A weather and climate modelling perspective on recent developments of the semi-Lagrangian method

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With thanks to...

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- **Andy White**
- **Elisabetta Cordero**
- **John Thuburn**
- **Mohamed Zerroukat**

Outline

- **The Met Office's Unified Model**
- **Advection of Scalars and Conservation**
- **Advection of Vectors and Departure Points**
- **Trajectories and Stability**

Met Office's Unified Model

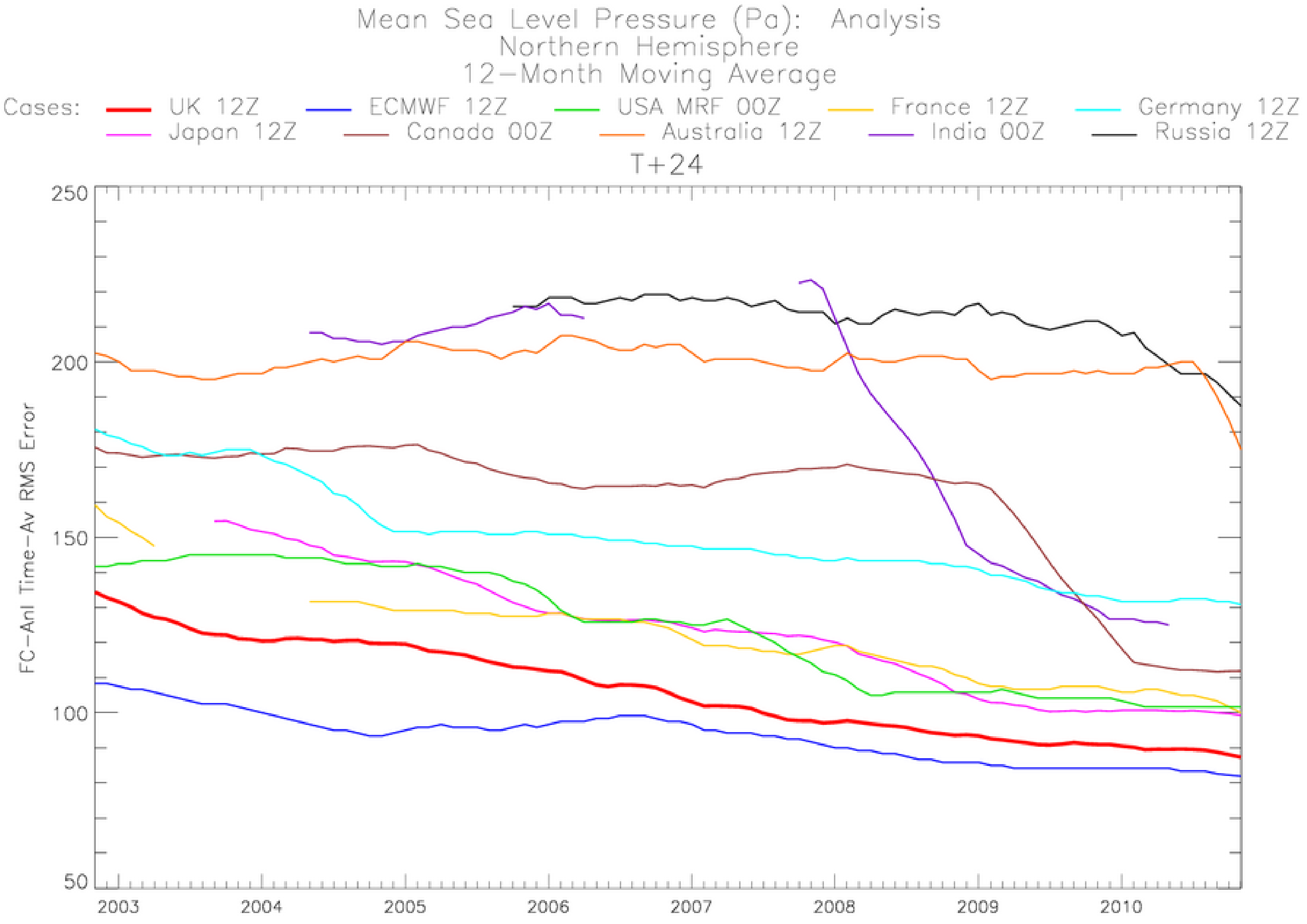
Unified Model (UM) in that it is a single model for:

- Operational weather forecasts at
  - Mesoscale (resolution approx. 12 km → 4 km → 1 km)
  - Global scale (resolution approx. 25 km)
- Global and regional climate predictions
  (resolution approx. 100km, run for 10-100 years)
- + Research mode (1km - 10m) and single column model
- 21 years old this year

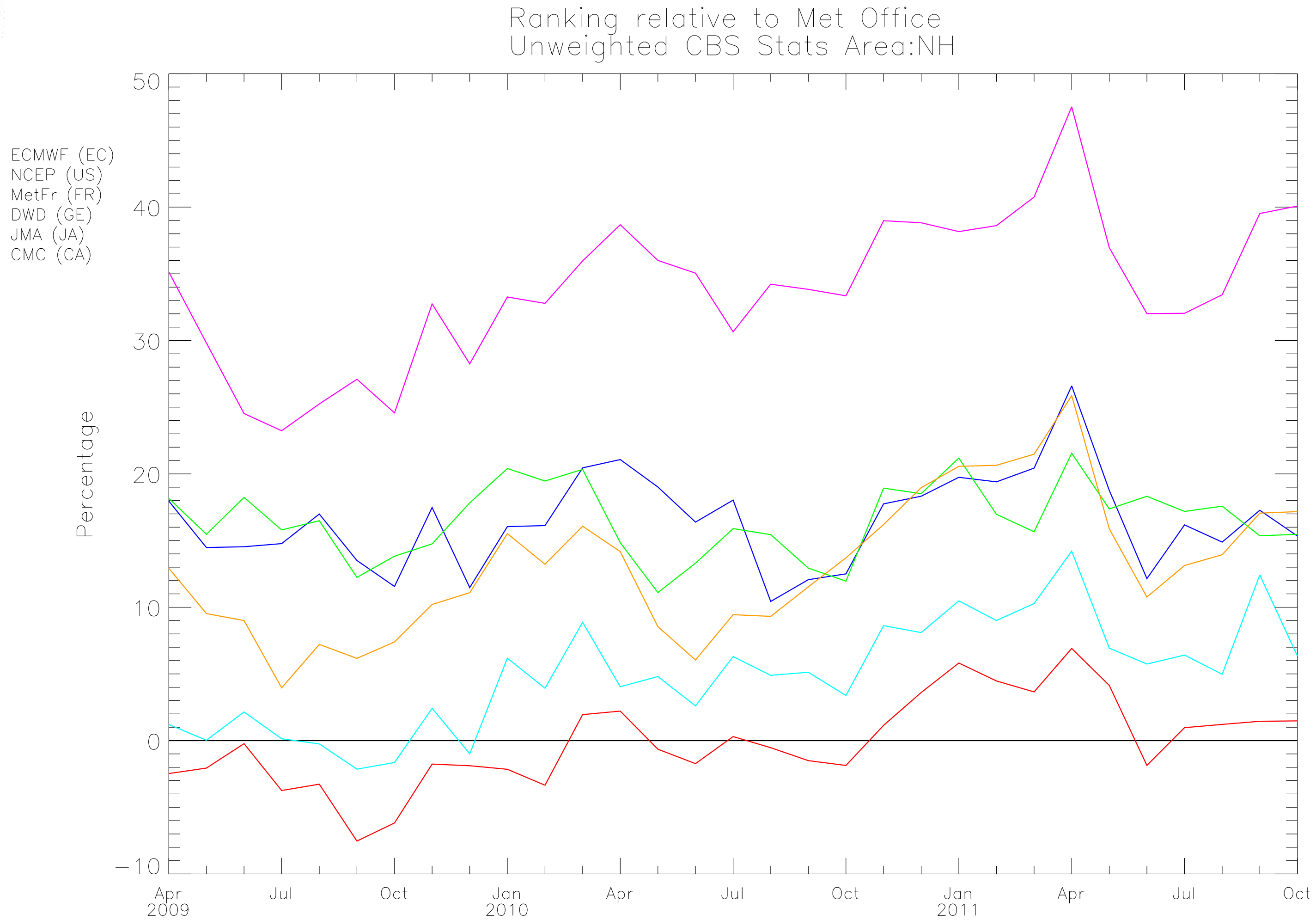
The Dynamical Core

- Deep-atmosphere nonhydrostatic
- Two-time-level semi-implicit time stepping
  (via predictor-corrector)
- C-grid in horizontal, Charney-Phillips in vertical
- Semi-Lagrangian advection of all variables...
  ...except Eulerian for dry mass conservation
- Sophisticated coupling with the physics
  (mixed sequential/parallel split - Dubal et al. (2006))

How does it compare?



And for a basket of measures...



Advection of scalars...

Semi-Lagrangian scalar advection

Consider

$$\frac{DF}{Dt} = R$$

Integrate along trajectory:

$$\int_t^{t+\Delta t} \frac{DF}{Dt} dt = \frac{F^{t+\Delta t}(x) - F^t(x - U\Delta t)}{\Delta t} = \int_t^{t+\Delta t} R dt \approx \frac{R^{t+\Delta t}(x) + R^t(x - U\Delta t)}{2}$$

i.e.

$$\left(F - \frac{\Delta t}{2} R\right)_A^{t+\Delta t} = \left(F + \frac{\Delta t}{2} R\right)_D^t$$

Pros and cons

- Semi-Lagrangian schemes allow:
  - enhanced stability and
  - accurate handling of meteorologically important slow modes

- But point-wise interpolation $\Rightarrow$ non-conservation

- Two approaches to obtaining conserving forms:
  - A posteriori correction schemes (more or less ad hoc)
  - Finite-volume approach
    $\Rightarrow$ **SLICE**: **S**emi-**L**agrangian **I**nherently **C**onserving and **E**fficient

Conservation...

SLICE

Two ingredients:

- Rewrite Eulerian flux form

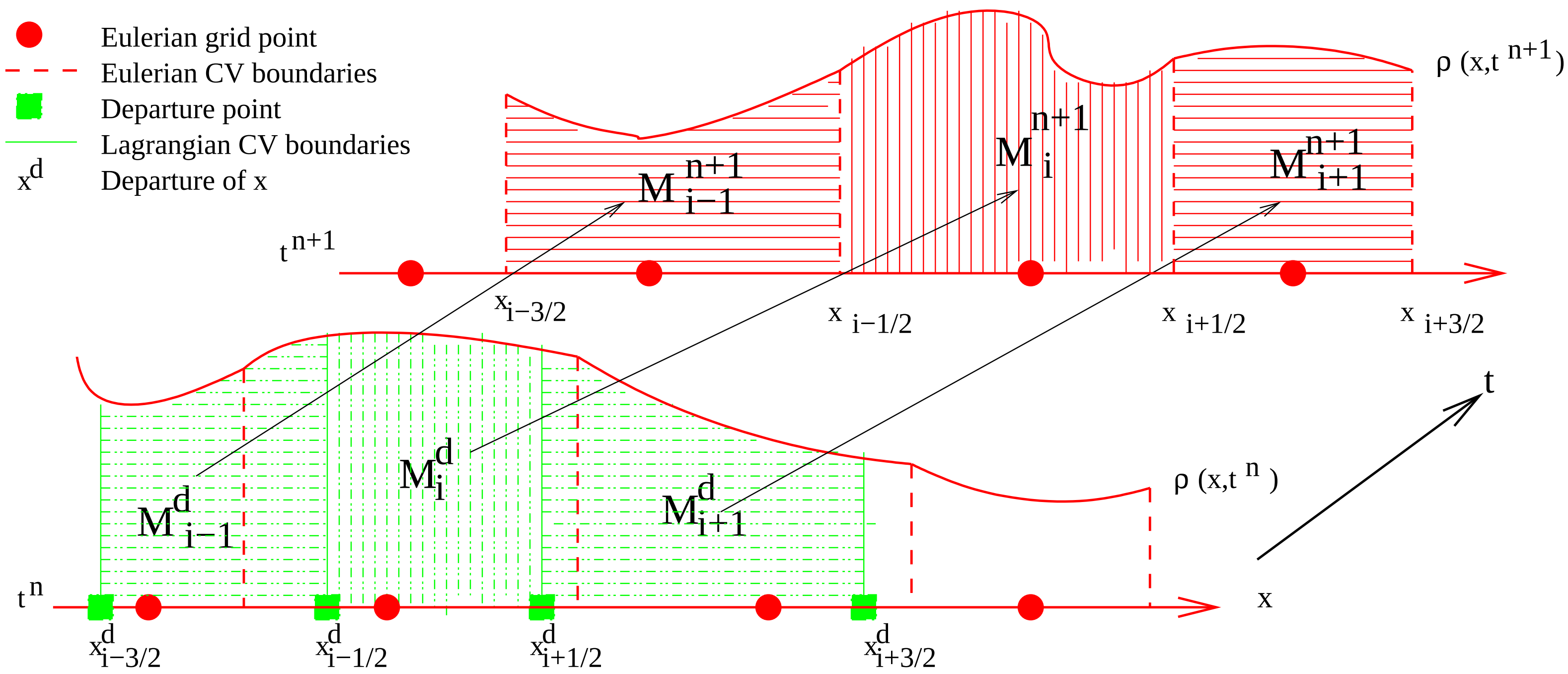
$$\frac{\partial F}{\partial t} + \nabla \cdot (\mathbf{u}F) = 0$$

in finite-volume Lagrangian form

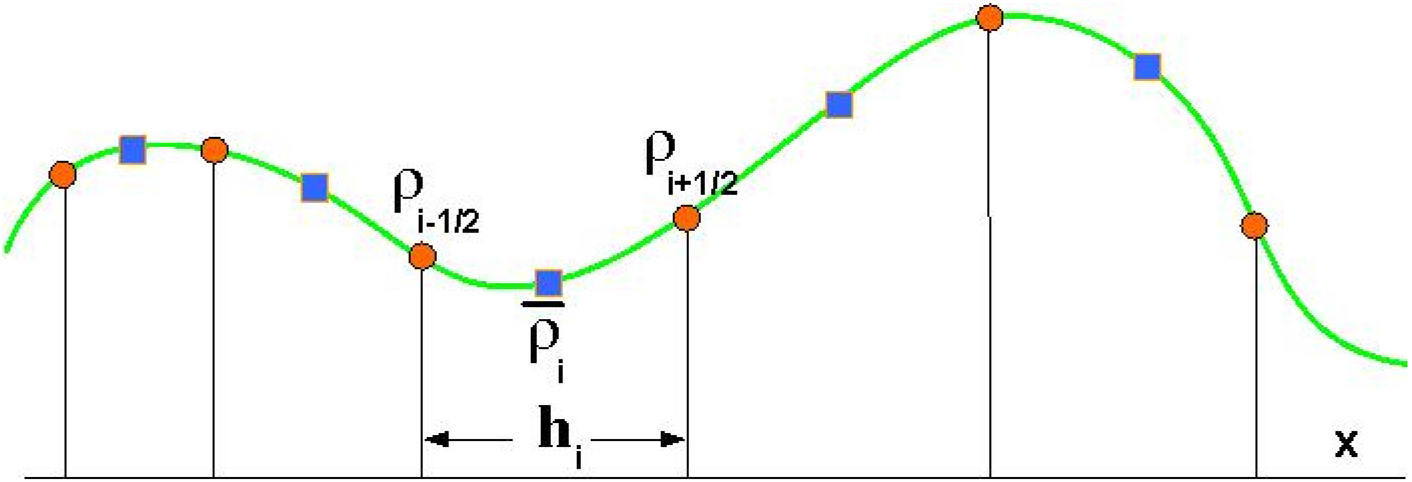
$$\frac{D}{Dt}\int_{\partial V} F\,dV = 0 \rightarrow F^{n+1} \times dV = \int_{\partial V_D} F^n dV$$

- Use Cascade remapping to enable split of $1 \times n$-dimensional redistribution into $n \times 1$-dimensional ones
  - [Cascade approach preserves characteristics of flow and hence minimises splitting error.]

SLICE-1D approach



Reconstruction



Reconstruct subgrid variation using one's favourite method:

- **Piecewise Parabolic Method**
- **Piecewise Cubic Method**
- **Parabolic Spline Method**
- **Quartic Spline Method**
- **...**

SLICE-2D

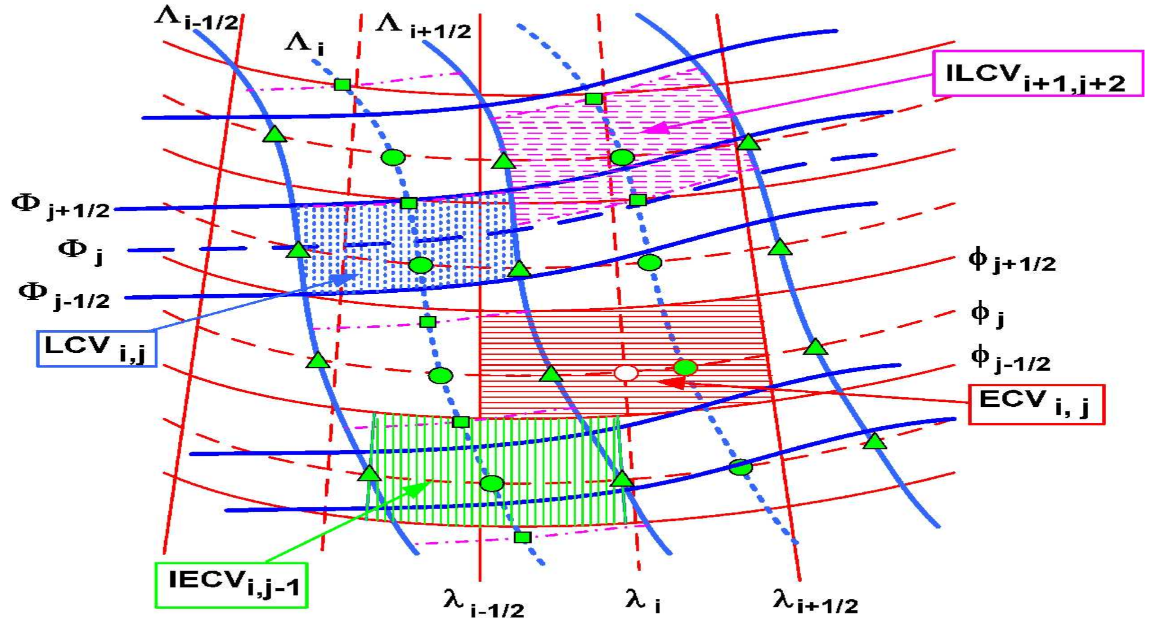
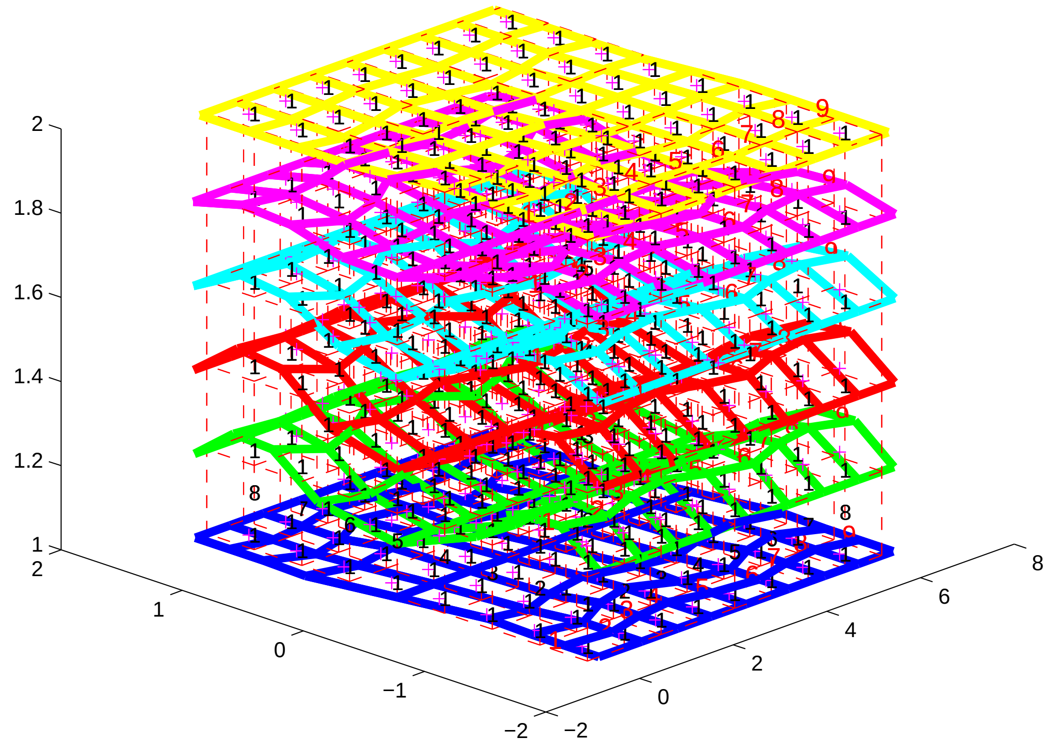


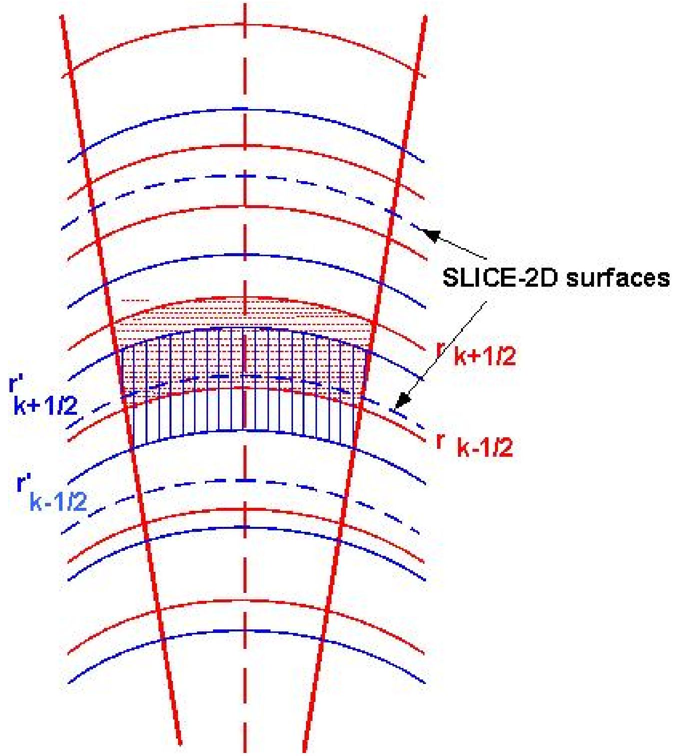
Figure 1: Superposition of Lagrangian and Eulerian grids away from poles.

SLICE-3D

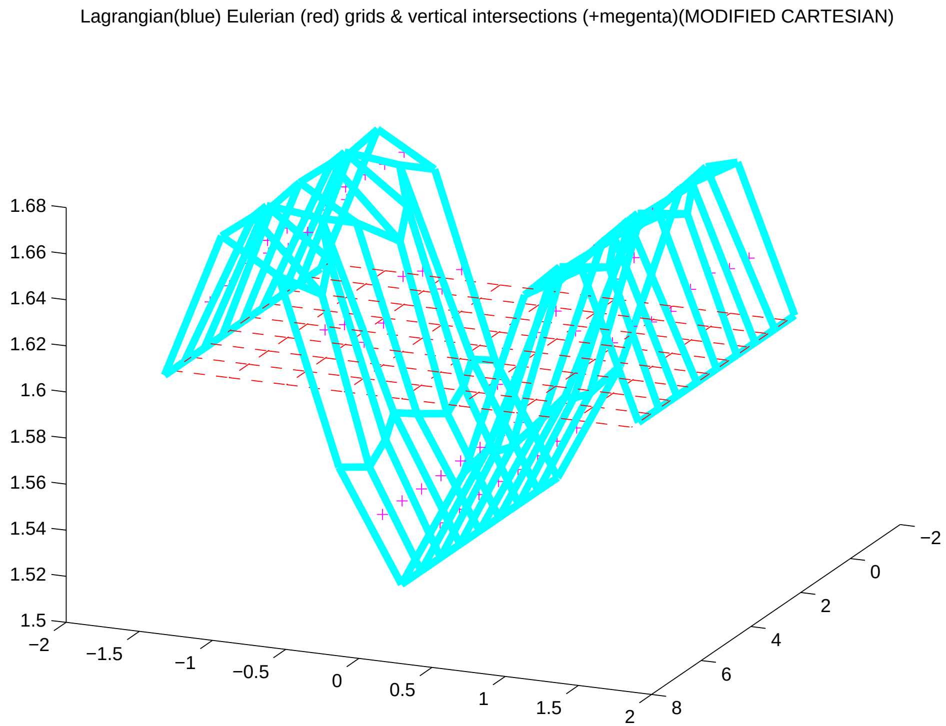
Lagrangian(blue) Eulerian (red) grids & vertical intersections (+megenta)(MODIFIED CARTESIAN)

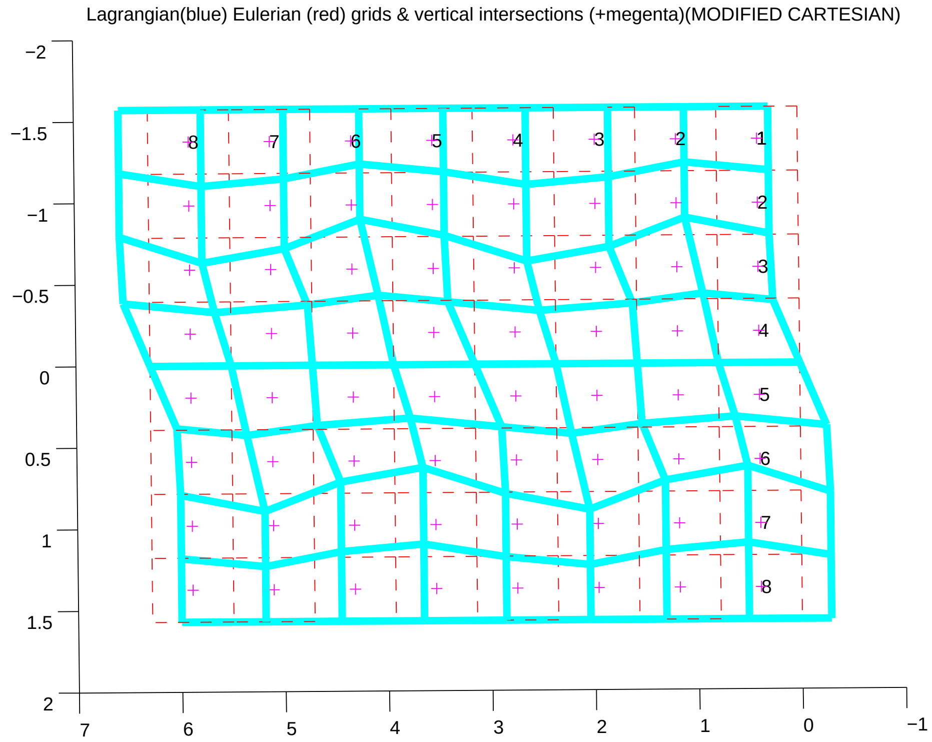


SLICE-3D (cont.)



SLICE-3D (cont.)

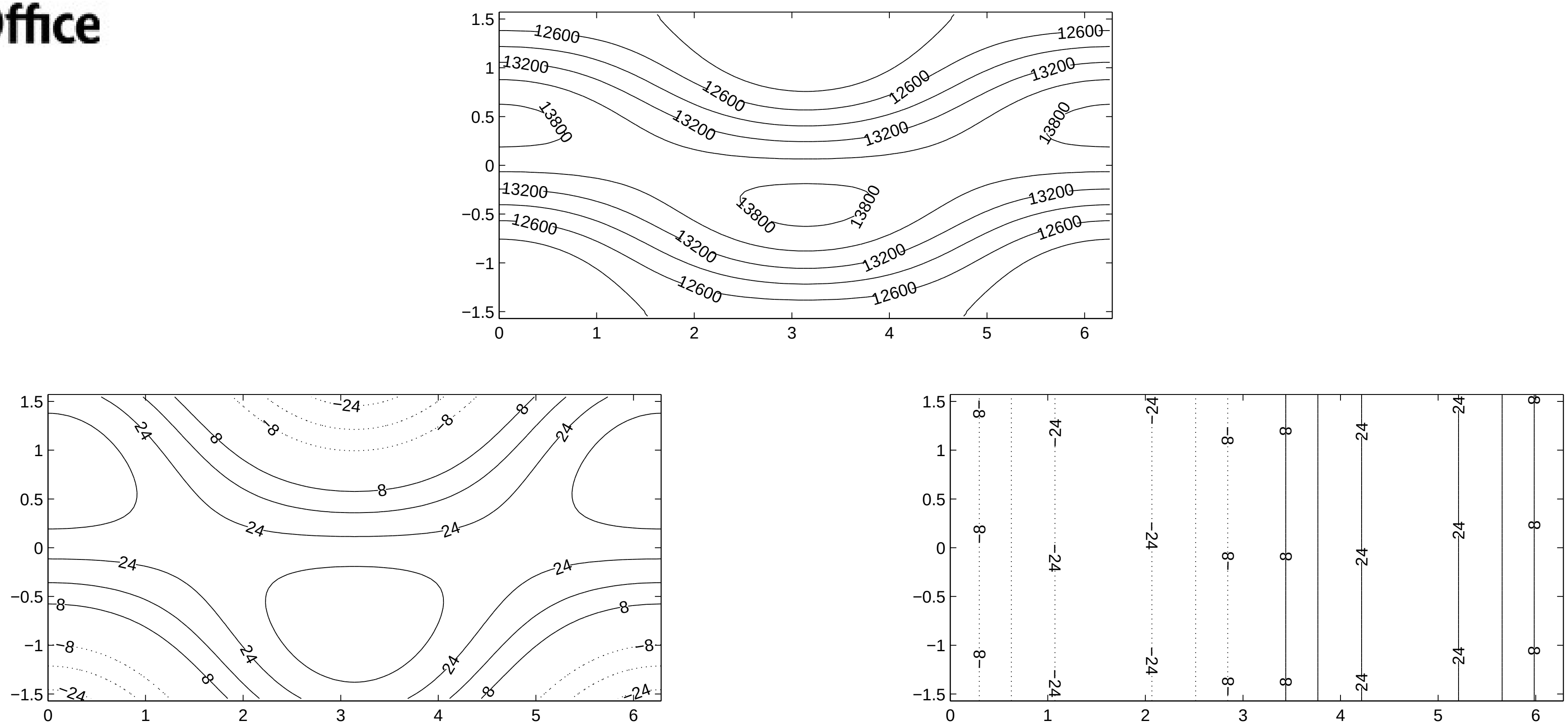




Aspects of SLICE-3D

1. No explicit calculation of complex geometry of irregular Lagrangian volumes
2. Uses C-grid staggering - essential for good wave propagation
3. Cascade in 1D in the vertical, then N SLICE-2D problems in the (deformed) horizontal
4. SLICE-2D in turn is a series of horizontal sweeps of the same 1D-remapping algorithm

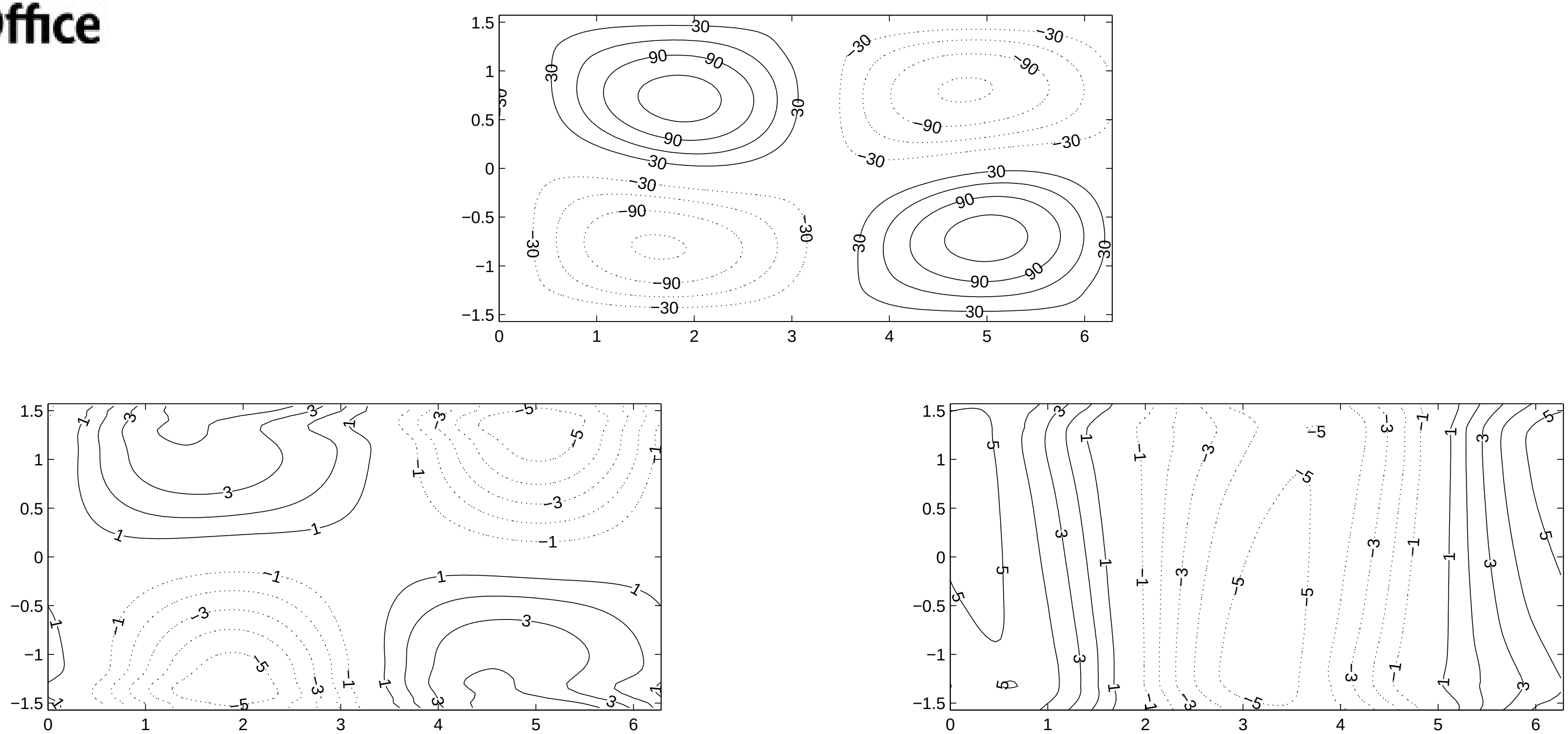
Läuter Exact Unsteady Flow - Initial Fields



(a) $\left(\Phi + \Phi^S\right)/g\ [300\,\mathrm{m}]$;

(b) $u\ \left[8\,\mathrm{m\,s^{-1}}\right]$; **(c)** $v\ \left[8\,\mathrm{m\,s^{-1}}\right]$.

Läuter Exact Unsteady Flow - Error Fields 5 Days



Mass-conserving. $\triangle t = 1\,\mathrm{h}$; $(I = 128, J = 64)$.

Contour intervals: $30\,\mathrm{m}$ **for** $\left(\Phi + \Phi^{S}\right)/g$; $1\,\mathrm{m\,s^{-1}}$ **for** u **and** v.

Läuter Exact Unsteady Flow - L2 Error Norms

$I \times J$	Δt	SLICE	SL	Δt	SLICE	SL
64×32	12	0.176E-02	0.179E-02	120	0.196E-01	0.201E-01
128×64	6	0.447E-03	0.447E-03	60	0.507E-02	0.510E-02
256×128	3	0.113E-03	0.111E-03	30	0.123E-02	0.123E-02

5 day integration:
Mass-conserving (SLICE) cf. non-conserving (SL)

Resolution given by $I \times J$; Δt in minutes

Advection of vectors...

Semi-Lagrangian vector advection in curved geometry

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For the vector equation

$$\frac{D\mathbf{v}}{Dt} = \mathbf{S}$$

Integrate along trajectory to obtain, as before:

$$\left(\mathbf{v} - \frac{\Delta t}{2}\mathbf{S}\right)_A^{t+\Delta t} = \left(\mathbf{v} + \frac{\Delta t}{2}\mathbf{S}\right)_D^{t}$$

All well and good...

But what of components?

The Problem of Curvilinear Coordinates...

The familiar problem that

$$\mathbf{i}.\frac{D\mathbf{v}}{Dt} \equiv \mathbf{i}.\frac{D}{Dt}\left(u\mathbf{i}+v\mathbf{j}+w\mathbf{k}\right) \neq \frac{D\left(\mathbf{i}.\mathbf{v}\right)}{Dt} \quad \left[= \frac{Du}{Dt}\right]$$

translates into the SL equivalent that

$$\boxed{\mathbf{i}_A.\left(\mathbf{v}_D\right) \neq \left(\mathbf{i}_A.\mathbf{v}_A\right)_D \quad [= u_D]}$$

In fact

$$\mathbf{v}_D = u_{\mathscr{D}}\mathbf{i}_D + v_{\mathscr{D}}\mathbf{j}_D + w_{\mathscr{D}}\mathbf{k}_D$$

so that, e.g.,

$$\mathbf{i}_A.\left(\mathbf{v}_D\right) = u_{\mathscr{D}}\left(\mathbf{i}_A.\mathbf{i}_D\right) + v_{\mathscr{D}}\left(\mathbf{i}_A.\mathbf{j}_D\right) + w_{\mathscr{D}}\left(\mathbf{i}_A.\mathbf{k}_D\right)$$

The Matrix Formulation

- Extending this to all three directions:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_D = \mathbf{M} \begin{pmatrix} u_{\mathscr{D}} \\ v_{\mathscr{D}} \\ w_{\mathscr{D}} \end{pmatrix}$$

where

$$\mathbf{M} \equiv \begin{pmatrix} \mathbf{i}_A.\mathbf{i}_D & \mathbf{i}_A.\mathbf{j}_D & \mathbf{i}_A.\mathbf{k}_D \\ \mathbf{j}_A.\mathbf{i}_D & \mathbf{j}_A.\mathbf{j}_D & \mathbf{j}_A.\mathbf{k}_D \\ \mathbf{k}_A.\mathbf{i}_D & \mathbf{k}_A.\mathbf{j}_D & \mathbf{k}_A.\mathbf{k}_D \end{pmatrix}$$

- $\mathbf{M}$ transforms
  - **from:** vector components in the departure-point frame
  - **to:** vectors in the arrival-point frame

The Consequence...

Component form of momentum equations can be written as:

$$\left(\mathbf{v} - \frac{\Delta t}{2}\mathbf{S}\right)^{t+\Delta t}_{\mathscr{A}} = \mathbf{M}\left(\mathbf{v} + \frac{\Delta t}{2}\mathbf{S}\right)^{t}_{\mathscr{D}}$$

where

$$\mathbf{X}_{\mathscr{A}} \equiv \left(X_{\mathscr{A}},\, Y_{\mathscr{A}},\, Z_{\mathscr{A}} \right)^T$$

$$\mathbf{X}_{\mathscr{D}} \equiv \left(X_{\mathscr{D}},\, Y_{\mathscr{D}},\, Z_{\mathscr{D}} \right)^T$$

- **No explicit metric terms**
- **No singularity at the pole**

Where have the Metric Terms Gone?

- SL form holds for finite displacements

- Reduces to Eulerian form as displacement ($\Delta\lambda \equiv \lambda_A - \lambda_D$) and $\Delta t \to 0$:

- In this limit:

$$\sin\Delta\lambda \to \Delta\lambda\,, \;\; \frac{\Delta\lambda}{\Delta t} \to \frac{u_{\mathscr{A}}}{r_A \cos\phi_A}, \;\; u_{\mathscr{D}} \to u_{\mathscr{A}}\,, \;\; v_{\mathscr{D}} \to v_{\mathscr{A}}$$

and so:

$$\boxed{M_{12} v_{\mathscr{D}} = v_{\mathscr{D}} \sin\phi_D \sin\Delta\lambda \to \frac{u_{\mathscr{A}} v_{\mathscr{A}} \tan\phi_A}{r_A}\Delta t}$$

- Each off-diagonal element of M generates one metric term

Departure points...

Rotation Matrix for Departure Points?

- They are also governed by a vector equation

$$\frac{D\mathbf{x}}{Dt} = \mathbf{v}$$

- So can we apply consistent approach (and avoid polar singularity issues)?

- Discrete vector form is

$$\mathbf{x}_D = \mathbf{x}_A - \frac{\Delta t}{2}\left(\mathbf{v}_A + \mathbf{v}_D\right)$$

Rotation Matrix for Departure Points II

As before component form can be written as

$$\mathbf{M}\mathbf{x}_{\mathscr{D}} = \mathbf{x}_{\mathscr{A}} - \frac{\Delta t}{2}\left(\mathbf{v}_{\mathscr{A}} + \mathbf{M}\mathbf{v}_{\mathscr{D}}\right)$$

Or, since $\mathbf{x}_{\mathscr{A}} = \left(0,\ 0,\ r_A\right)^T$ **and** $\mathbf{x}_{\mathscr{D}} = \left(0,\ 0,\ r_D\right)^T$**:**

$$M_{13}r_{\mathscr{D}} = -\frac{\Delta t}{2}\left[u_{\mathscr{A}} + \left(M_{11}u_{\mathscr{D}} + M_{12}v_{\mathscr{D}} + M_{13}w_{\mathscr{D}}\right)\right]$$

$$M_{23}r_{\mathscr{D}} = -\frac{\Delta t}{2}\left[v_{\mathscr{A}} + \left(M_{21}u_{\mathscr{D}} + M_{22}v_{\mathscr{D}} + M_{23}w_{\mathscr{D}}\right)\right]$$

$$M_{33}r_{\mathscr{D}} = r_A - \frac{\Delta t}{2}\left[w_{\mathscr{A}} + \left(M_{31}u_{\mathscr{D}} + M_{32}v_{\mathscr{D}} + M_{33}w_{\mathscr{D}}\right)\right]$$

A Local Cartesian Transform Approach

The result can be cast into a local Cartesian form:

$$X_{DA} = -\frac{\Delta t}{2}(U_{\mathscr{A}} + U_{DA})$$

$$Y_{DA} = -\frac{\Delta t}{2}(V_{\mathscr{A}} + V_{DA})$$

$$Z_{DA} = r_A - \frac{\Delta t}{2}(W_{\mathscr{A}} + W_{DA})$$

where

$$\mathbf{X}_{DA} \equiv \mathbf{M}\mathbf{x}_{\mathscr{D}},$$

and

$$\mathbf{V}_{DA} \equiv \mathbf{M}\mathbf{v}_{\mathscr{D}}$$

are Departure point coordinates and velocities as seen in the Arrival-point Cartesian system

But...

Consider

$$\mathbf{x}_D = \mathbf{x}_A - \frac{\Delta t}{2}(\mathbf{v}_A + \mathbf{v}_D)$$

At the equator and for $v = 0$

$$\left(r_{\mathscr{A}} - \frac{\Delta t}{2} w_{\mathscr{A}}\right)^2 + \left(\frac{\Delta t}{2} u_{\mathscr{A}}\right)^2 = \left(r_{\mathscr{D}} + \frac{\Delta t}{2} w_{\mathscr{D}}\right)^2 + \left(\frac{\Delta t}{2} u_{\mathscr{D}}\right)^2$$

Consider small perturbations

$$(u, v, w) = (U + u', 0, w')$$

then

$$r_{\mathscr{A}} - r_{\mathscr{D}} \simeq \frac{\Delta t}{2}(w_{\mathscr{A}} + w_{\mathscr{D}}) + \boxed{\left(\frac{\Delta t}{2}\right)^2 \frac{2U}{r_{\mathscr{A}}}(u_{\mathscr{D}} - u_{\mathscr{A}})}$$

Fuller analysis of compressible case shows instability $\propto UN^2$

So...

Project onto the arrival spherical shell, i.e. write

$$\frac{D\mathbf{r}}{Dt} = \frac{D}{Dt}(r\mathbf{k}) = \mathbf{k}\frac{Dr}{Dt} + r\frac{D\mathbf{k}}{Dt}, \quad \text{where} \quad \mathbf{k} \equiv \mathbf{r}/r$$

This leads to

$$\frac{dr}{dt} = w$$

and

$$\frac{d\mathbf{k}}{dt} = \frac{\mathbf{v}^H}{r}$$

Solve by constraining departure point to remain on arrival sphere:

$$\mathbf{x}_D = \mathbf{x}_A - \frac{\Delta t}{2}(\mathbf{v}_A + \mathbf{v}_D) + \mathscr{B}(\mathbf{x}_A + \mathbf{x}_D)$$

Procedure

1. Cartesian transformation as before

$$\mathbf{X}_{DA} \equiv \mathbf{M}\mathbf{x}_{\mathscr{D}}$$

2. and

$$\mathbf{X}^H_{\mathscr{D}} = -\left(\frac{1+M_{33}}{2}\right)\frac{\Delta t}{2}\left(\mathbf{V}^H_{\mathscr{A}} + \mathbf{V}^H_{\mathscr{D}}\right)$$

3. but velocities transform using momentum shallow-atmosphere rotation matrix

$$\mathbf{V}_{DA} \equiv \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \mathbf{v}_{\mathscr{D}}$$

where

$$p = \frac{M_{11}+M_{22}}{1+M_{33}}, \quad q = \frac{M_{12}-M_{21}}{1+M_{33}}$$

What does it all mean?

Single set of equations:

$$\boxed{(u_i - \alpha\Delta t\Psi_i)^{n+1} = M_{ij}(u_j + \beta\Delta t\Psi_j)^n_D}$$

with similar ones for departure point equations

Identical numerical system encompassing hierarchy of model geometries:

1. **Spheroidal**
2. **Spherical**
3. **Shallow-atmosphere approximation (non-Euclidean geometry)**
4. **Cartesian**

Trajectories and stability...

A prototypical problem

Consider coupled problem

$$\frac{DW}{Dt} = B - \gamma z \ \text{ and } \ \frac{DB}{Dt} = 0$$

Linearize about

$$(W, B) = (0, \gamma z) + (w, b)$$

to obtain

$$\frac{\partial w}{\partial t} = b \text{ and } \frac{\partial b}{\partial t} + \gamma w = 0$$

Solution is given by

$$\boxed{\frac{\partial^2 w}{\partial t^2} + \gamma w = 0}$$

Semi-Implicit Semi-Lagrangian discretization

Semi-implicit scheme "linearizes" about specified state, i.e.

$$B(z,t) = b^*(z,t) + \gamma^* z$$

so that

$$b^* = b + (\gamma - \gamma^*) z$$

Discretize ($D/Dt \to \mathscr{D}/\mathscr{D}t$, $\partial/\partial t \to \delta/\delta t$, $F \to \overline{F}^t$ **average along trajectory)**

$$\frac{DB}{Dt} = 0 \xrightarrow{\text{SISL}} \frac{\mathscr{D}b^*}{\mathscr{D}t} + \gamma^* \overline{w}^t = 0$$

$$\frac{\mathscr{D}b^*}{\mathscr{D}t} + \gamma^* \overline{w}^t = 0 \xrightarrow{\text{linear}} \frac{\delta b}{\delta t} + (\gamma - \gamma^*) \left(\frac{z_A - z_D}{\Delta t} \right) + \gamma^* \overline{w}^t = 0$$

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Departure point equation

Requires mid-point velocity

$$\frac{z_A - z_D}{\Delta t} = w^M = w^M\left(w^{n+1}, w^n, w^{n-1}, \ldots\right)$$

So finally

$$\frac{\delta w}{\delta t} = \overline{b}^t$$

and

$$\frac{\delta b}{\delta t} + (\gamma - \gamma^*)\, w^M + \gamma^* \overline{w}^t = 0$$

Solution is given by

$$\boxed{\frac{\delta^2 w}{\delta t^2} + \gamma^* \overline{w}^{tt} + \overline{(\gamma - \gamma^*)\, w^M}^t = 0}$$

Consequence

When $\gamma^* \neq \gamma$ estimate w^M for mid-point velocity in trajectory calculation is critical to accuracy and stability.

AOK if

$$w^M \equiv \overline{w}^t$$

but not for extrapolations such as:

$$w^M \equiv w^n$$

or

$$w^M \equiv \frac{3}{2}w^n - \frac{1}{2}w^n$$

An example from Cordero et al (2005)

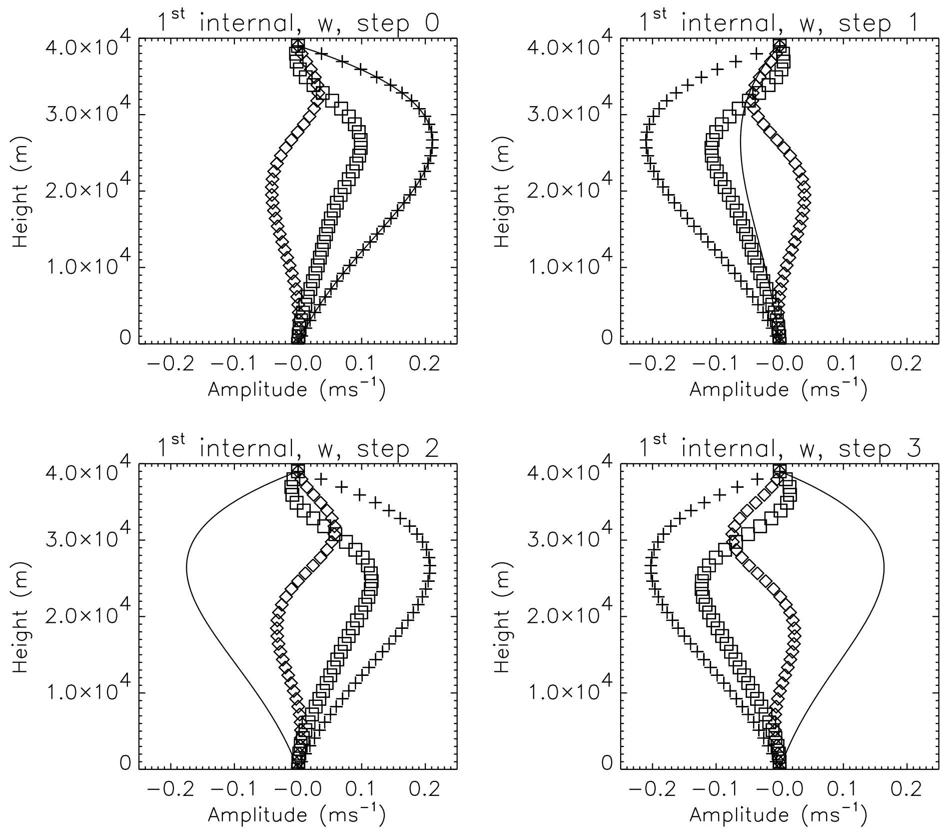

Figure 2: Linear evolution of the w component of the first internal mode of the operational problem computed for $T_s = 340$K. The solid curve denotes the analytic mode, squares and diamonds denote the mode computed using extrapolated trajectory schemes t_1 (1-term) and t_2 (2-term) respectively. For reference, the mode computed using the interpolated trajectory scheme is also plotted and it is denoted by plus symbols.

Summary

- Semi-Lagrangian scheme is powerful, especially when combined with the semi-implicit scheme

- Conservation is an issue - schemes exist (SLICE) but they come at a price

- A particular attraction is SL applied to vectors - ‘hides’ metric terms and makes for straightforwardly switchable code

- But! In terms of stability scheme is not as innocuous as may seem...

Thank you!

Questions?

Further details can be found in:

- Impact of semi-Lagrangian trajectories on the discrete normal modes of a non-hydrostatic vertical column model
  Cordero, Wood and Staniforth. 2005 Q.J.R.Meteorol.Soc. 131 pp 93-108
- SLICE-S: a semi-Lagrangian Inherently Conserving and Efficient scheme for transport problems on the Sphere
  Zerroukat, Wood and Staniforth. 2004 Q.J.R.Meteorol.Soc. 130 pp 2649-2664
- Rotation matrix treatment of vector equations in semi-Lagrangian models of the atmosphere I: Momentum equation
  Staniforth, White and Wood. 2010 Q.J.R.Meteorol.Soc. 136 pp 497-506
- Rotation matrix treatment of vector equations in semi-Lagrangian models of the atmosphere II: Kinematic equation
  Wood, White and Staniforth. 2010 Q.J.R.Meteorol.Soc. 136 pp 507-516
- A geometrical view of the shallow-atmosphere approximation, with application to the semi-Lagrangian departure point calculation
  Thuburn and White 2012 submitted to Q.J.R.Meteorol.Soc.