Johnson argument does not apply to the case of advection-diffusion!

Compare:

Semi-Lagrange:

$$\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}^*_{-\tau}\omega_h^{n-1}, \eta\right)_\Omega,$$

Explicit Euler:

$$\frac{1}{\tau}\left(\omega_h^n - \omega_h^{n-1}, \eta\right)_\Omega + \mathsf{a}\left(\omega_h^{n-1}, \eta\right).$$

advection
bilinear form

# Advection-Diffusion I

Goal: error $= O(\tau + \tau^{-\frac{1}{2}} h^{r+1})$

Johnson argument does not apply to the case of advection-diffusion!

Compare:

Semi-Lagrange:

$$\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}_{-\tau}^{*}\omega_h^{n-1}, \eta\right)_\Omega,$$

Explicit Euler:

$$\frac{1}{\tau}\left(\omega_h^n - \omega_h^{n-1}, \eta\right)_\Omega + \mathsf{a}\left(\omega_h^{n-1}, \eta\right).$$

advection bilinear form

But with $\omega_h^{n-1}, \eta \in \Lambda_h^k(\mathcal{T})$:

$$\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}_{-\tau}^{*}\omega_h^{n-1}, \eta\right)_\Omega = \frac{1}{\tau}\left(\omega_h^n - \omega_h^{n-1}, \eta\right)_\Omega + \underbrace{\frac{1}{\tau}\left(\omega_h^{n-1} - \mathbf{X}_{-\tau}^{*}\omega_h^{n-1}, \eta\right)_\Omega}$$

# Advection-Diffusion I

Goal: error $= O(\tau + \tau^{-\frac{1}{2}} h^{r+1})$

Johnson argument does not apply to the case of advection-diffusion!

Compare:

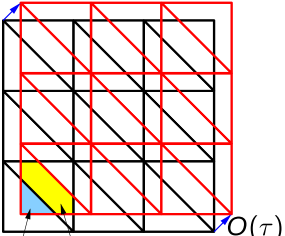
Semi-Lagrange: $\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}_{-\tau}^* \omega_h^{n-1}, \eta\right)_\Omega$,

Explicit Euler: $\frac{1}{\tau}\left(\omega_h^n - \omega_h^{n-1}, \eta\right)_\Omega + \mathsf{a}\left(\omega_h^{n-1}, \eta\right)$.

advection bilinear form

But with $\omega_h^{n-1}, \eta \in \Lambda_h^k(\mathcal{T})$:

$$\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}_{-\tau}^* \omega_h^{n-1}, \eta\right)_\Omega = \frac{1}{\tau}\left(\omega_h^n - \omega_h^{n-1}, \eta\right)_\Omega + \underbrace{\frac{1}{\tau}\left(\omega_h^{n-1} - \mathbf{X}_{-\tau}^* \omega_h^{n-1}, \eta\right)_\Omega}_{\to\ \mathsf{a}\left(\omega_h^{n-1}, \eta\right) \text{ for } \tau \to 0}$$



# Advection-Diffusion I

Goal: error $= O(\tau + \tau^{-\frac{1}{2}} h^{r+1})$

Johnson argument does not apply to the case of advection-diffusion!

Compare:

Semi-Lagrange:

$$\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}_{-\tau}^* \omega_h^{n-1}, \eta\right)_\Omega,$$

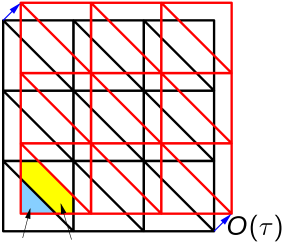
Explicit Euler:

$$\frac{1}{\tau}\left(\omega_h^n - \omega_h^{n-1}, \eta\right)_\Omega + \mathrm{a}\left(\omega_h^{n-1}, \eta\right).$$

advection bilinear form

But with $\omega_h^{n-1}, \eta \in \Lambda_h^k(\mathcal{T})$:

$$\frac{1}{\tau}\left(\omega_h^n - \mathbf{X}_{-\tau}^* \omega_h^{n-1}, \eta\right)_\Omega = \frac{1}{\tau}\left(\omega_h^n - \omega_h^{n-1}, \eta\right)_\Omega + \underbrace{\frac{1}{\tau}\left(\omega_h^{n-1} - \mathbf{X}_{-\tau}^* \omega_h^{n-1}, \eta\right)_\Omega}_{\to\ \mathrm{a}\left(\omega_h^{n-1}, \eta\right) \text{ for } \tau \to 0}$$




*For small $\tau$ Semi-Lagrange is close to explicit Euler, and error $= O(\tau + h^{r+\frac{1}{2}})$ for explicit Euler!*

# Advection-Diffusion II

Main tool is analysis of characteristic methods for stationary advection:

$$a_\tau(\omega_h, \eta_h) := \frac{1}{\tau}(\omega_h, \eta_h)_\Omega - \frac{1}{\tau}\left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_\Omega = l(\eta_h) \ ,$$

# Advection-Diffusion II

Main tool is analysis of characteristic methods for stationary advection:

$$a_\tau(\omega_h, \eta_h) := \frac{1}{\tau}(\omega_h, \eta_h)_\Omega - \frac{1}{\tau}\left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_\Omega = \mathsf{l}(\eta_h) \ ,$$

$\tau$ artificial parameter

# Advection-Diffusion II

Main tool is analysis of characteristic methods for stationary advection:

$$a_\tau(\omega_h,\eta_h) := \frac{1}{\tau}(\omega_h,\eta_h)_\Omega - \frac{1}{\tau}\left(\mathbf{X}^*_{-\tau}\omega_h,\eta_h\right)_\Omega = \mathsf{l}(\eta_h)\ ,$$

$\tau$ artificial parameter

in some mesh and $\tau$-dependent norm:

$$\|\omega\|^2_{h,\tau} := \|\omega\|^2_{L^2\Lambda^k} + \frac{1}{2\tau}\left\|\omega - \mathbf{X}^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k} \quad \stackrel{\tau\to 0}{\to} \quad \|\omega\|^2_{\mathrm{DG}} \geq \|\omega\|^2_{L^2\Lambda^k}\ .$$

# Advection-Diffusion II

Main tool is analysis of characteristic methods for stationary advection:

$$a_\tau(\omega_h, \eta_h) := \frac{1}{\tau}(\omega_h, \eta_h)_\Omega - \frac{1}{\tau}\left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_\Omega = \mathsf{l}(\eta_h) \,,$$

$\tau$ artificial parameter

in some mesh and $\tau$-dependent norm:

$$\|\omega\|^2_{h,\tau} := \|\omega\|^2_{L^2\Lambda^k} + \frac{1}{2\tau}\left\|\omega - \mathbf{X}^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k} \quad \stackrel{\tau\to 0}{\to} \quad \|\omega\|^2_{\mathrm{DG}} \geq \|\omega\|^2_{L^2\Lambda^k} \,.$$

characteristic methods for stationary advection:

stability: $C_1 \|\eta_h\|^2_{h,\tau} \leq a_\tau(\eta_h, \eta_h)\,,$

continuity: $a_\tau(\omega, \eta_h) \leq C_2\tau^{-\frac{1}{2}} \|\omega\|_{L^2\Lambda^k} \|\eta_h\|_{h,\tau} \,,$

consistency: $|a_\tau(\omega, \eta_h) - (\mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta_h)_\Omega| \leq C_3\tau \|\omega\|_{H^2\Lambda^k} \|\eta_h\|_{L^2\Lambda^k} \,,$

convergence: $\|\omega - \omega_h\|_{h,\tau} \leq C\left(h^{r+1}\tau^{-\frac{1}{2}} + \tau\right)\|\omega\|_{H^{r+1}\Lambda^k(\Omega)}\,,$

if $\frac{1}{2\tau}(\omega, \omega)_\Omega - \frac{1}{2\tau}\left(\mathbf{X}^*_{-\tau}\omega, \mathbf{X}^*_{-\tau}\omega\right)_\Omega$ positive .

# Advection-Diffusion II

Main tool is analysis of characteristic methods for stationary advection:

$$a_\tau(\omega_h, \eta_h) := \frac{1}{\tau}(\omega_h, \eta_h)_\Omega - \frac{1}{\tau}\left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_\Omega = \mathsf{l}(\eta_h)\,,$$

$\tau$ artificial parameter

in some mesh and $\tau$-dependent norm:

$$\|\omega\|^2_{h,\tau} := \|\omega\|^2_{L^2\Lambda^k} + \frac{1}{2\tau}\left\|\omega - \mathbf{X}^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k} \quad \overset{\tau\to 0}{\to} \quad \|\omega\|^2_{\mathrm{DG}} \geq \|\omega\|^2_{L^2\Lambda^k}\,.$$

characteristic methods for stationary advection:

e.g. $k = 0$: $\operatorname{div}\boldsymbol{\beta} > 0$

stability: $C_1\|\eta_h\|^2_{h,\tau} \leq a_\tau(\eta_h, \eta_h)\,,$

continuity: $a_\tau(\omega, \eta_h) \leq C_2\tau^{-\frac{1}{2}}\|\omega\|_{L^2\Lambda^k}\|\eta_h\|_{h,\tau}\,,$

consistency: $|a_\tau(\omega, \eta_h) - (\mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta_h)_\Omega| \leq C_3\tau\|\omega\|_{H^2\Lambda^k}\|\eta_h\|_{L^2\Lambda^k}\,,$

convergence: $\|\omega - \omega_h\|_{h,\tau} \leq C\left(h^{r+1}\tau^{-\frac{1}{2}} + \tau\right)\|\omega\|_{H^{r+1}\Lambda^k(\Omega)}\,,$

if $\frac{1}{2\tau}(\omega,\omega)_\Omega - \frac{1}{2\tau}\left(\mathbf{X}^*_{-\tau}\omega, \mathbf{X}^*_{-\tau}\omega\right)_\Omega$ positive $\overset{\tau\to 0}{\to}$ $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ positive.

# Advection-Diffusion II

Main tool is analysis of characteristic methods for stationary advection:

$$a_\tau(\omega_h, \eta_h) := \frac{1}{\tau}(\omega_h, \eta_h)_\Omega - \frac{1}{\tau}\left(\mathbf{X}^*_{-\tau}\omega_h, \eta_h\right)_\Omega = \mathrm{l}(\eta_h) \,,$$

$\tau$ artificial parameter

in some mesh and $\tau$-dependent norm:

$$\|\omega\|^2_{h,\tau} := \|\omega\|^2_{L^2\Lambda^k} + \frac{1}{2\tau}\left\|\omega - \mathbf{X}^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k} \quad \overset{\tau\to 0}{\to} \quad \|\omega\|^2_{\mathrm{DG}} \geq \|\omega\|^2_{L^2\Lambda^k} \,.$$

characteristic methods for stationary advection:

e.g. $k = 0$: div $\beta > 0$

stability: $C_1 \|\eta_h\|^2_{h,\tau} \leq a_\tau(\eta_h, \eta_h)\,,$

continuity: $a_\tau(\omega, \eta_h) \leq C_2 \tau^{-\frac{1}{2}} \|\omega\|_{L^2\Lambda^k} \|\eta_h\|_{h,\tau}\,,$

consistency: $|a_\tau(\omega, \eta_h) - (\mathrm{L}_{\boldsymbol\beta}\,\omega, \eta_h)_\Omega| \leq C_3 \tau \|\omega\|_{H^2\Lambda^k} \|\eta_h\|_{L^2\Lambda^k}\,,$

convergence: $\|\omega - \omega_h\|_{h,\tau} \leq C\left(h^{r+1}\tau^{-\frac{1}{2}} + \tau\right)\|\omega\|_{H^{r+1}\Lambda^k(\Omega)}\,,$

if $\frac{1}{2\tau}(\omega,\omega)_\Omega - \frac{1}{2\tau}\left(\mathbf{X}^*_{-\tau}\omega, \mathbf{X}^*_{-\tau}\omega\right)_\Omega$ positive $\overset{\tau\to 0}{\to}$ $\mathrm{L}_{\boldsymbol\beta} + \mathcal{L}_{\boldsymbol\beta}$ positive.

Ritz-Galerkin projector with respect to characteristic methods yields,

SL for advection-diffusion of $k$-forms $k > 0$: error = $\left(h^{r+1}\tau^{-\frac{1}{2}} + \tau\right)$

# Summary and Conclusions

Summary:

- Lagrangian timestepping schemes for differential forms;
- Analysis of fully discrete SL-schemes;
- Proof of uniform convergence for advection-diffusion;

Outlook:

- Analysis of interpolation based Semi-Lagrangian schemes;
- Constraint preserving schemes for hyperbolic problems;

Literatur


- Johnson; Comput. Methods Appl. Mech. Engrg. 1992; vol. 100;
- Baranger, Machmoum; M2AN 1999; vol. 33;
- Heumann, Hiptmair, Li, Xu; SAM-Report 2011-12, ETH Zürich;
- Heumann, Hiptmair; SAM-Report 2011-47, ETH Zürich;
- Heumann, Hiptmair; SAM-Report 2011-60, ETH Zürich;