Mean field games equations with quadratic Hamiltonian: a specific approach

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Rome - May 12th 2011

Mean field games equations with quadratic Hamiltonian

MFG equations with quadratic Hamiltonian on the domain $[0, T] \times \Omega$, Ω standing for $(0,1)^d$:

$$\text{(HJB)} \qquad \partial_t u + \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = -f(x,m)$$

$$\text{(K)} \qquad \partial_t m + \nabla \cdot (m \nabla u) = \frac{\sigma^2}{2}\Delta m$$

- Boundary conditions: $\frac{\partial u}{\partial n} = \frac{\partial m}{\partial n} = 0$ on $(0, T) \times \partial\Omega$
- Terminal condition: $u(T, \cdot) = u_T(\cdot)$ a given payoff.
- Initial condition: $m(0, \cdot) = m_0(\cdot) \geq 0$ a positive function in $L^1(\Omega)$, typically a probability distribution function.

Change of variable

Proposition ($u = \sigma^2 \ln(\phi)$, $m = \phi\psi$)

Let's consider a smooth solution (ϕ, ψ) of the following system $(\mathcal{S})$ with $\phi > 0$:

$$\partial_t \phi + \frac{\sigma^2}{2}\Delta\phi = -\frac{1}{\sigma^2} f(x, \phi\psi)\phi \qquad (E_\phi)$$

$$\partial_t \psi - \frac{\sigma^2}{2}\Delta\psi = \frac{1}{\sigma^2} f(x, \phi\psi)\psi \qquad (E_\psi)$$

with:

- *Boundary conditions:* $\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n} = 0$ *on* $(0, T) \times \partial\Omega$
- *Terminal condition:* $\phi(T, \cdot) = \exp\left(\frac{u_T(\cdot)}{\sigma^2}\right)$.
- *Initial condition:* $\psi(0, \cdot) = \frac{m_0(\cdot)}{\phi(0, \cdot)}$

Then $(u, m) = (\sigma^2 \ln(\phi), \phi\psi)$ *is a solution of (MFG).*

Hypotheses

- $\forall x, \xi \mapsto f(x, \xi)$ is a continuous and decreasing function. *Similar to the hypothesis in the usual proof of uniqueness.*
- $f \in L^\infty$
- $f \leq 0$
  This is not a restriction since f is bounded...
  $f \leftarrow f - \|f\|_\infty \Rightarrow u \leftarrow u - \|f\|_\infty t.$
- $u_T \in L^\infty(\Omega)$
- $m_0 \in L^2(\Omega)$

Notations

We define $\mathcal{P} \subset C([0,T], L^2(\Omega))$ with:

$$g \in \mathcal{P}$$

$$\iff$$

$$g \in L^2(0,T,H^1(\Omega)) \quad \text{and} \quad \partial_t g \in L^2(0,T,H^{-1}(\Omega))$$

We also define:

$$\mathcal{P}_\epsilon = \{g \in \mathcal{P}, g \geq \epsilon\}$$

Equation (E_ϕ)

Proposition (Well-posedness)

$\forall \psi \in \mathcal{P}_0$, there is a unique weak solution ϕ to the following equation (E_ϕ):

$$\partial_t \phi + \frac{\sigma^2}{2}\Delta\phi = -\frac{1}{\sigma^2} f(x, \phi\psi)\phi \qquad (E_\phi)$$

with $\frac{\partial \phi}{\partial n} = 0$ on $(0, T) \times \partial\Omega$ and $\phi(T, \cdot) = \exp\left(\frac{u_T(\cdot)}{\sigma^2}\right)$.
Hence $\Phi : \psi \in \mathcal{P}_0 \mapsto \phi \in \mathcal{P}$ is well defined.

Equation (E_ϕ) (continued)

Two results:

Proposition (Uniform lower bound)

$\forall \psi \in \mathcal{P}_0, \phi = \Phi(\psi) \in \mathcal{P}_\epsilon$ *for* $\epsilon = \exp\left(-\frac{1}{\sigma^2}\left(\|u_T\|_\infty + \|f\|_\infty T\right)\right)$

This uniform bound will allow to define $\psi(0,\cdot)$*.*

Proposition (Monotonicity)

$$\forall \psi_1 \leq \psi_2 \in \mathcal{P}_0, \Phi(\psi_1) \geq \Phi(\psi_2)$$

This monotonicity result will be central in the constructive scheme.

Equation (E_ψ)

Proposition (Well-posedness)

Let's fix $\epsilon > 0$ as above.
$\forall \phi \in \mathcal{P}_\epsilon$, *there is a unique weak solution ψ to the following equation (E_ψ):*

$$\partial_t \psi - \frac{\sigma^2}{2}\Delta\psi = \frac{1}{\sigma^2} f(x, \phi\psi)\psi \qquad (E_\psi)$$

with $\frac{\partial\psi}{\partial n} = 0$ on $(0, T) \times \partial\Omega$ and $\psi(0, \cdot) = \frac{m_0(\cdot)}{\phi(0,\cdot)}$.
Hence $\Psi : \phi \in \mathcal{P}_\epsilon \mapsto \psi \in \mathcal{P}$ is well defined.

Equation (E_ψ) (continued)

Two results:

Proposition (Positiveness)

$$\forall \phi \in \mathcal{P}_\epsilon, \psi = \Psi(\phi) \in \mathcal{P}_0$$

Proposition (Monotonicity)

$$\forall \phi_1 \leq \phi_2 \in \mathcal{P}_\epsilon, \Psi(\phi_1) \geq \Psi(\phi_2)$$

This monotonicity result will be central in the constructive scheme.

Constructive scheme - Definition

The scheme we consider involves two sequences $(\phi^{n+\frac{1}{2}})_n$ and $(\psi^n)_n$ that are built using the following recursive equations:

$$\begin{aligned}
\psi^0 &= 0 \\
\partial_t \phi^{n+\frac{1}{2}} + \frac{\sigma^2}{2}\Delta\phi^{n+\frac{1}{2}} &= -\frac{1}{\sigma^2} f(x, \phi^{n+\frac{1}{2}}\psi^n)\phi^{n+\frac{1}{2}} \\
\partial_t \psi^{n+1} - \frac{\sigma^2}{2}\Delta\psi^{n+1} &= \frac{1}{\sigma^2} f(x, \phi^{n+\frac{1}{2}}\psi^{n+1})\psi^{n+1}
\end{aligned}$$

with:

- Boundary conditions: $\frac{\partial \phi^{n+\frac{1}{2}}}{\partial \vec{n}} = \frac{\partial \psi^{n+1}}{\partial \vec{n}} = 0$ on $(0, T) \times \partial\Omega$
- Terminal condition: $\phi^{n+\frac{1}{2}}(T, \cdot) = \exp\left(\frac{u_T(\cdot)}{\sigma^2}\right)$.
- Initial condition: $\psi^{n+1}(0, \cdot) = \frac{m_0(\cdot)}{\phi^{n+\frac{1}{2}}(0,\cdot)}$

Constructive scheme - Definition

In other words, the constructive scheme is defined as:

$$\psi^0 = 0$$

$$\forall n \in \mathbb{N}, \phi^{n+\frac{1}{2}} = \Phi(\psi^n)$$

$$\forall n \in \mathbb{N}, \psi^{n+1} = \Psi(\phi^{n+\frac{1}{2}})$$

Constructive scheme

Theorem

The above scheme has the following properties:

- $(\phi^{n+\frac{1}{2}})_n$ *is a decreasing sequence of* $\mathcal{P}_\epsilon$.
- $(\psi^n)_n$ *is an increasing sequence of* $\mathcal{P}_0$*, bounded from above in* $\mathcal{P}$ *by* $\Psi(\epsilon)$
- $(\phi^{n+\frac{1}{2}}, \psi^n)_n$ *converges for almost every* $(t, x) \in (0, T) \times \Omega$*, and in* $L^2(0, T, L^2(\Omega))$ *towards a couple* (ϕ, ψ).
- $(\phi, \psi) \in \mathcal{P}_\epsilon \times \mathcal{P}_0$ *is a weak solution of* $(\mathcal{S})$.

It's noteworthy that there is nothing like mass conservation, except asymptotically.

Introduction

- Uniform subdivision $(t_0, \ldots, t_I)$ of $(0, T)$ where $t_i = i\Delta t$
- Uniform subdivision $(x_0, \ldots, x_J)$ of $(0, 1)$ where $x_j = j\Delta x$
- Finite difference scheme: $\hat{\psi}^n_{i,j}$ and $\hat{\phi}^{n+\frac{1}{2}}_{i,j}$
- Neumann conditions: $\hat{\psi}^n_{i,-1} = \hat{\psi}^n_{i,1}$ and $\hat{\psi}^n_{i,J+1} = \hat{\psi}^n_{i,J-1}$
- Neumann conditions: $\hat{\phi}^{n+\frac{1}{2}}_{i,-1} = \hat{\phi}^{n+\frac{1}{2}}_{i,1}$ and $\hat{\phi}^{n+\frac{1}{2}}_{i,J+1} = \hat{\phi}^{n+\frac{1}{2}}_{i,J-1}$

$$\mathcal{M} = M_{I+1,J+1}(\mathbb{R})$$

$$\mathcal{M}_\epsilon = \{(m_{i,j})_{i,j} \in \mathcal{M}, \quad \forall i,j, m_{i,j} \geq \epsilon\}$$

Numerical scheme

$$\hat{\psi}^0_{i,j}=0$$

Completely implicit scheme for $\hat{\phi}^{n+\frac{1}{2}}$:

$$\frac{\hat{\phi}^{n+\frac{1}{2}}_{i+1,j}-\hat{\phi}^{n+\frac{1}{2}}_{i,j}}{\Delta t}+\frac{\sigma^2}{2}\frac{\hat{\phi}^{n+\frac{1}{2}}_{i,j+1}-2\hat{\phi}^{n+\frac{1}{2}}_{i,j}+\hat{\phi}^{n+\frac{1}{2}}_{i,j-1}}{(\Delta x)^2}=-\frac{1}{\sigma^2}f(x_j,\hat{\phi}^{n+\frac{1}{2}}_{i,j}\hat{\psi}^{n}_{i,j})\hat{\phi}^{n+\frac{1}{2}}_{i,j}$$

$$\hat{\phi}^{n+\frac{1}{2}}_{I,j}=\exp\left(\frac{u_T(x_j)}{\sigma^2}\right)$$

Completely implicit scheme for $\hat{\psi}^{n+1}$:

$$\frac{\hat{\psi}^{n+1}_{i+1,j}-\hat{\psi}^{n+1}_{i,j}}{\Delta t}-\frac{\sigma^2}{2}\frac{\hat{\psi}^{n+1}_{i+1,j+1}-2\hat{\psi}^{n+1}_{i+1,j}+\hat{\psi}^{n+1}_{i+1,j-1}}{(\Delta x)^2}=\frac{1}{\sigma^2}f(x_j,\hat{\phi}^{n+\frac{1}{2}}_{i+1,j}\hat{\psi}^{n+1}_{i+1,j})\hat{\psi}^{n+1}_{i+1,j}$$

$$\hat{\psi}^{n+1}_{0,j}=\frac{m_0(x_j)}{\hat{\phi}^{n+\frac{1}{2}}_{0,j}}$$

Well-posedness I

Proposition (Well-posedness)

$\forall \hat{\psi} \in \mathcal{M}_0$, there is a unique solution $\hat{\phi} \in \mathcal{M}$ to the following equation:

$$\frac{\hat{\phi}_{i+1,j} - \hat{\phi}_{i,j}}{\Delta t} + \frac{\sigma^2}{2} \frac{\hat{\phi}_{i,j+1} - 2\hat{\phi}_{i,j} + \hat{\phi}_{i,j-1}}{(\Delta x)^2}$$

$$= -\frac{1}{\sigma^2} f(x_j, \hat{\phi}_{i,j}\hat{\psi}_{i,j})\hat{\phi}_{i,j}$$

with $\hat{\phi}_{I,j} = \exp\left(\frac{u_T(x_j)}{\sigma^2}\right)$ and the conventions $\hat{\phi}_{i,-1} = \hat{\phi}_{i,1}$, $\hat{\phi}_{i,J+1} = \hat{\phi}_{i,J-1}$.
Hence $\Phi_d : \hat{\psi} \in \mathcal{M}_0 \mapsto \hat{\phi} \in \mathcal{M}$ is well defined.

Uniform lower bound and Monotonicity

Proposition (Uniform lower bound)

$\forall \hat{\psi} \in \mathcal{M}_0, \hat{\phi} = \Phi_d(\hat{\psi}) \in \mathcal{M}_\epsilon$ *for the same* ϵ *as in the continuous case.*

Proposition (Monotonicity)

$$\forall \hat{\psi}_1 \leq \hat{\psi}_2 \in \mathcal{M}_0, \Phi_d(\hat{\psi}_1) \geq \Phi_d(\hat{\psi}_2)$$

Well-posedness II

Proposition (Well-posedness)

Let's fix $\epsilon > 0$ as above.
$\forall \hat{\phi} \in \mathcal{M}_\epsilon$, there is a unique solution $\hat{\psi} \in \mathcal{M}$ to the following equation:

$$\frac{\hat{\psi}_{i+1,j} - \hat{\psi}_{i,j}}{\Delta t} - \frac{\sigma^2}{2} \frac{\hat{\psi}_{i+1,j+1} - 2\hat{\psi}_{i+1,j} + \hat{\psi}_{i+1,j-1}}{(\Delta x)^2}$$

$$= \frac{1}{\sigma^2} f(x_j, \hat{\phi}_{i+1,j}\hat{\psi}_{i+1,j})\hat{\psi}_{i+1,j}$$

with $\hat{\psi}_{0,j} = \frac{m_0(x_j)}{\hat{\phi}_{0,j}}$ and the conventions $\hat{\psi}_{i,-1} = \hat{\psi}_{i,1}$, $\hat{\psi}_{i,J+1} = \hat{\psi}_{i,J-1}$.
Hence $\Psi_d : \hat{\phi} \in \mathcal{M}_\epsilon \mapsto \hat{\psi} \in \mathcal{M}$ is well defined.

Positiveness and monotonicity

Proposition (Positiveness)

$$\forall \hat{\phi} \in \mathcal{M}_\epsilon, \hat{\psi} = \Psi_d(\hat{\phi}) \in \mathcal{M}_0$$

Proposition (Monotonicity)

$$\forall \hat{\phi}_1 \leq \hat{\phi}_2 \in \mathcal{M}_\epsilon, \Psi_d(\hat{\phi}_1) \geq \Psi_d(\hat{\phi}_2)$$

Monotonicity of the scheme and limit behavior

Theorem

Assume that m_0 is bounded. The numerical scheme verifies the following properties:

- $(\hat{\phi}^{n+\frac{1}{2}})_n$ *is a decreasing sequence of* $\mathcal{M}_\epsilon$.
- $(\hat{\psi}^n)_n$ *is an increasing sequence of* $\mathcal{M}_0$, *bounded from above, independently of the subdivision.*
- $(\hat{\phi}^{n+\frac{1}{2}}, \hat{\psi}^n)_n$ *converges towards a couple* $(\hat{\phi}, \hat{\psi}) \in \mathcal{M}_\epsilon \times \mathcal{M}_0$.

Convergence of the scheme I

Definition of the norm $|||\cdot|||$

$$\forall m = (m_{i,j})_{i,j} \in \mathcal{M}, |||m|||^2 = \sup_{0\le i\le I} \frac{1}{J+1}\sum_{j=0}^{J} m_{i,j}^2$$

Hypothesis

- We suppose that f, u_T and m_0 are bounded.
- We also suppose that f is Lipschitz with respect to ξ (Lipschitz constant: K)

Convergence of the scheme II

$$\phi_{i,j}^{n+\frac{1}{2}} = \phi^{n+\frac{1}{2}}(t_i, x_j)$$

$$\psi_{i,j}^{n+1} = \psi^{n+1}(t_i, x_j)$$

Consistency errors

$$\eta_{i,j}^{n+\frac{1}{2}} = \frac{\phi_{i+1,j}^{n+\frac{1}{2}} - \phi_{i,j}^{n+\frac{1}{2}}}{\Delta t} + \frac{\sigma^2}{2}\frac{\phi_{i,j+1}^{n+\frac{1}{2}} - 2\phi_{i,j}^{n+\frac{1}{2}} + \phi_{i,j-1}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{1}{\sigma^2} f(x_j, \phi_{i,j}^{n+\frac{1}{2}}\psi_{i,j}^{n})\phi_{i,j}^{n+\frac{1}{2}}$$

$$\eta_{i,j}^{n+1} = \frac{\psi_{i+1,j}^{n+1} - \psi_{i,j}^{n+1}}{\Delta t} - \frac{\sigma^2}{2}\frac{\psi_{i+1,j+1}^{n+1} - 2\psi_{i+1,j}^{n+1} + \psi_{i+1,j-1}^{n+1}}{(\Delta x)^2} - \frac{1}{\sigma^2} f(x_j, \phi_{i+1,j}^{n+\frac{1}{2}}\psi_{i+1,j}^{n+1})\psi_{i+1,j}^{n+1}$$

Convergence of the scheme III

Theorem (Stability bounds)

If $\frac{1}{\Delta t} > 1 + \frac{K}{\sigma^2} \max\left(e^{2\frac{\|u_T\|_\infty}{\sigma^2}}, \|\psi\|_\infty^2\right)$, *then* $\forall n \in \mathbb{N}$, $\exists C_{n+\frac{1}{2}}, C_{n+1}, D_{n+\frac{1}{2}}, D_{n+1}$ *such that:*

$$|||\hat{\phi}^{n+\frac{1}{2}} - \phi^{n+\frac{1}{2}}||| \le C_{n+\frac{1}{2}} |||\hat{\psi}^n - \psi^n||| + D_{n+\frac{1}{2}} |||\eta^{n+\frac{1}{2}}|||$$

$$|||\hat{\psi}^{n+1} - \psi^{n+1}||| \le C_{n+1} |||\hat{\phi}^{n+\frac{1}{2}} - \phi^{n+\frac{1}{2}}||| + D_{n+1} |||\eta^{n+1}|||$$

Convergence of the scheme IV

Theorem (Convergence)

Let's suppose that u_T, m_0 and f are so that $\forall n \in \mathbb{N}, \phi^{n+\frac{1}{2}}, \psi^n \in C^{1,2}([0,T] \times [0,1])$ and $\phi, \psi \in C^{1,2}([0,T] \times [0,1])$ and still f a Lipschitz function with respect to ξ.

Then:

$$\lim_{\Delta t, \Delta x \to 0} \lim_{n \to \infty} |||\hat{\phi}^{n+\frac{1}{2}} - \phi||| = 0$$

$$\lim_{\Delta t, \Delta x \to 0} \lim_{n \to \infty} |||\hat{\psi}^{n+1} - \psi||| = 0$$

A first framework

People willing to live at the center, but not together.

$$\Omega = (0,1), \quad T = 2, \quad \sigma = 1$$

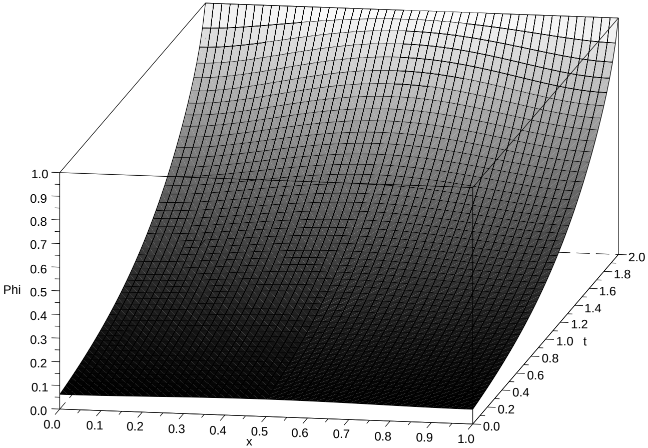
$$f(x,\xi) = -16(x-1/2)^2 - 0.1\min(5, \max(0,\xi))$$

$$m_0(x) = 1 + 0.2\cos\left(\pi\left(2x - \frac{3}{2}\right)\right)^2 \quad u_T(x) = 0$$

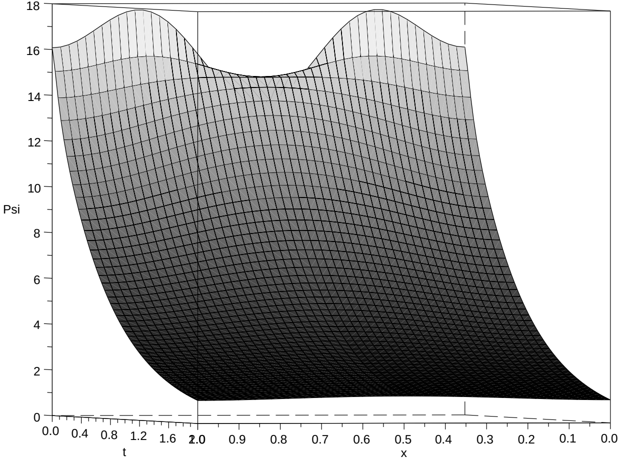
51 points in time and 51 points in space.

Convergence after 7 iterations for n.

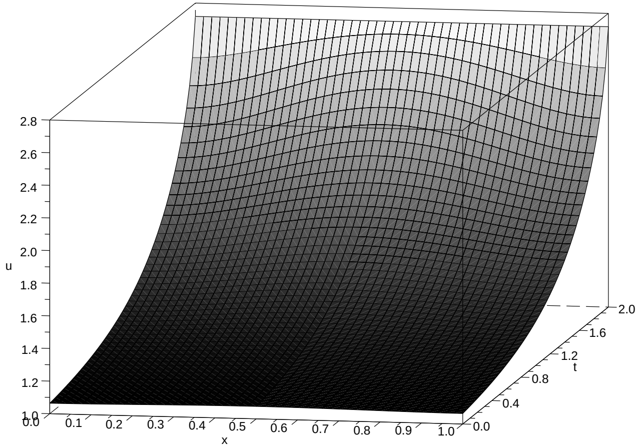
Solution ϕ



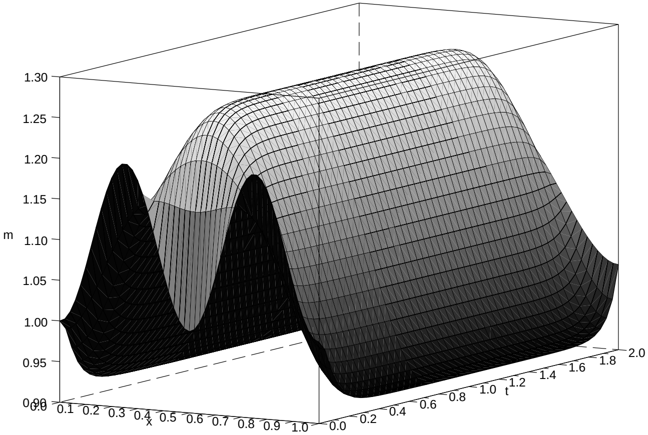
Solution ψ



Solution u


2.8
2.6
2.4
2.2
2.0
u
1.8
1.6
1.4
1.2
1.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x
0.0
0.4
0.8
1.2
t
1.6
2.0

Solution m


m
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1.25
1.20
1.15
1.10
1.05
1.00
0.95
0.90
x
0.0
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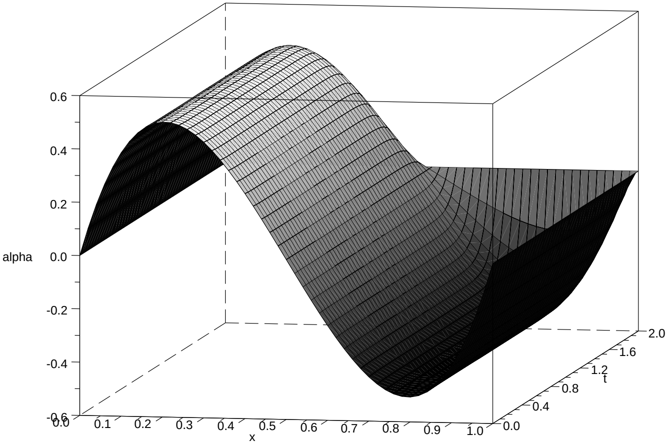
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
t
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0

Solution optimal control $\alpha = \nabla u$



A first framework

People willing to live at $x = \frac{1}{4}$ or $x = \frac{3}{4}$ during the game and at the center at the end, but never together.

$$\Omega = (0,1), \quad T = 2, \quad \sigma = 1$$

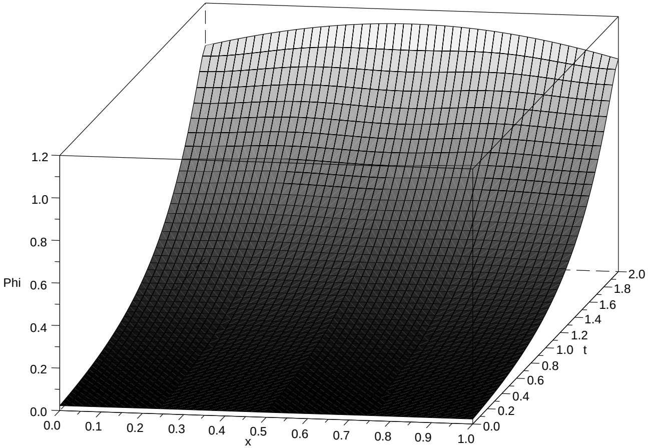
$$f(x,\xi) = 2\cos\left(\pi\left(2x - \frac{3}{2}\right)\right)^2 - 2 - \min(5, \max(0,\xi))$$

$$m_0(x) = 1, \qquad u_T = \frac{1}{2}x(1-x)$$

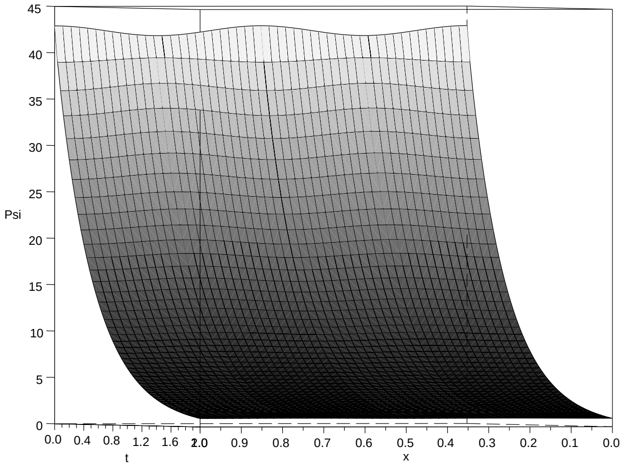
51 points in time and 51 points in space.

Convergence after 28 iterations for n.

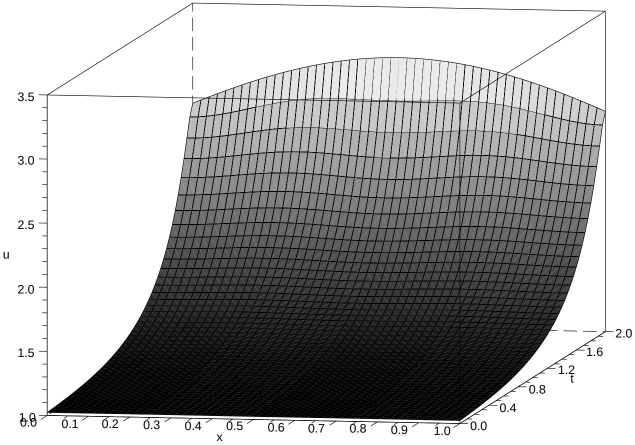
Solution ϕ



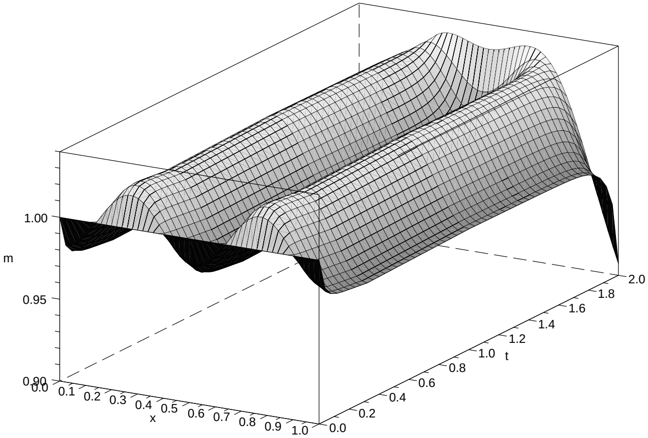
Solution ψ



Solution u



Solution m


1.00
0.95
0.90
m
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
t

Solution optimal control $\alpha = \nabla u$