Evolution Models for Mass Transportation Problems

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We present a dynamical formulation of mass transportation problems, using a Benamou-Brenier approach which consists in the minimization of a suitable functional depending on the density and on the velocity of the transport flow, coupled with the continuity equation.

The goal is to include in this formulation the cases of congestion and concentration effects, which occur in several models from the applications.

• Congestion effects for instance occur in the simulation of traffic flows with high density and of movement of crowds under panic effects.

Due to the congestion, the transport rays tend to be far each other.

• Concentration effects for instance occur in several models of branching transportation, as roots of trees, roads, communication networks, delta of rivers, blood vessels.

The main tool is a good comprehension of lower semicontinuous functionals defined on the space of measures, studied in a series of papers by Bouchitté-Buttazzo:

- Nonlinear Anal. 1990
- Ann.IHP Anal.NonLin. 1992
- Ann.IHP Anal.NonLin. 1993

Applications to mass transportation problems:
Brancolini-Buttazzo-Santambrogio JEMS 2006
Buttazzo-Jimenez-Oudet SIAM JCO 2009
Brasco-Buttazzo-Santambrogio preprint 2010
available at `http://cvgmt.sns.it`

Example 1 - Lebesgue For L^p measures $\mu = u\,dx$ define

$$F(\mu) = \int_\Omega |u|^p\,dx \qquad p > 1.$$

Example 2 - Dirac For discrete measures $\mu = \sum m_k \delta_{x_k}$ define

$$F(\mu) = \sum_k |m_k|^\alpha = \int_\Omega |\mu(x)|^\alpha\,d\#(x) \qquad \alpha < 1.$$

Example 3 - Mumford-Shah For measures with no Cantor part $\mu = u\,dx + \sum m_k \delta_{x_k}$ define

$$F(\mu) = \int_\Omega |u|^p\,dx + \int_\Omega |\mu(x)|^\alpha\,d\#(x) \quad p > 1,\ \alpha < 1.$$

A full classification of all weakly* l.s.c. functionals on $\mathcal{M}(\Omega)$ (translation invariant for simplicity), which are local, is the following

$$
\begin{aligned}
F(\mu) = \ & \int_\Omega f(\mu^a)\, dm(x) && \text{Lebesgue part} \\
& + \int_\Omega f^\infty(\mu^c) && \text{Cantor part} \\
& + \int_\Omega g(\mu(x))\, d\#(x) && \text{Dirac part}
\end{aligned}
$$

where f is convex, f^∞ is its recession function, g is subadditive, and the compatibility condition $f^\infty = g^0$ holds.

In Example 1 $f(z) = |z|^p$, $g(z) \equiv +\infty$;
In Example 2 $f(z) \equiv +\infty$, $g(z) = |z|^\alpha$;
In Example 3 $f(z) = |z|^p$, $g(z) = |z|^\alpha$.

Previous attempts have been made to model concentration/congestion effects:

- Q. Xia (2003) through the minimization of a suitable functional defined on currents;
- V. Caselles, J. M. Morel, S. Solimini, . . . (2002) through a kind of analogy of fluid flow in thin tubes;
- A. Brancolini, G. Buttazzo, F. Santambrogio (2006) through geodesic curves in the space of measures.

The path functionals approach consists in studying the evolution of densities as a curve in the space of probabilities $\mathcal{P}(\Omega)$ endowed with the Wasserstein distance, which minimizes a kind of length functional:

$$\mathcal{L}(\mu) = \int_0^1 J(\mu(t))|\mu'(t)|_W \, dt.$$

Here $|\mu'|_W$ is the metric derivative in the Wasserstein space. In a general (X, d) space the definition of the metric derivative is

$$|x'(t)|_X = \lim_{\varepsilon \to 0} \frac{d\Big(x(t+\varepsilon), x(t)\Big)}{\varepsilon} .$$

Theorem *Let X be a compact metric space (or closed bounded subsets of X are compact), let $x_0, x_1 \in X$ and consider*

$$\mathcal{L}(\phi) = \int_0^1 J(\phi(t))|\phi'(t)|_X \, dt.$$

Assume that
i) J is lower semicontinuous in X;
ii) $J \geq c$ with $c > 0$;
iii) $\mathcal{L}(\phi) < +\infty$ for at least one curve ϕ joining x_0 to x_1.
Then there exists an optimal path for the problem

$$\min \left\{ \mathcal{L}(\phi) \ : \ \phi(0) = x_0, \ \phi(1) = x_1 \right\}.$$

Take now X the Wasserstein space of probabilities on Ω (a compact subset of $\mathbf{R}^N$).

In the diffusion/congestion case

$$J(\mu) = \int_\Omega |u|^p \, dx \text{ for } \mu = u \, dx, \quad p > 1$$

• Two measures μ_0, μ_1 with L^p densities can be joined by a path $\mu(t)$ of finite minimal cost $\mathcal{L}$.

• If $p < 1 + 1/N$ every μ_0, μ_1 can be joined by a path $\mu(t)$ of finite minimal cost $\mathcal{L}$, with counterexamples if $p \geq 1 + 1/N$.

In the concentration/branching case:

$$J(\mu) = \sum_k |m_k|^\alpha \text{ for } \mu = \sum m_k \delta_{x_k}, \quad \alpha < 1$$

• Two discrete measures μ_0, μ_1 can be joined by a path $\mu(t)$ of finite minimal cost $\mathcal{L}$.

• If $\alpha > 1 - 1/N$ every μ_0, μ_1 can be joined by a path $\mu(t)$ of finite minimal cost $\mathcal{L}$, with counterexamples if $\alpha \leq 1 - 1/N$.

A coefficient $J(\mu)$ of Lebesgue type then provides a congestion functional, while $J(\mu)$ of Dirac type gives a model for describing concentrations.

Some refinements of the path theory approach have been made in:

L. Brasco, F. Santambrogio DCDS (2011)

L. Brasco Ann. Mat. Pura Appl. (2010)

L. Brasco Ph.D. Thesis, U.Pisa + U.Paris-Dauphine, 2010.

In this presentation however we adopt a different point of view, introduced by Brenier to give a dynamic formulation of mass transportation problems. The unknowns ρ (density) and v (velocity) solve the minimum problem

$$\min\left\{ \int_0^1 \int_\Omega \rho|v|^2\,dx\,dt \;:\; \rho_t + \mathrm{div}_x(\rho v) = 0 \right\}$$

under the initial/terminal conditions $\rho|_{t=0} = \rho_0$ and $\rho|_{t=1} = \rho_1$.

The minimal value coincides with the Wasserstein distance $W_2^2(\rho_0, \rho_1)$.

Setting $\rho v = q$ the continuity equation becomes linear:

$$\rho_t + \mathrm{div}_x\, q = 0$$

and the cost functional (representing the kinetic energy) becomes convex:

$$\int_0^1 \int_\Omega \frac{|q|^2}{\rho}\, dx\, dt.$$

To be precise, the correct meaning has to be given in terms of measures:

$$\int_0^1 \int_\Omega \left|\frac{dq}{d\rho}\right|^2 d\rho(x)\, dt.$$

Setting $Q = [0,T] \times \Omega$, $\sigma = (\rho, q)$, and $f = \delta_T(t) \otimes \rho_1(x) - \delta_0(t) \otimes \rho_0(x)$ the problem above can be written in the form

$$\min \left\{ \Psi(\sigma) \ : \ -\operatorname{div} \sigma = f \text{ in } Q,\ \sigma \cdot \nu = 0 \text{ on } \partial Q \right\}$$

where $\Psi(\sigma)$ is a functional defined on $\mathcal{M}(Q)$.

Theorem *If Ψ is a weakly* l.s.c. functional on $\mathcal{M}(Q)$ and $f \in \mathcal{M}(Q)$, then the minimum problem*

$$\min \left\{ \Psi(\sigma) \ : \ -\operatorname{div} \sigma = f \text{ in } Q,\ \sigma \cdot \nu = 0 \text{ on } \partial Q \right\}$$

has a solution, provided $\int_Q df = 0$ and Ψ is coercive, i.e. $\Psi(\sigma) \geq c|\sigma| - c_1$.

The functionals Ψ we have in mind are of the form

$$\Psi(\sigma) = \int_0^T J\Big(\sigma(t)\Big)\, dt$$

and again J of Lebesgue type would provide congestion models, while J of Dirac type would provide concentration models.

The congestion case is simpler, because the functional J is convex. The concentration case, on the contrary, requires some extra analysis, due to concavity effects.

Dual formulation (in the convex case):

$$\sup \Big\{ \langle f, \phi \rangle - \Psi^*(D\phi) \ : \ \phi \in C^1(Q) \Big\}.$$

Primal-dual relation:

$$\Psi(\sigma_{opt}) + \Psi^*(D\phi_{opt}) = \langle \sigma_{opt}, D\phi_{opt} \rangle.$$

The point is that the maximizer in the dual formulation is not of class C^1 in general. A relaxation formula is then needed for Ψ^* to extend it to its natural space.

The natural spaces for functionals like Ψ^* are the Sobolev spaces $W^{1,p}_\mu$ with respect to a measure μ, defined by relaxation of the energies

$$\int |Du|^p \, d\mu.$$

All the usual properties known for the standard Sobolev spaces continue to hold, provided the gradient is replaced by the tangential gradient $D_\mu u$ suitably defined.

We do not enter in the details of this rather delicate theory, referring to Bouchitté-Buttazzo-Seppecher (Calc.Var. 1997).

The numerical approximation has been performed in [BJO] following the scheme used in Benamou-Brenier, through an augmented Lagrangian algorithm. The following animations deal with a domain Ω not convex (a kind of subway gate) and with the cases:

- $J(\rho, q) = \frac{|q|^2}{\rho}$ in which the transportation simply follows the Wasserstein geodesics.
- $J(\rho, q) = \frac{|q|^2}{\rho} + c\rho^2$ in which the Wasserstein transportation is perturbed by the addition of a diffusion term (panic effect).
- $J(\rho, q) = \frac{|q|^2}{\rho} + \chi_{\{\rho \leq M\}}$ in which there is the additional constraint that two different individual do not want to stay too close.

In the concentration case we take in [Brasco-B-Santambrogio]

$$\Psi(\rho, q) = \int_0^T J\Big(\rho(t), q(t)\Big)\, dt$$

under the continuity equation for (ρ, q) and

$$J(\rho, q) := \begin{cases} \sum |v_i| \rho_i^\alpha & \text{if } q = v \cdot \rho \text{ is atomic,} \\ +\infty & \text{otherwise} \end{cases}$$

with $0 < \alpha < 1$.

Theorem *The minimum problem for* Ψ *(under the continuity equation) admits a solution.*

This result is obtained by the direct methods of the calculus of variations, consisting in proving lower semicontinuity and coercivity with respect to a suitable convergence.

We notice that the weak* convergence of (ρ, q) in $Q = [0, T] \times \Omega$ does not imply the lower semicontinuity. On the other hand, if

$$(\rho_t^n, q_t^n) \rightharpoonup (\rho_t, q_t), \quad \text{for a.e. } t \in [0, T],$$

by Fatou's Lemma we obtain the semicontinuity property.

We use a convergence stronger than the weak* convergence on Q, but weaker than weak* convergence for a.e. $t \in [0, T]$.

Definition We say that (ρ^n, q^n) τ-converges to (ρ, q) if it weakly* converges on Q and

$$\sup_{n \in \mathbf{N},\ t \in [0,1]} J(\rho_t^n, q_t^n) < +\infty.$$

The existence of an evolution path (ρ, q) follows from:

- (coercivity) If $\Psi(\rho^n, q^n) \leq C$ (and continuity equation), then up to a time reparametrization, (ρ^n, q^n) is τ-compact.

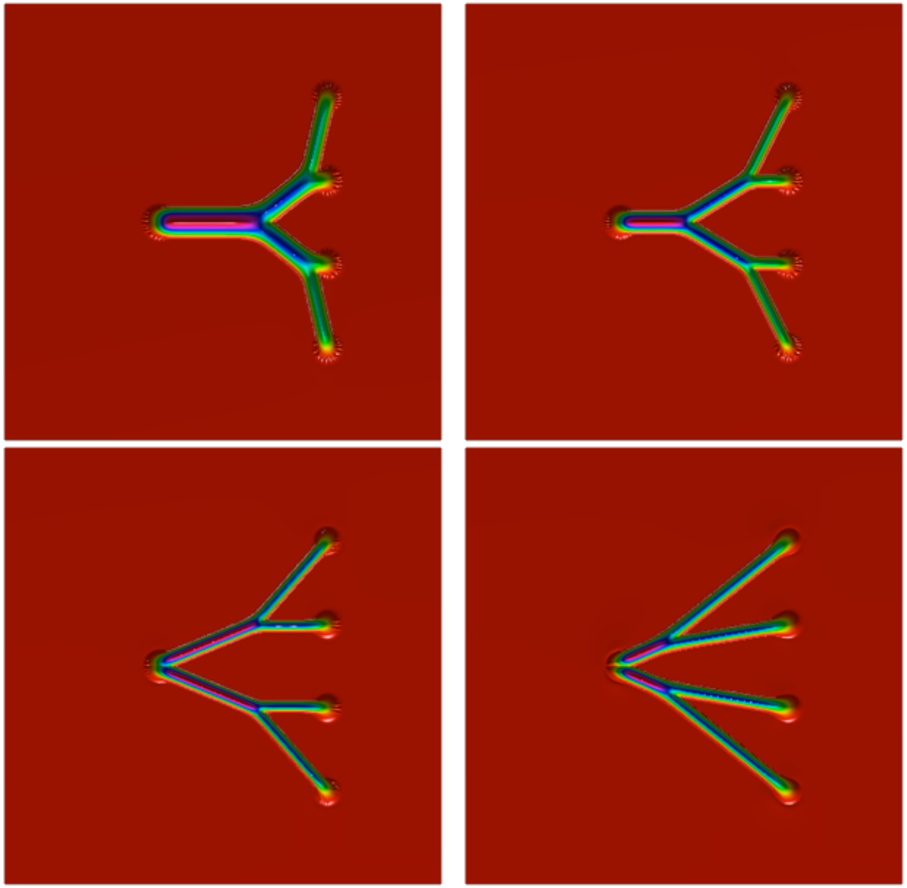
- (semicontinuity) If (ρ^n, q^n) τ-converges to (ρ, q) (and continuity equation), then

$$\Psi(\rho, q) \leq \liminf_{n\to\infty} \Psi(\rho^n, q^n).$$

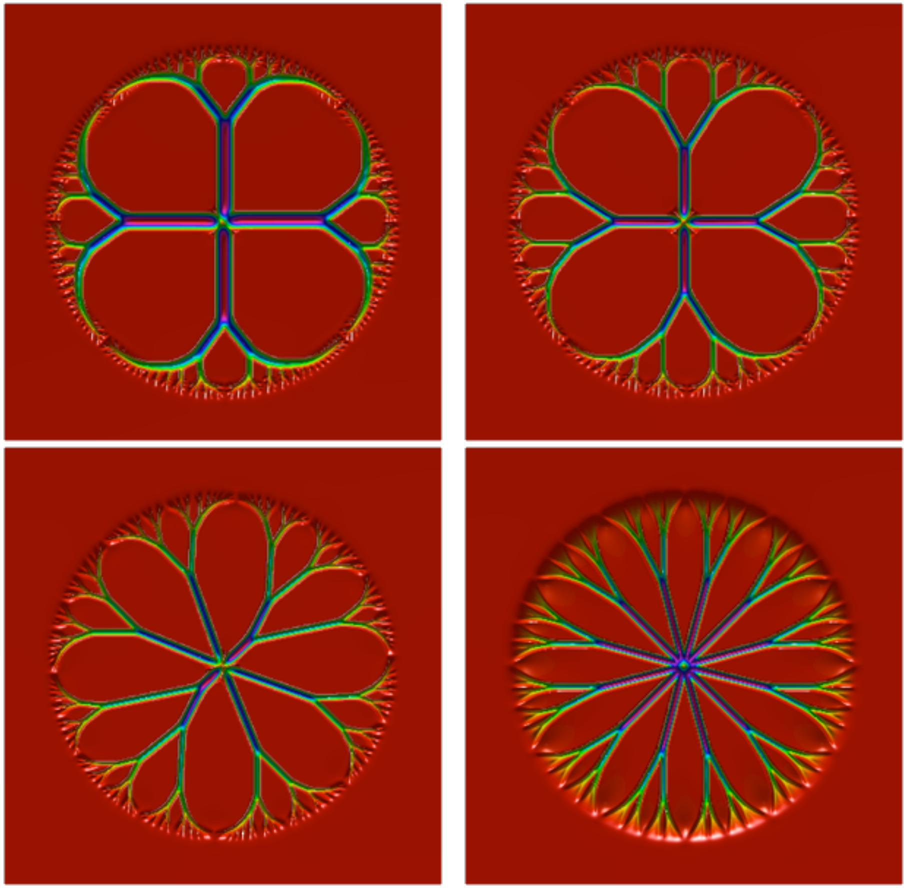
The evolution model above is "equivalent" to the static ones by Gilbert, Xia, Bernot-Caselles-Morel, in the sense that the two minima coincide and there is a natural way to pass from a dynamic minimizer of our problem to a static minimizer of the previous models.

Some numerical computations have been made by E. Oudet and can be found on his web page:

`http://www.lama.univ-savoie.fr/~oudet/`



Branched transport of a point in 4 points: $\alpha = 0.6,\ 0.75,\ 0.85,\ 0.95$



Branched transport of a point in a circle: $\alpha = 0.6,\ 0.75,\ 0.85,\ 0.95$