Three Lectures on:
Control of Coupled Fast and Slow Dynamics

Zvi Artstein

Ravello, September 2012

Control of Coupled Fast and Slow Dynamics

Zvi Artstein

Plan:

Modeling
Variational Limits
Classical Approach to slow-fast dynamics
What limits are appropriate? Young Measures
Modern Approach to slow-fast dynamics
Other chattering limits and averaging techniques
Control Invariant Measures
Stabilization
Optimal Control
Some special cases
Computations, error estimates
A Future Direction

Plan:

Modeling

Variational Limits

Classical Approach to slow-fast dynamics

What limits are appropriate? Young Measures

Modern Approach to slow-fast dynamics

Other chattering limits and averaging techniques

Control Invariant Measures

Stabilization

Optimal Control

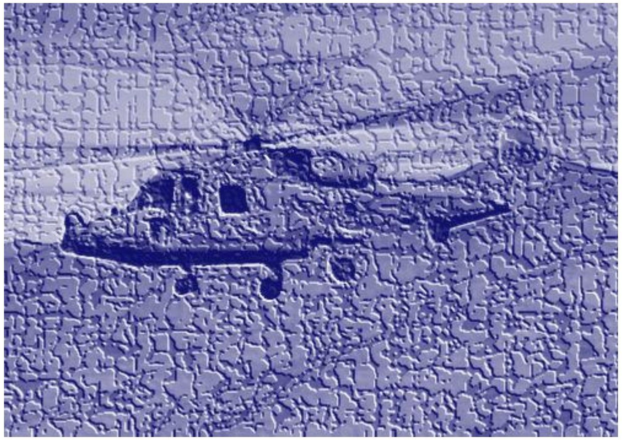
Some special cases

Computations, error estimates

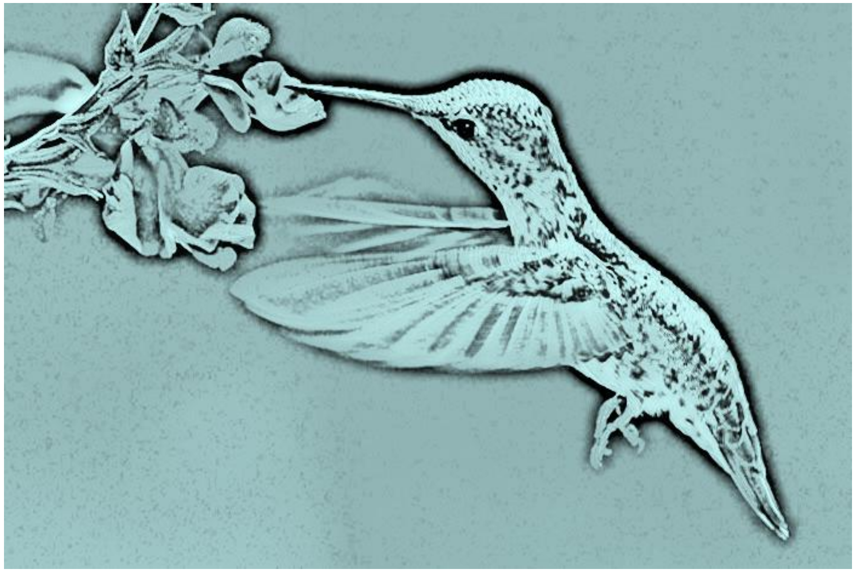
A Future Direction

Example from real life: Airplane

Example from real life: Helicopter



Example from real life: The hummingbird



The framework ODE:

An ordinary differential equation

$$\frac{dx}{dt} = f(x)$$

$$\frac{dx}{dt} = f(x,t)$$

$x \in R^n$

$x(t_0) = x_0$

trajecrory in R^n

initial condition

The framework CONTROL:

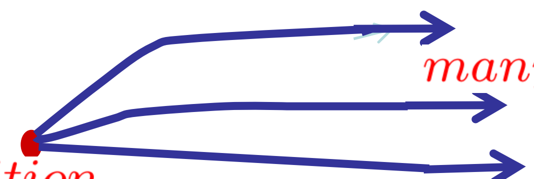
A control equation

$$\frac{dx}{dt} = f(x,u)$$

$$\frac{dx}{dt} = f(x,t,u)$$

$$u \in U$$

$$x(t_0) = x_0$$



Possible goals:

Bolza Problem $\quad minimize \quad \int_{t_0}^{T} c(x(t),t,u(t))dt$

Mayer Problem $\quad minimize \quad C(x(T))$

A reduction of Bolza to Mayer:

The goal

$$minimize \quad \int_0^T c(x(t), t, u(t))dt$$

can be reduced to

$$minimize \quad C(x(T))$$

by adding a coordinate and an equation

$$x_{n+1} = c(x, t, u)$$

<u>and seeking minimizing the additional coordinate</u>

Adolf Mayer
1839 - 1907

Oskar Bolza
1857 - 1942

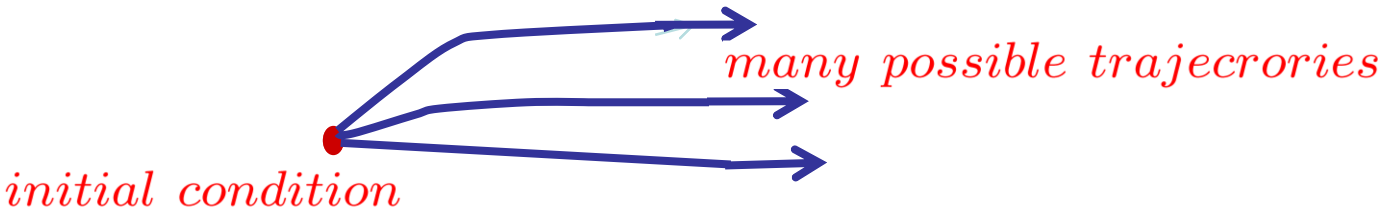
The equivalent differential inclusion:

A control equation

$$\frac{dx}{dt} = f(x,t,u), \qquad u \in U$$

can be written as:

$$\frac{dx}{dt} \in F(x,t) \qquad F(x,t) = \{f(x,t,u) : u \in U\}$$



How to model coupled slow and fast motions?

Tikhonov's Singular Perturbations model of coupled slow and fast motions

The perturbed system:

$$\frac{dx}{dt} = f(x, y) \qquad x(a) = x_0$$

$$\frac{dy}{dt} = \frac{1}{\epsilon}\, g(x, y) \qquad y(a) = y_0$$

The fast part can be written as:

$$\epsilon \frac{dy}{dt} = g(x, y)$$

Where: x in R^n the slow and y in R^m the fast, variables
We are interested in the **limit behavior** of the system as $\epsilon \to 0$

A mathematical example capturing reality: An elastic structure in a rapidly flowing nearly invicid fluid (with Marshall Slemrod)

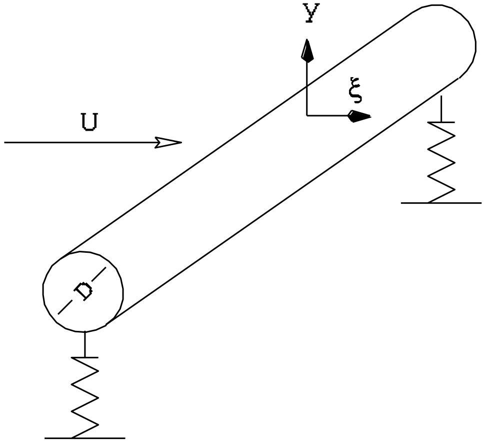


Figure 1

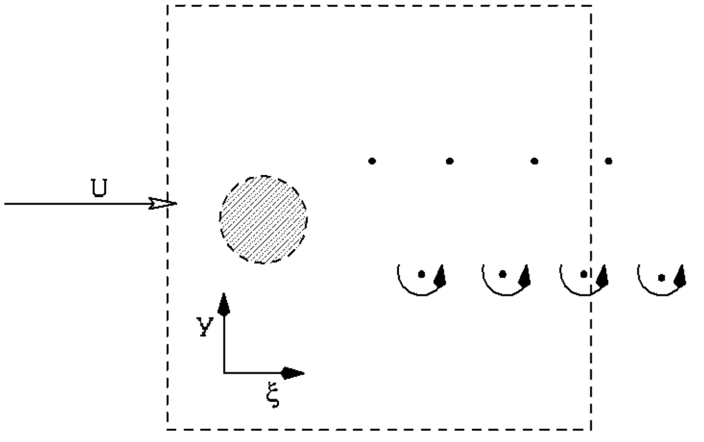


Figure 2

To make the long story short:

Based on a model of Iwan/Belvins and Dowel/Ilgamov, the limit (after normalization) equations:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\alpha_1 x_1 - \alpha_2 x_2 \beta_3 \theta_1 + \beta_4 F(\gamma\theta_2)$$

$$\epsilon\frac{d\theta_1}{dt} = \theta_2$$

$$\epsilon\frac{d\theta_2}{dt} = -\beta_1\theta_1 + \beta_3 F(\gamma\theta_2) - \alpha_3 x_1 - \alpha_4 x_2$$

With $F(\theta)$ a generator of a van der Pol oscillator

An example – Relaxation oscillation

$$\frac{dx}{dt} = y$$

$$\epsilon \frac{dy}{dt} = -x + y - y^3$$

Singularly perturbed optimal control systems:

$$minimize \quad \boxed{\int_a^b c(x,y,u)dt}$$

$$subject\ to \quad \boxed{\begin{aligned} \frac{dx}{dt} &= f(x,y,u) \\ \epsilon\frac{dy}{dt} &= g(x,y,u) \end{aligned}}$$

$$u \in U$$

Where: x in R^n the slow and y in R^m the fast, variables

Of interest: The behavior of the system as $\epsilon \to 0$

Applications

A variety of natural phenomena and engineering design.

The latter include: Regulation, LQ-Systems, Feedback Design, Stabilization, Robustness, Stochastics, Filters, Optimal Control, Hydropower Production, Nuclear Reactions, Aircraft Design, Flight Control, and many more !

The classical approach to handle the applications is the model-reduction - after Levinson-Tikhonov in the differential equations trait and after Kokotovic in the control Setting

Other coupled slow and fast dynamics

$$\frac{dx}{dt} = g(x) + \frac{1}{\epsilon} f(x)$$

$$\frac{dx}{dt} = g(x, u) + \frac{1}{\epsilon} f(x, u)$$

Plan:

√ Modeling

Variational Limits

Classical Approach to slow-fast dynamics

What limits are appropriate? Young Measures

Modern Approach to slow-fast dynamics

Other chattering limits and averaging techniques

Control Invariant Measures

Stabilization

Optimal Control

Some special cases

Computations, error estimates

A Future Direction

The definition of a **variational limit**:

Given a system (an equation or a control equation) with a parameter that tends to a limit. **A variational limit** is a system whose solutions capture the limit behavior of the solutions of the parameterized system, as the parameter tends to its limit,

Capture = limit of the trajectories, limit of the optimal controls, limit of the values

Plan:

√ Modeling

√ Variational Limits

Classical Approach to slow-fast dynamics

What limits are appropriate? Young Measures

Modern Approach to slow-fast dynamics

Other chattering limits and averaging techniques

Control Invariant Measures

Stabilization

Optimal Control

Some special cases

Computations, error estimates

A Future Direction

The classical Tikhonov order reduction approach

Write the perturbed system as:

$$\frac{dx}{dt} = f(x,y)$$

$$\epsilon\frac{dy}{dt} = g(x,y)$$

The limit behavior as $\epsilon \to 0$ is captured by the system:

$$\frac{dx}{dt} = f(x,y)$$

$$0 = g(x,y)$$

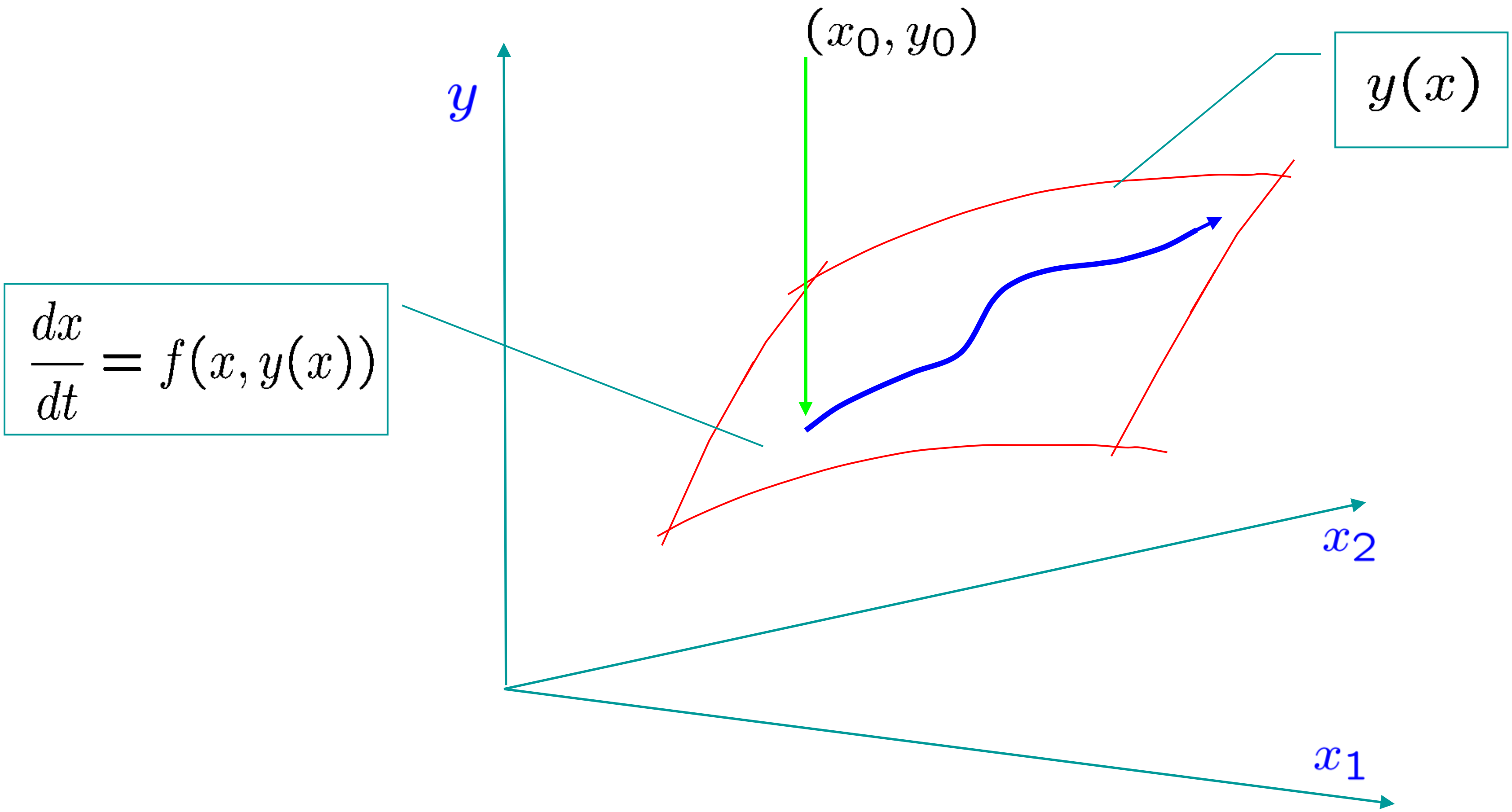
Andrei Nikolayevich Tikhonov

1906 -1993

Norman Levinson

1912 - 1975

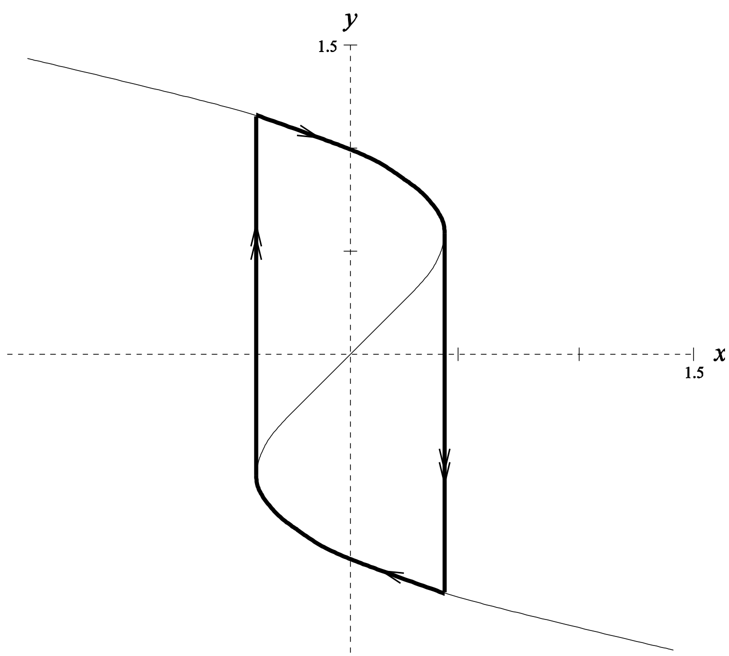
The geometry of the solution:



An example – Relaxation oscillation

$$\frac{dx}{dt} = y$$

$$\epsilon\frac{dy}{dt} = -x + y - y^3$$



Recall: Singularly perturbed control systems:

$$\textit{minimize} \quad \boxed{\int_a^b c(x,y,u)dt}$$

$$\textit{subject to} \quad \boxed{\begin{aligned} \frac{dx}{dt} &= f(x,y,u) \\ \epsilon\frac{dy}{dt} &= g(x,y,u) \end{aligned}}$$

Of interest: The behavior of the system as $\epsilon \to 0$

What is the variational limit?

The limit of:

$$\text{minimize} \quad \int_a^b c(x,y,u)\,dt$$

$$\text{subject to} \quad \frac{dx}{dt} = f(x,y,u)$$

$$\epsilon\frac{dy}{dt} = g(x,y,u)$$

is

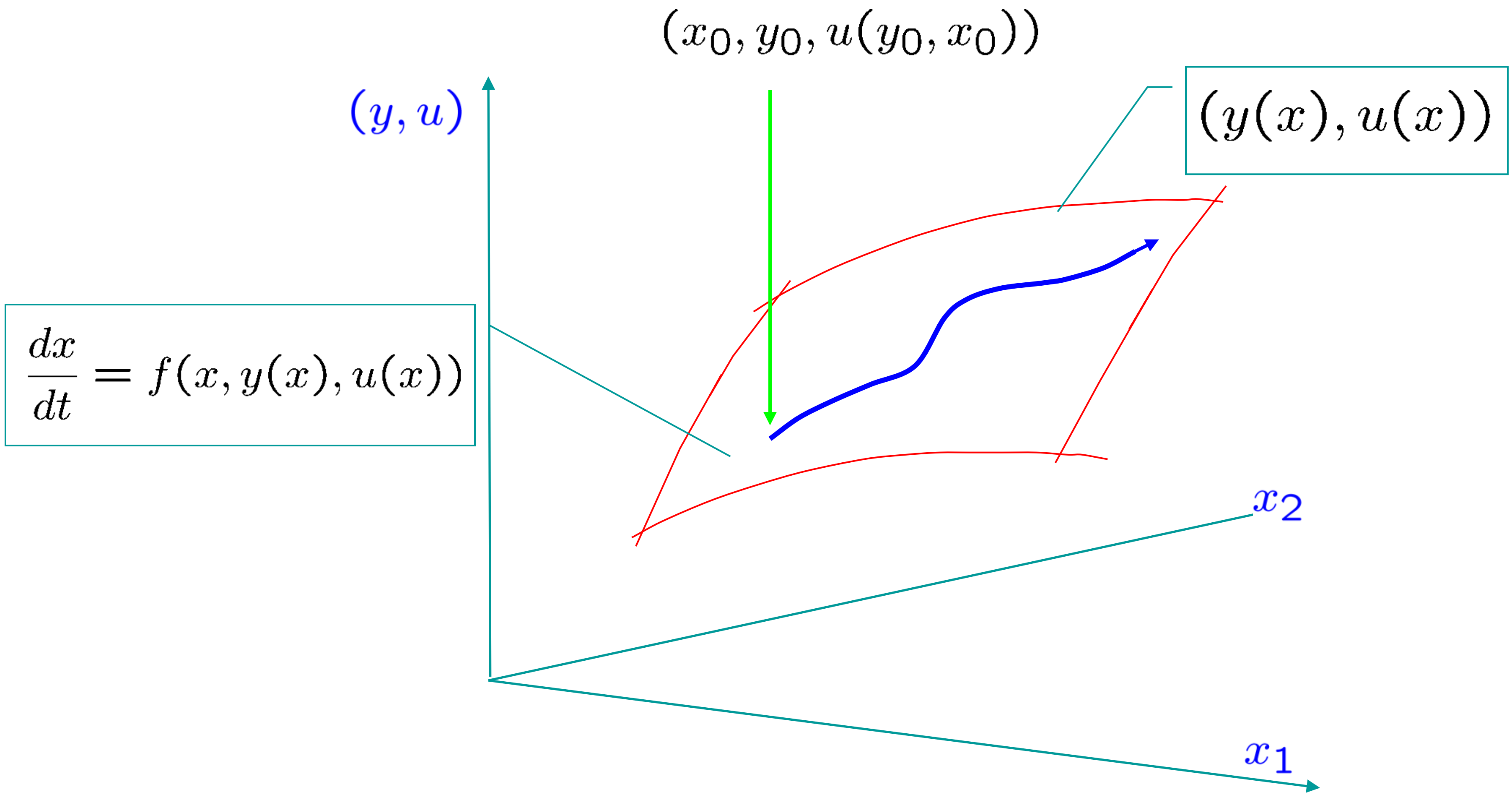
$$\text{minimize} \quad \int_a^b c(x,y,u)\,dt$$

$$\text{subject to} \quad \frac{dx}{dt} = f(x,y,u)$$

$$0 = g(x,y,u)$$

Petar Kokotovic

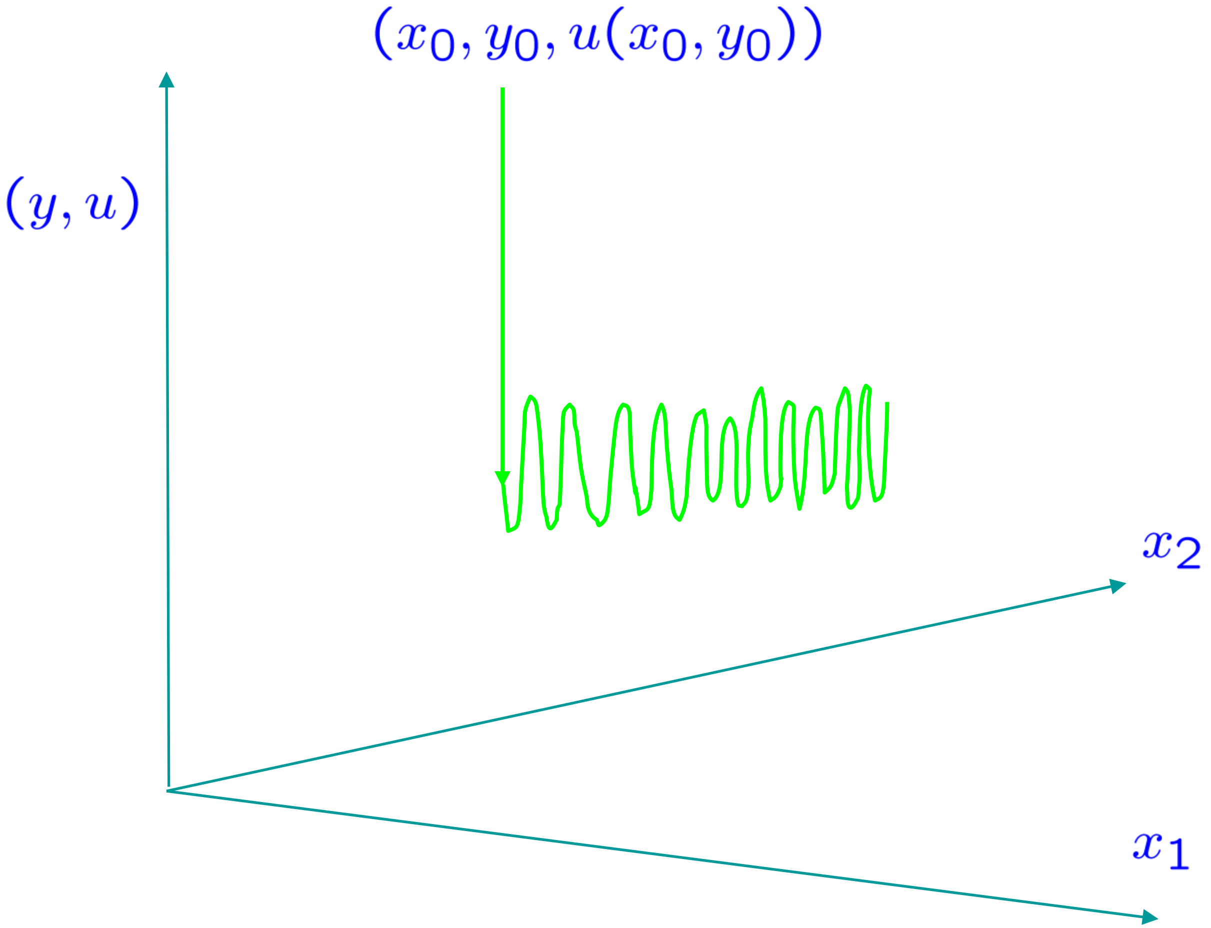
The geometry of the solution:



BUT

The general situation:

There is no reason why the optimal fast solution will converge and not, say, oscillate!



This was pointed out in the mid 1980's, independently,

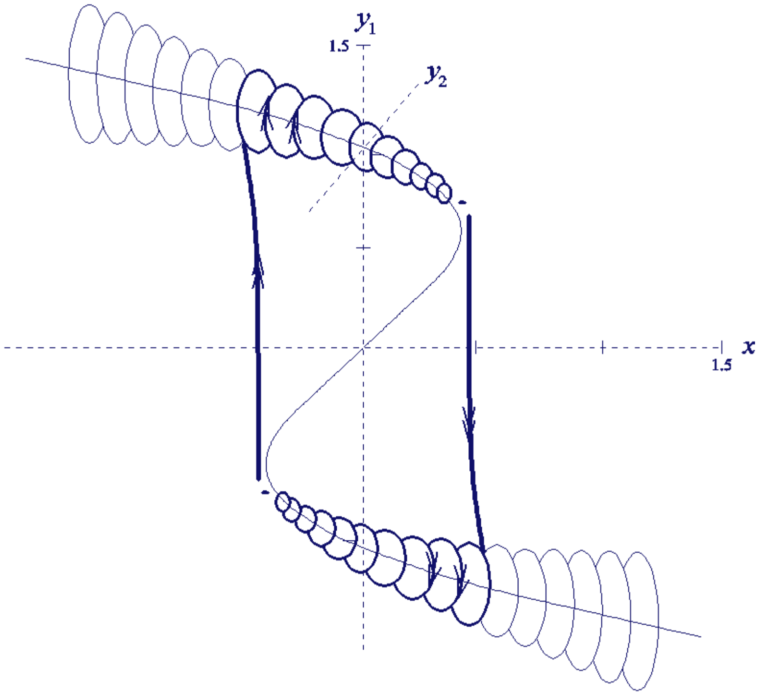
By

Assen Dontchev and Valadimir Veliov

And by

Vladimir Gaitsgory

A mathematical illustration: Non-stationary relaxation oscillation



Recall: A mathematical example capturing reality: An elastic structure in a rapidly flowing nearly invicid fluid

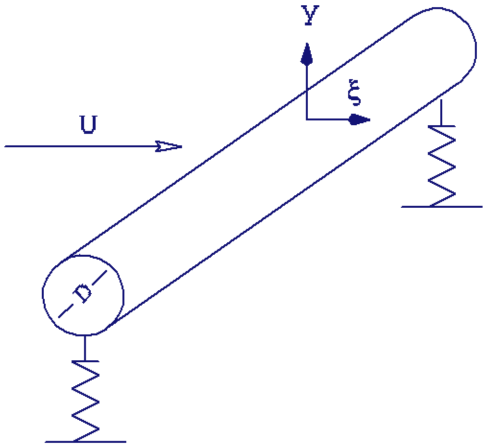


Figure 1

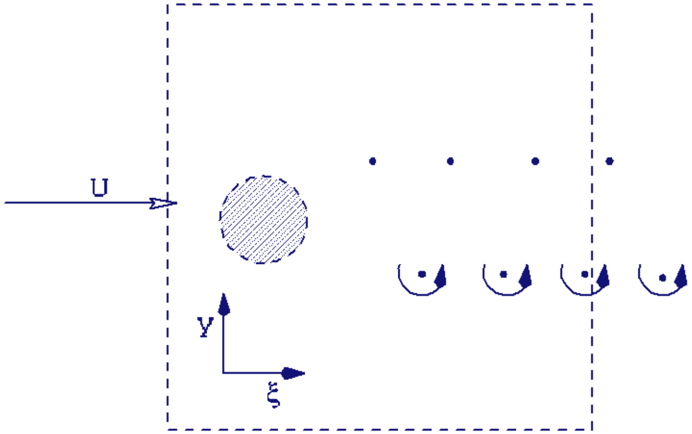
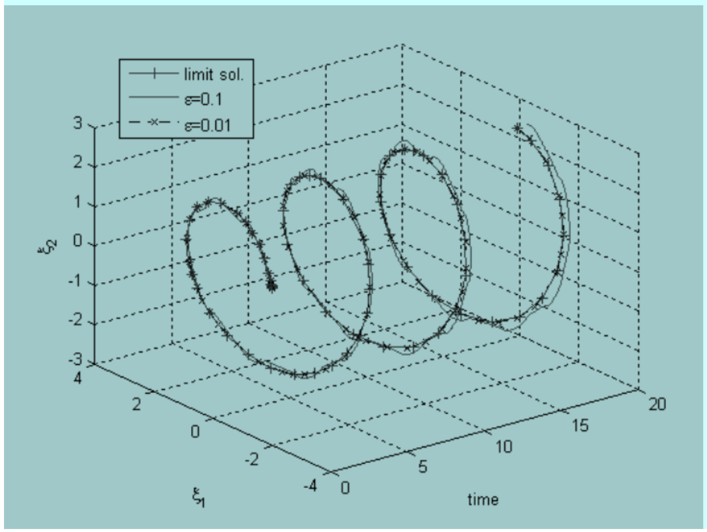
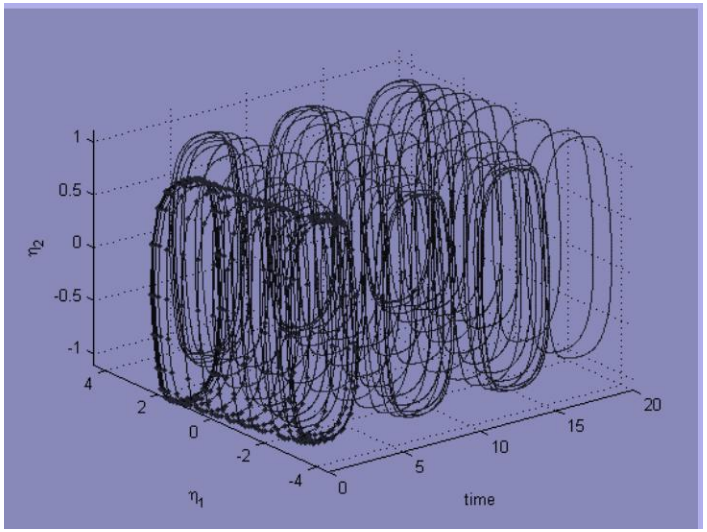


Figure 2

Numerical results:





The slow dynamics

The fast dynamics

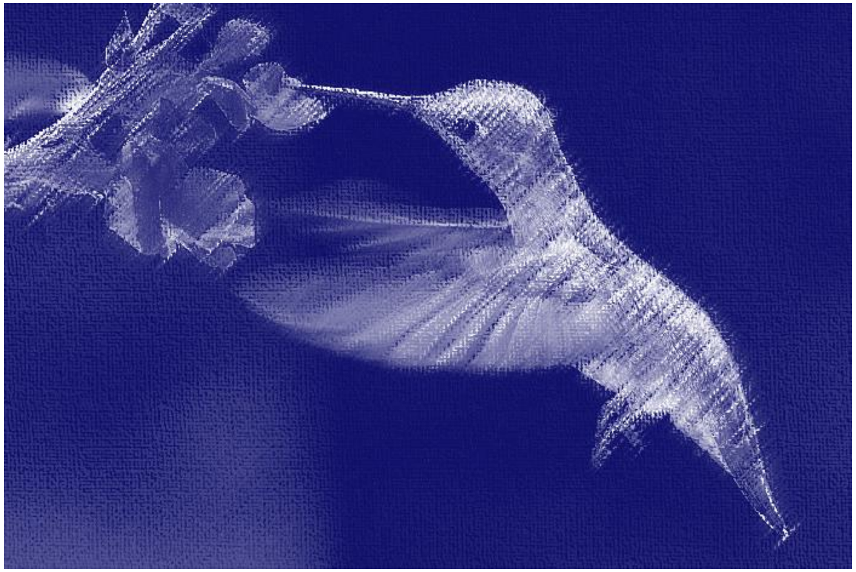
Computations by:

Zvi Artstein

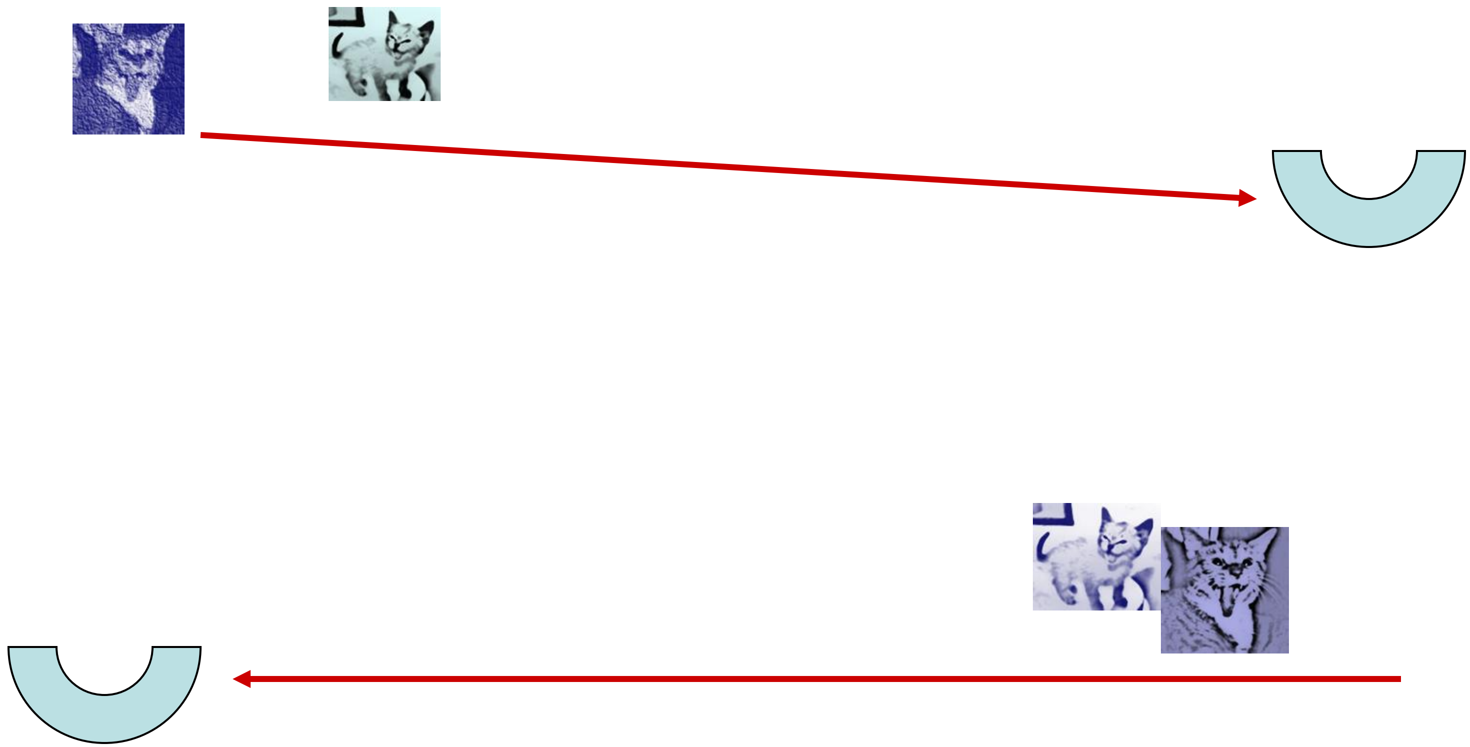
Jasmine Linshiz

Edriss Titi:

Also in Nature: The hummingbird



An illustration of a control problem:



<u>The questions</u>: when should the switch be made? How should this be carried out when the speed is very fast?

An example – after V. Veliov 1996

$$maximize \quad \int_0^1 |y_1(t) - 2y_2(t)|dt$$

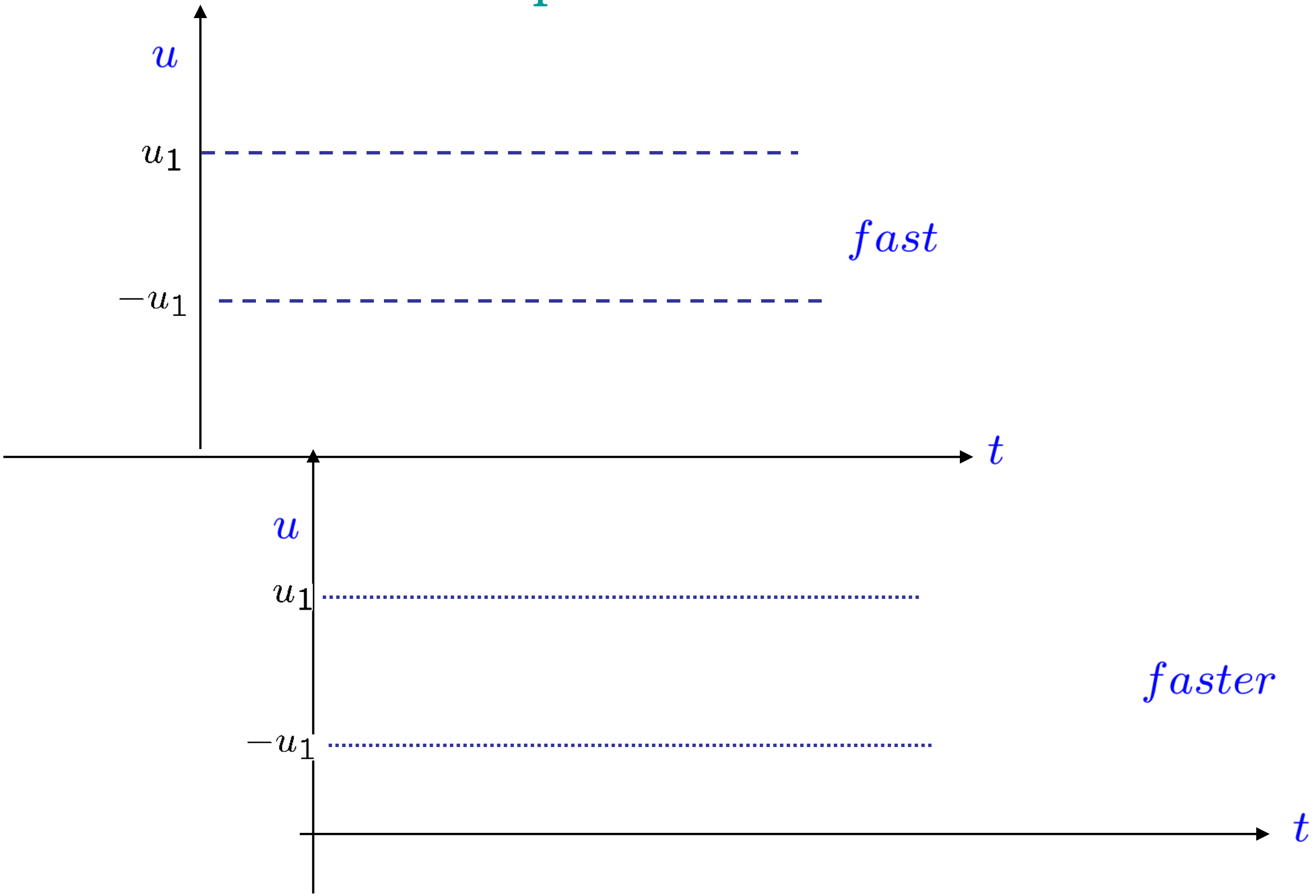
$$subject\ to \quad \epsilon \frac{dy_1}{dt} = -y_1 + u$$

$$\epsilon \frac{dy_2}{dt} = -2y_2 + u$$

$$u \in [-1, 1]$$

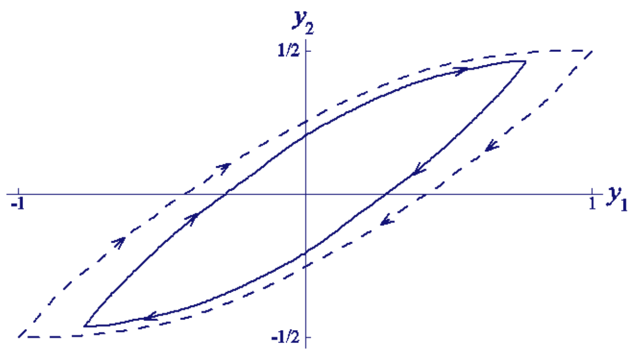
Applying an order reduction (i.e. plugging $\epsilon = 0$) yields zero value. Clearly one can do better!

No equilibrium in the limit:

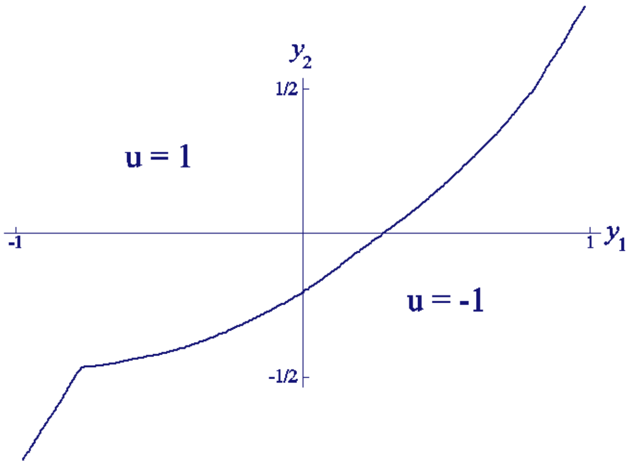


The limit solution:

The limit strategy as $\epsilon \to 0$ can be expressed as a bang-bang feedback $u(y_1, y_2)$ resulting in:



Limit trajectories



The bang-bang feedback

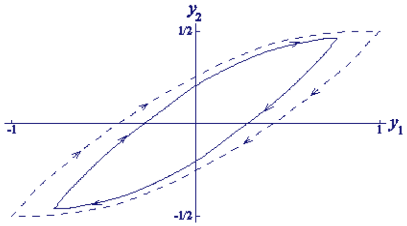
Plan:

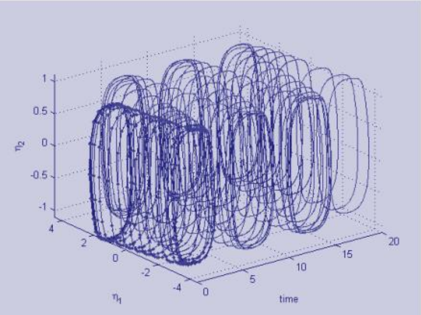
- √ Modeling
- √ Variational Limits
- √ Classical Approach to slow-fast dynamics
- **What limits are appropriate? Young Measures**
- Modern Approach to slow-fast dynamics
- Other chattering limits and averaging techniques
- Control Invariant Measures
- Stabilization
- Optimal Control
- Some special cases
- Computations, error estimates
- A Future Direction

Strong limit on a space of functions:

Recall the strong limit in, say L_2 :

The strong limit may not work for the variational limit of singular perturbations – recall the examples.





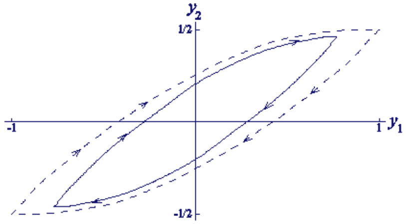
Weak limit on a space of functions:

The L_2 weak-limit:

The sequence $f(\cdot)_j$ converges weakly to $f(\cdot)_0$ if

$$\int_0^T f_j(t) \cdot g(t)\ dt \to \int_0^T f_0(t) \cdot g(t)\ dt$$

The weak limit may not work for the variational limit of singular perturbations – recall the examples.



On a space of parameters:

Consider an ordinary differential equation with parameters $h(\cdot)_j$

$$\frac{dx}{dt} = f(x, t, h_j(t))$$

Strong convergence of the parameters implies continuous dependence of solution but **is not compact**

Weak convergence of the parameters is compact but **does not imply continuous dependence**

Observation:

Continuous dependence and compactness are opposing properties !

Can we construct a convergence that will have both properties: continuous dependence of solution and compactness ?

To the rescue

Laurence Chisholm Young

July 14, 1905 - December 24, 2000
Cambridge, England,   Madison WI, USA

The price:
The limit function will be out of the original space

It is called: **A Young measure**

<u>An implicit</u> Definition of a Young Measure:

Let $h(\cdot)_j$ be a sequence of parameter functions (say bounded from an interval I to R^m.

There exist a subsequence (say the sequence itself) and a family of probability measures $\mu_t(dy)$ on R^m parameterized by $t \in I$ such that for every right hand side $f(x,t,y)$

$f(x,t,h_j(t))$ **converges weakly to** $\int_{R^n} f(x,t,y)\mu_t(dy)$

Proof

Based on (simple) functional analysis arguments (incorporating weak* convergence and Alaoglu compactness Theorem).

A consequence:

Solutions of the ordinary differential equation with parameters $h(\cdot)_j$

$$\frac{dx}{dt} = f(x, t, h_j(t))$$

converge to solution of the ordinary differential equation with the Young measure

$$\frac{dx}{dt} = \int_{R^n} f(x, t, y)\mu_t(dy)$$

A constructive Definition of a Young Measure:

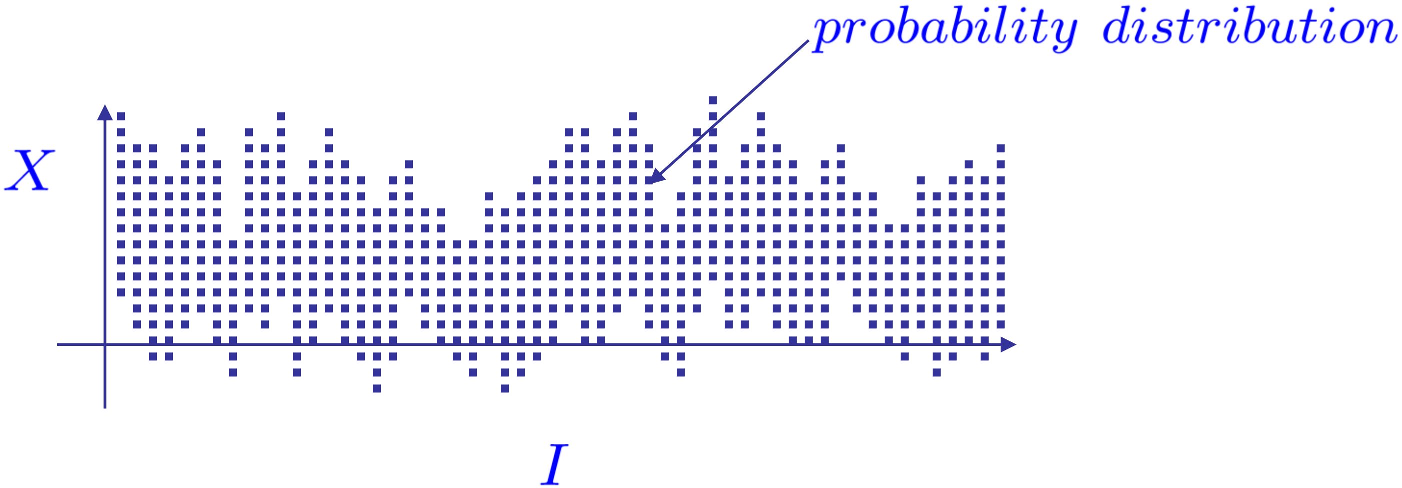
Let X be a metric space

Denote by $P(X)$ the family of probability measures on X

Let I be another metric space endowed with a measure (say Lebesgue measure on an interval)

Definition: A mapping from I to $P(X)$ is a Young Measure

A Pictorial Definition:



The structure of the space $P(X)$:

The elements: σ-additive set-functions from the Borel subsets onto the unit interval.

Convergence of $\mu_j \to \mu_0$ if

$$\int_X h(x)\mu_j(dx) \to \int_X h(x)\mu_0(dx)$$

For every $h(x) : X \to R$ continuous and bounded.

Prohorov metric $Proh(\nu,\mu)$ between measures μ and ν is the smallest η such that for every Borel set B

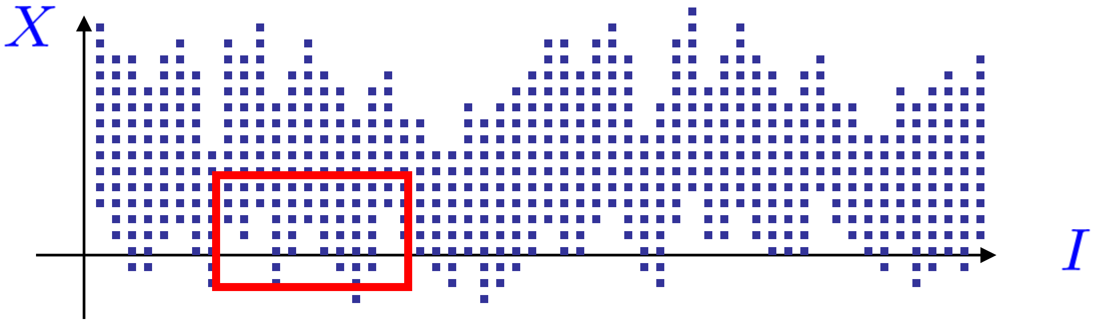
$$\mu(B) \leq \nu(B^\eta) + \eta \quad \text{and} \quad \nu(B) \leq \mu(B^\eta) + \eta$$

Consequences concerning $P(X)$:

If X is complete and separable so is $P(X)$

If X is compact so is $P(X)$

The structure of Young measures :



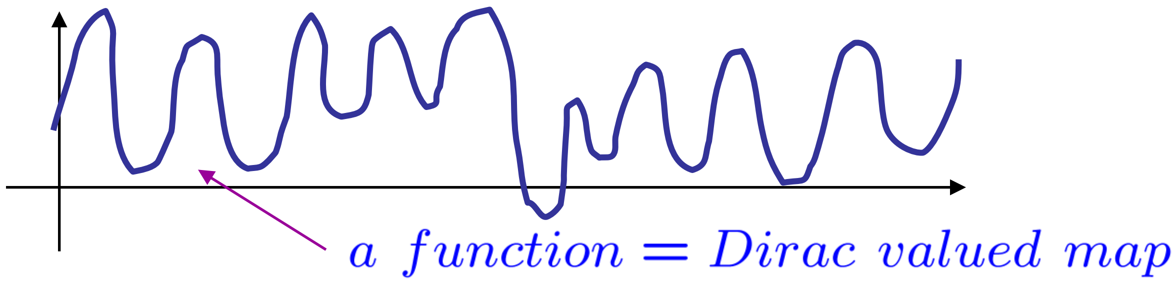
Can be viewed as a "probability" measure on $I \times X$

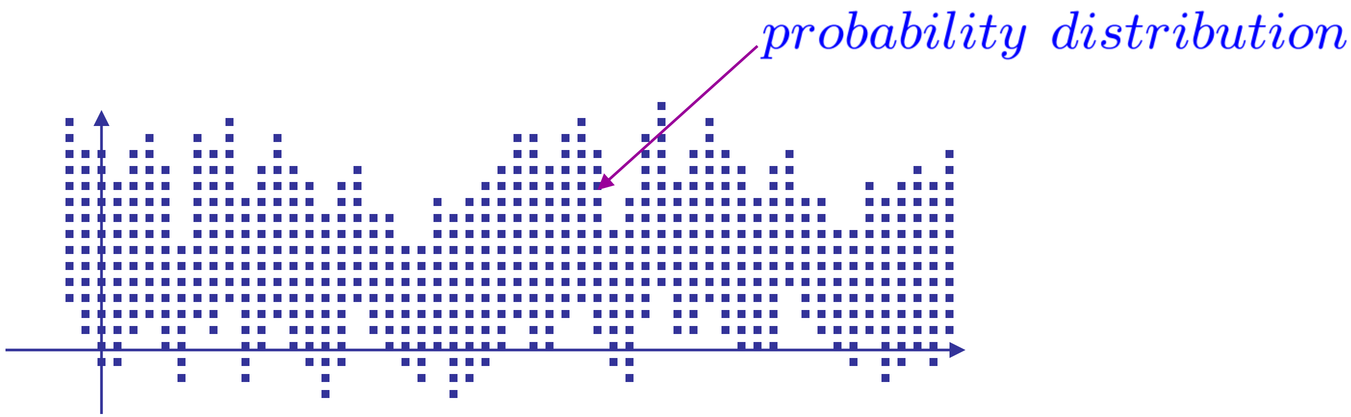
Consequences: If $I \times X$ is complete and separable so is the space of Young measures

If $I \times X$ is compact so is the space of Young measures

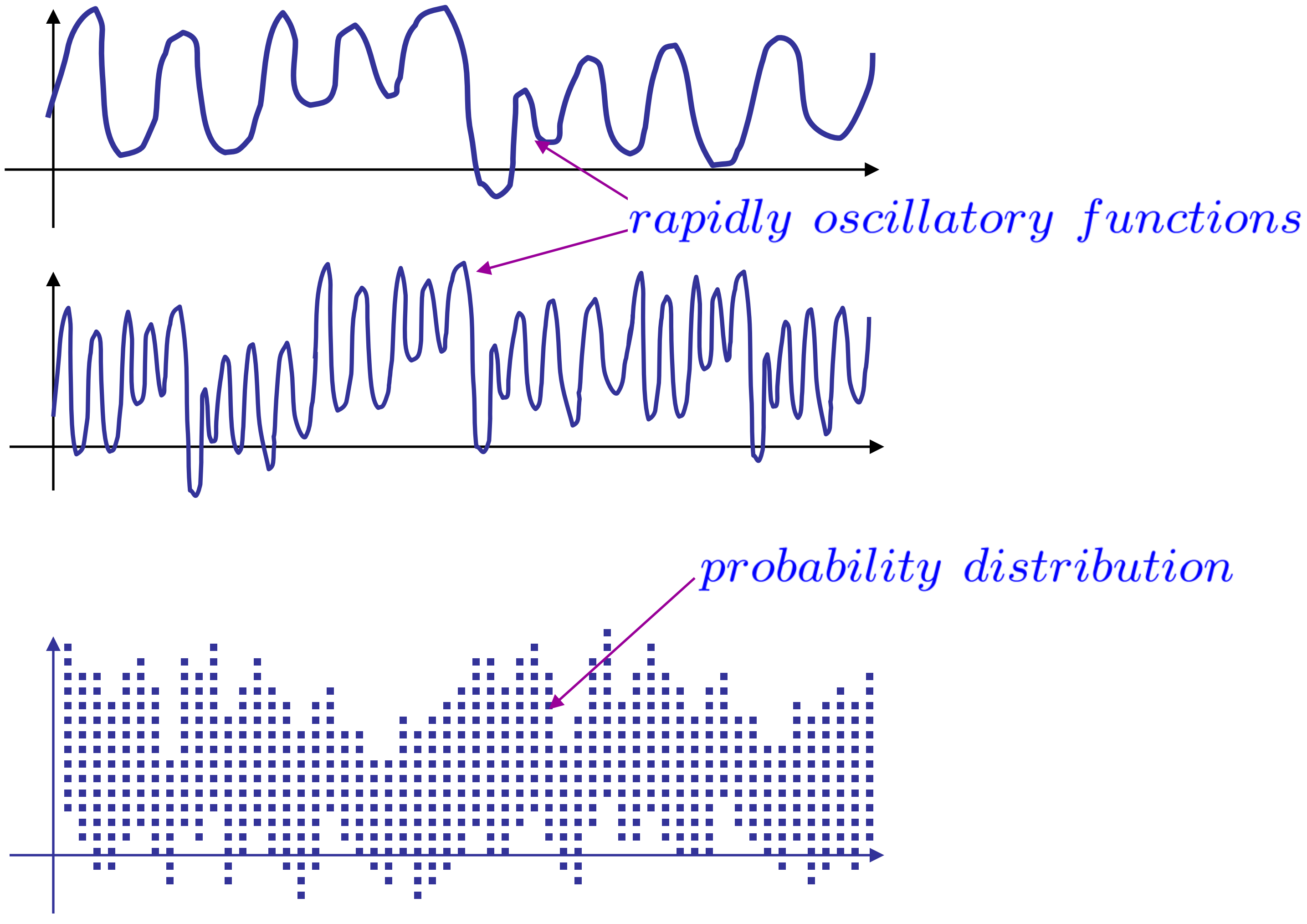
A major property of Young measures:

An ordinary function can be viewed as a Dirac-valued Young measure.





The nature of the convergence



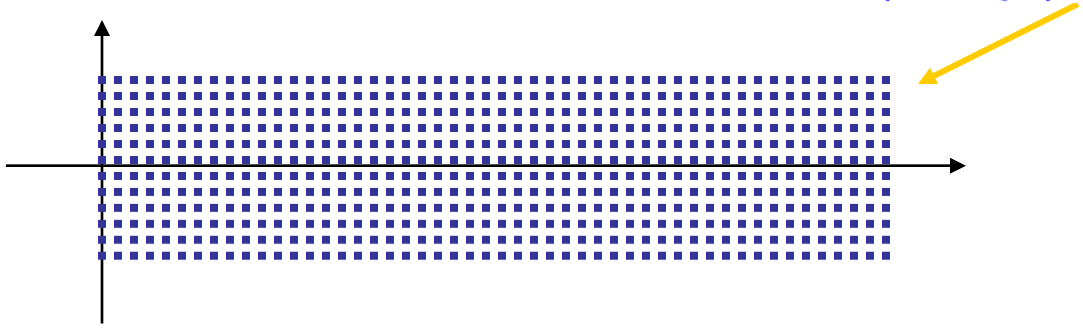
For instance:

The sequence

$$f_j(s) \;=\; sin(js)$$

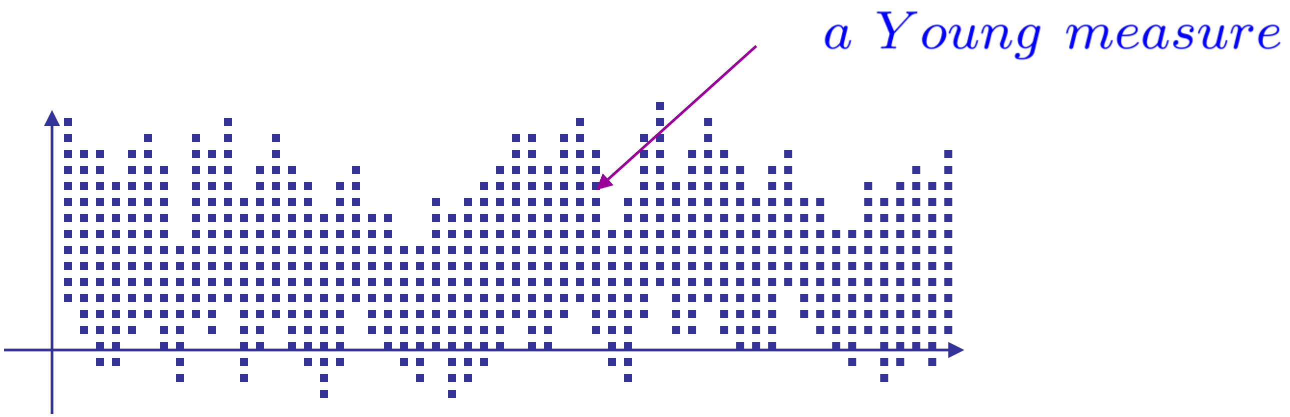
converges to a Young measure with a constant value, namely the measure on $[-1,1]$ given by

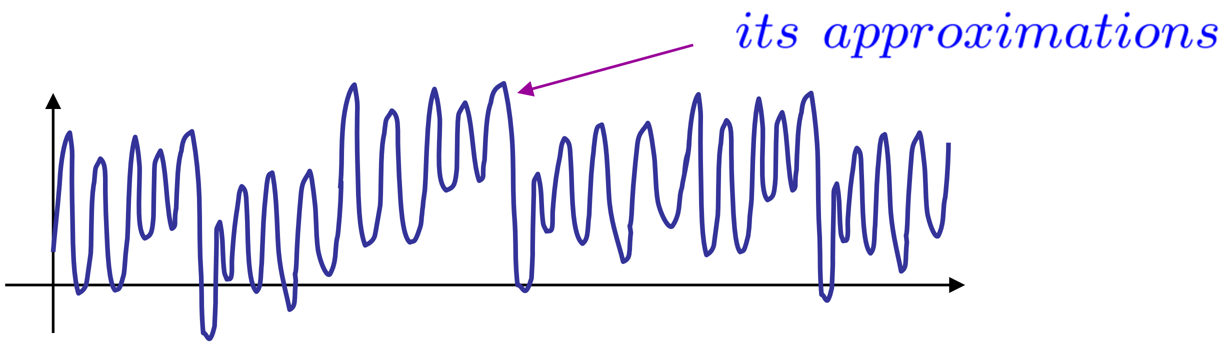
$$d\mu = \frac{1}{\pi(1-\xi^2)^{1/2}} d\xi$$



Functions are dense in the space of Young Measures !

When the underlying space I is without atoms then any Young measure can be approximated by a function

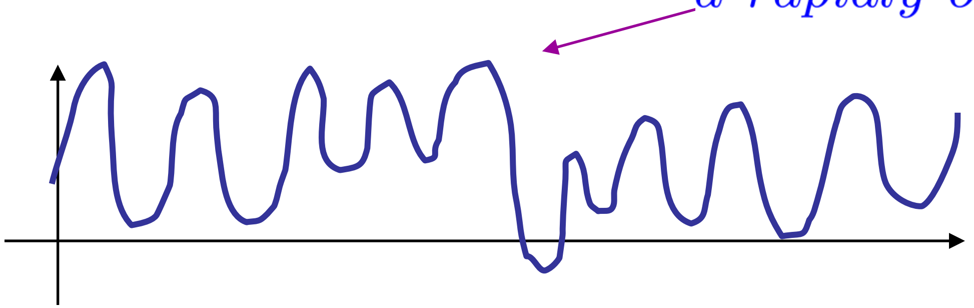




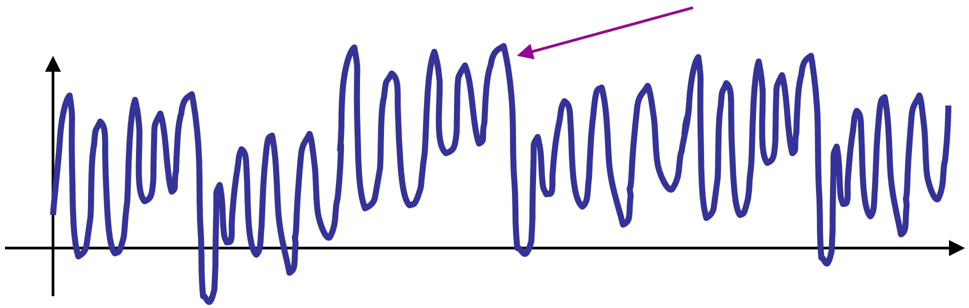
When the limit is a function:

The space of Young Measures completes the space of functions. What convergence does it reflect if the limit Young measure happens to be a function?

a rapidly oscillatory function



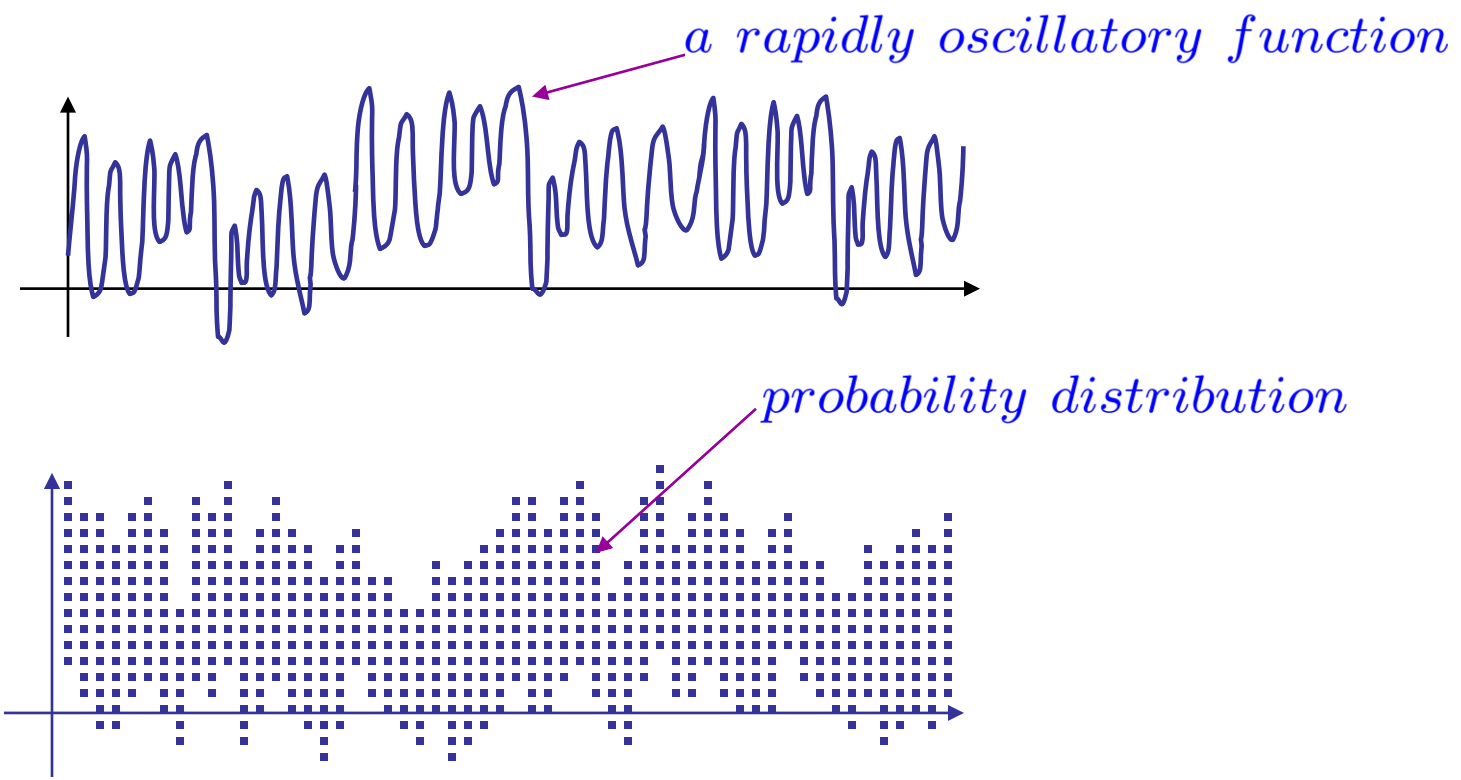
the limit probability distribution that is a function



The limit is then strong (L_1, L_2, but not L_∞)

Key properties:

- ✓ Existence of the limit
- ✓ Keeping information about the location of the values
- ✓ Possibility to approximate by an ordinary function



Recall the case of a space of parameters:

Consider an ordinary differential equation with parameters $h(\cdot)_j$

$$\frac{dx}{dt} = f(x, t, h_j(t))$$

Strong convergence implies continuous dependence of solution but **is not compact**

Weak convergence is compact but **does not imply continuous dependence**

What happens if $h(\cdot)_j$ converges to a Young measure?

Definition:

If $F(z) : Z \to R^n$ and $\mu(dz)$ is a probability measure then

$$F(\mu) = \int_Z F(z)\mu(dz)$$

Likewise, for an ordinary differential equation

$$\frac{dx}{dt} = f(x, t, \mu(t))$$

we mean

$$\frac{dx}{dt} = \int_{R^n} f(x, t, z)\mu(t)(dz)$$

Main application:

Consider an ordinary differential equation with parameters $h(\cdot)_j$

$$\frac{dx}{dt} = f(x, t, h_j(t))$$

And $h(\cdot)_j$ converges to a Young measure $\mu(\cdot)_0$

Then the solutions of the odes with parameters converge to the solution of the ode with the Young measure

Thus, the convergence to the Young measure is both **compact** and **implies continuous dependence**

The key tool:

Consider the right hand side of the ordinary differential equation with parameters

$$f(x, t, h_j(t))$$

and $h(\cdot)_j$ converges to a Young measure $\mu(\cdot)_0$

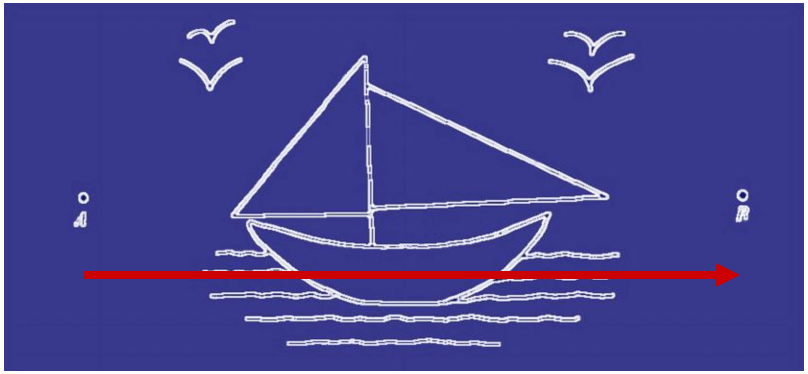
Then $f(x, t, h_j(t))$ converges weakly to $f(x, t, \mu_0(t))$

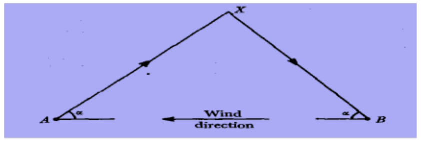
Plan:

- √ Modeling
- √ Variational Limits
- √ Classical Approach to slow-fast dynamics
- √ What limits are appropriate? Young Measures
- **Modern Approach to slow-fast dynamics**
- Other chattering limits and averaging techniques
- Control Invariant Measures
- Stabilization
- Optimal Control
- Some special cases
- Computations, error estimates
- Future Directions

Applications to Control and the Calculus of Variations

An application (after L.C. Young):

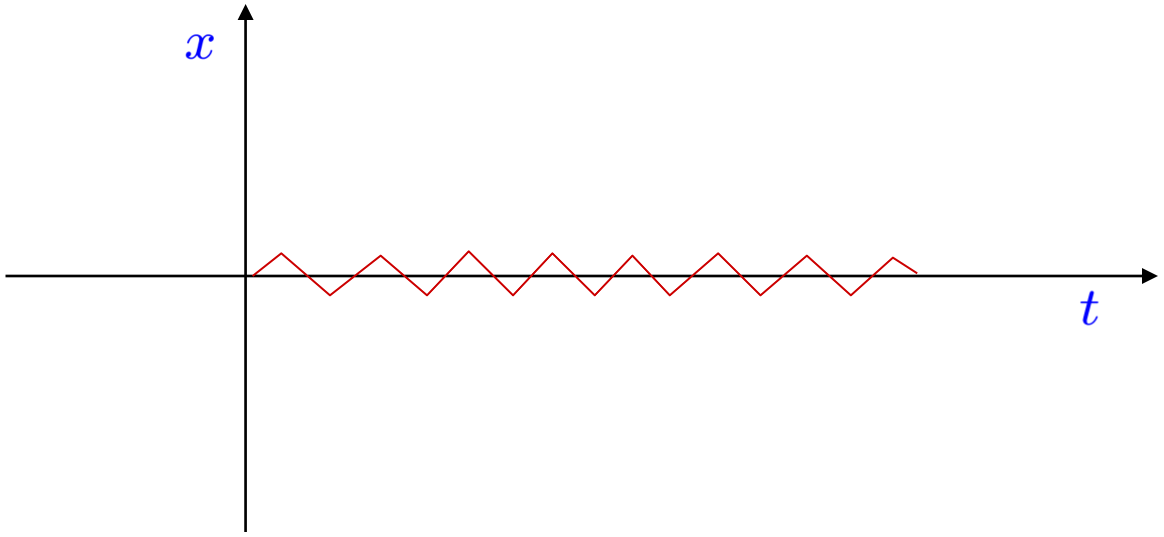




A problem without a solution:

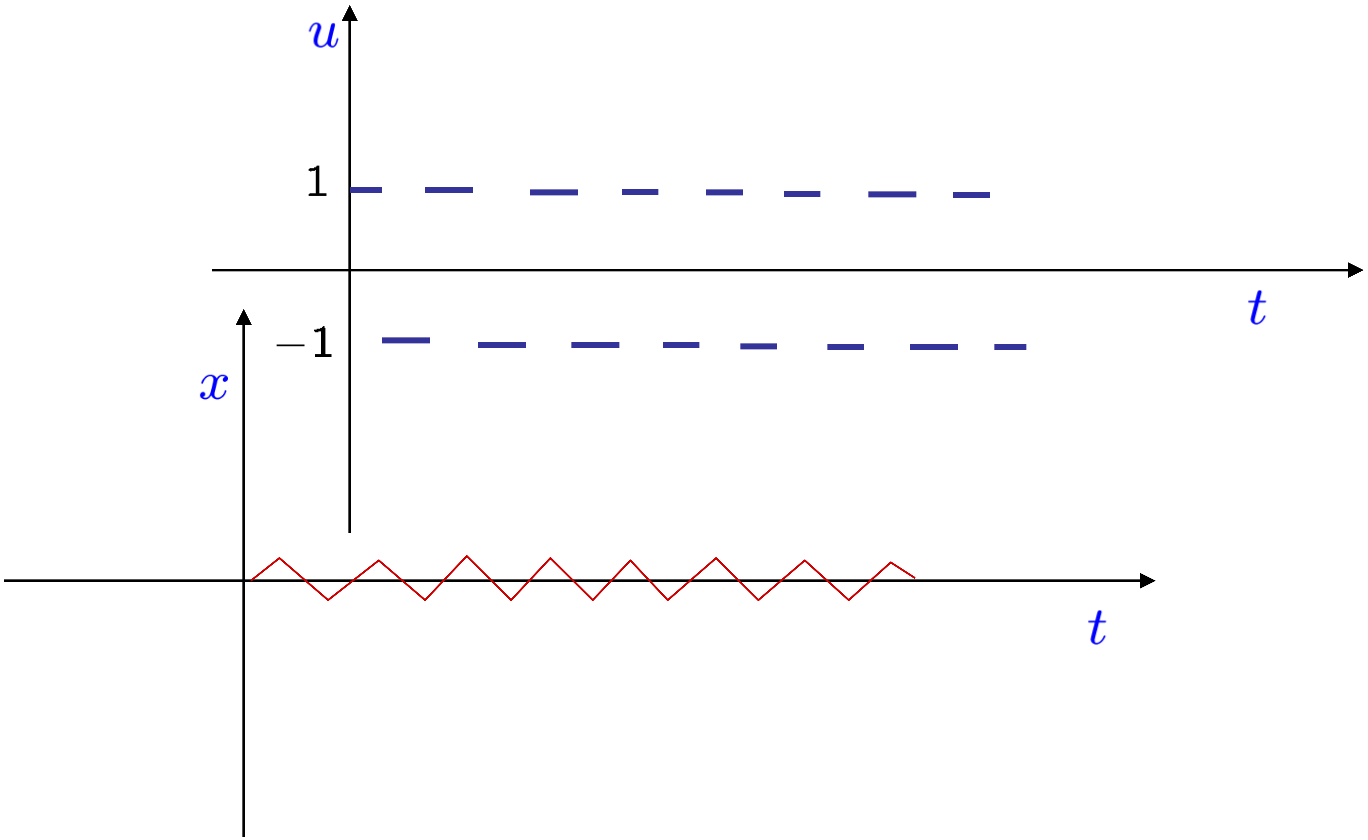
$$\text{minimize} \quad \int_0^1 (x(t)^2 + (1 - u(t)^2)^2)dt$$

$$\text{subject to} \quad \frac{dx}{dt} = u, \quad x(0) = 0$$



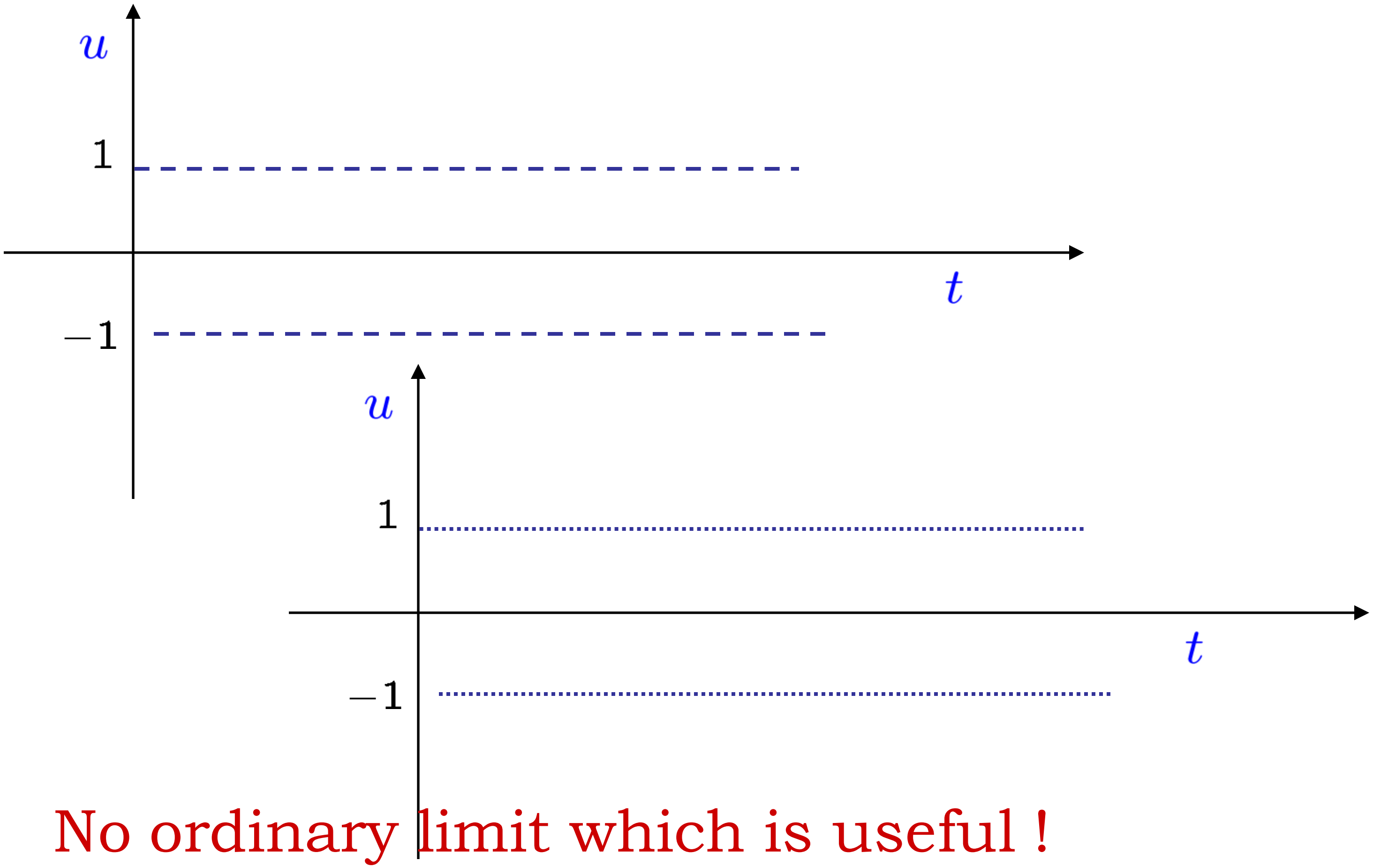
Approximate solutions:

minimize $\int_0^1 (x(t)^2 + (1 - u(t)^2)^2) dt$



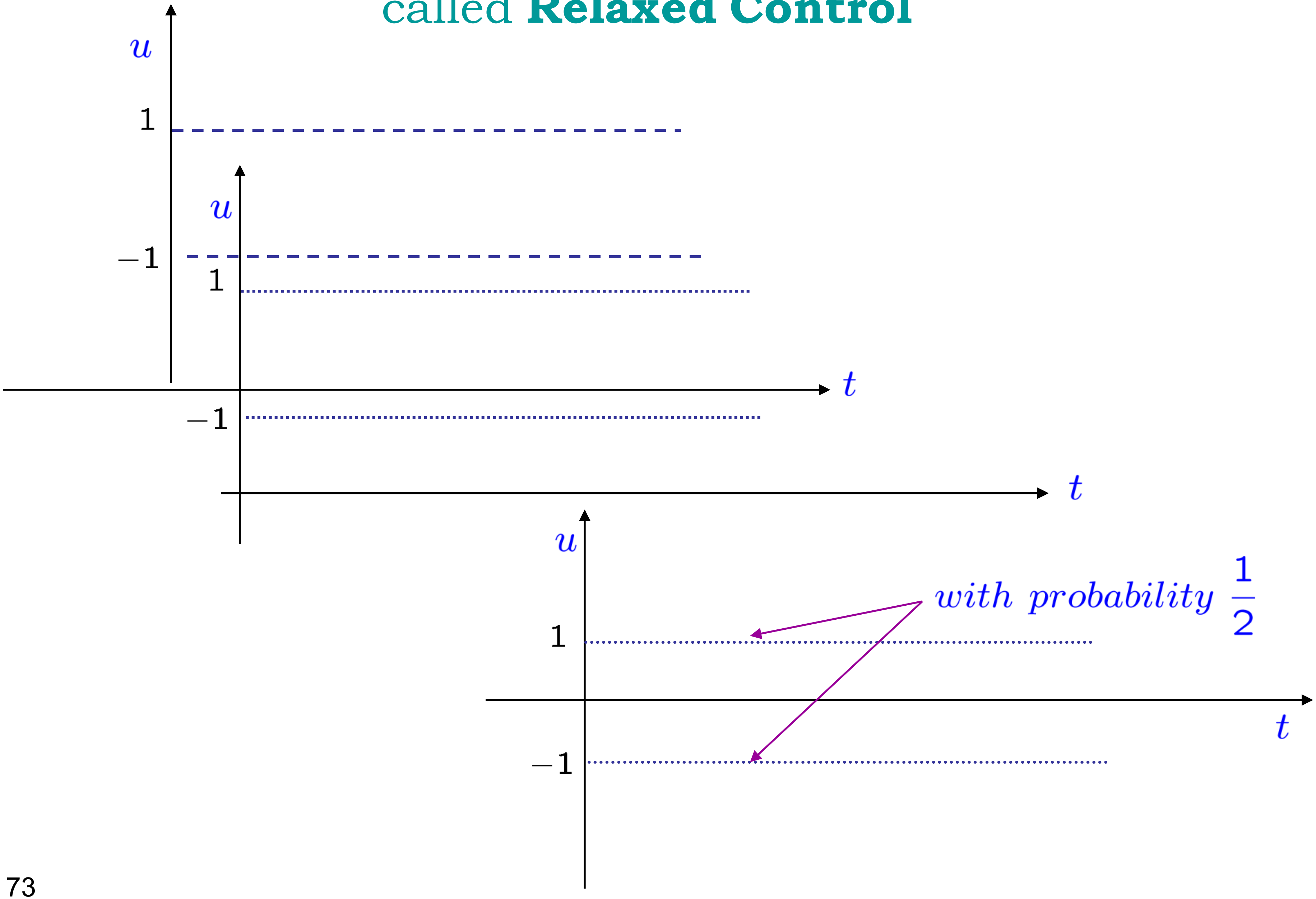
Better approximations:

minimize $\int_0^1 (x(t)^2 + (1 - u(t)^2)^2) dt$



No ordinary limit which is useful !

A generalized limit of a Young measure type: called **Relaxed Control**



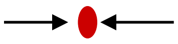
The effect of Relaxed Control:
Convexification of the vector field

$$\frac{dx}{dt} = u$$

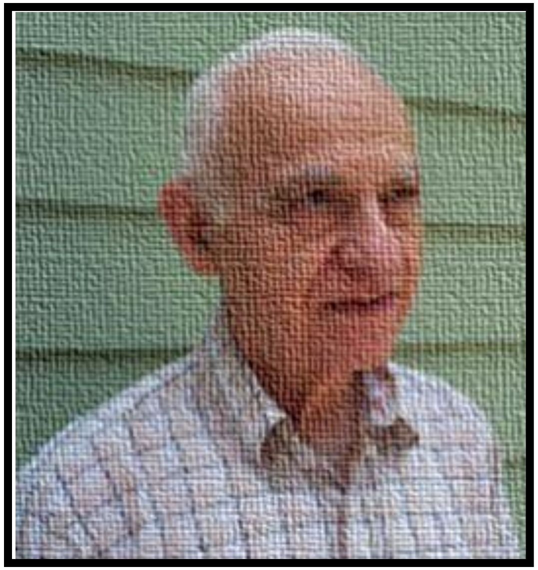
$$u \in \{-1, 1\}$$

Is there a control that makes $x(t) \equiv 0$ a solution?

Yes, the control that averages +1 and -1.



Jack Warga



1922 - 2011

A prior appearance in the calculus of variations:

The general problem:

$$\text{minimize} \quad \int_0^1 L(x(t), \dot{x}(t), t)dt$$

$$s.\ t. \qquad x(0) = x_0, \quad x(1) = x_1$$

The particular problem without a solution:

$$\text{minimize} \quad \int_0^1 (x(t)^2 + (1 - \dot{x}(t)^2)^2)dt$$

$$s.\ t. \qquad x(0) = 0$$

Generalized curves in the calculus of variations:

For the problem:

$$\text{minimize} \quad \int_0^1 L(x(t), \dot{x}(t), t)\ dt$$

$$s.\ t. \qquad x(0) = x_0, \quad x(1) = x_1$$

A **generalized curve** is a pair:

$$(x(t), \mu(t))$$

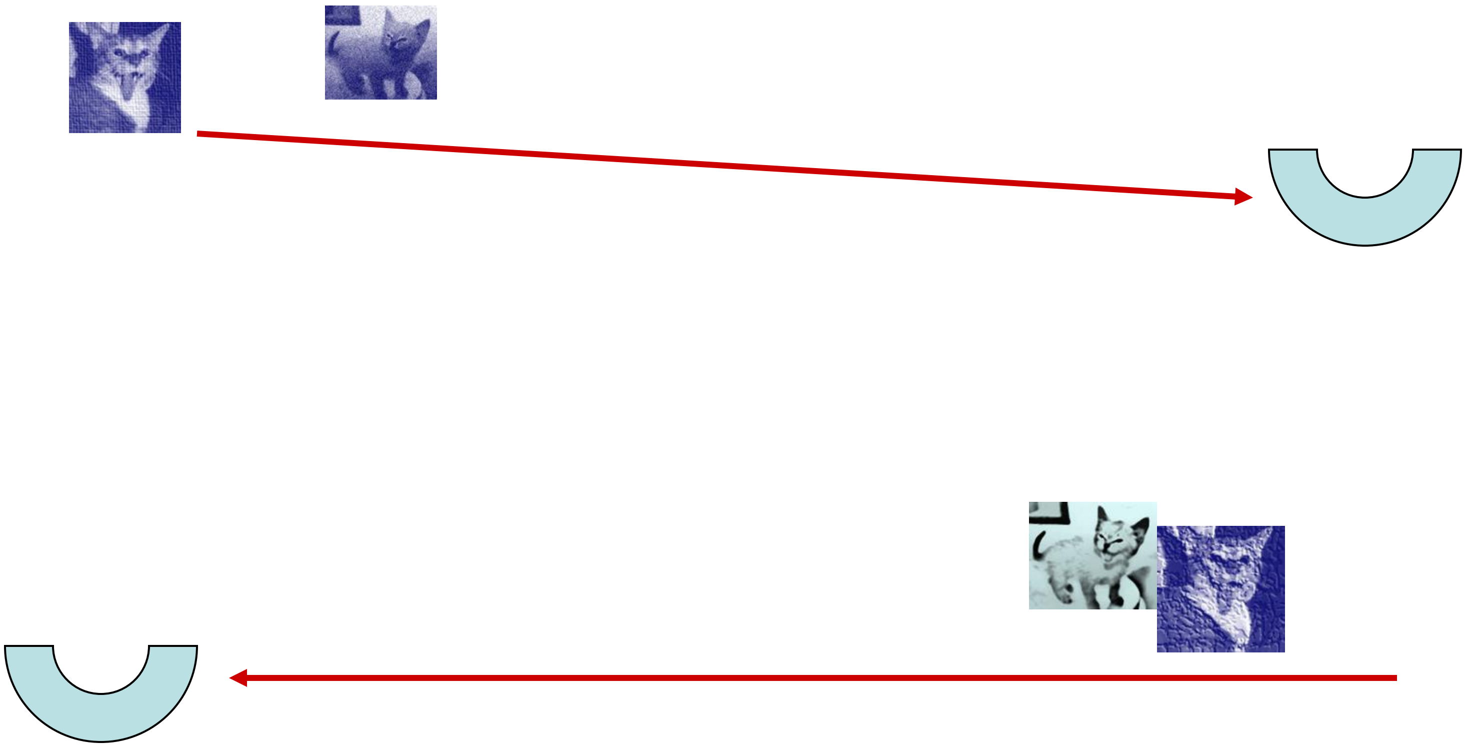
$x(t)$: *curve*; $\mu(t)$: *probability distribution*

satisfying: $\dot{x}(t) = E(\mu(t))$

The goal:

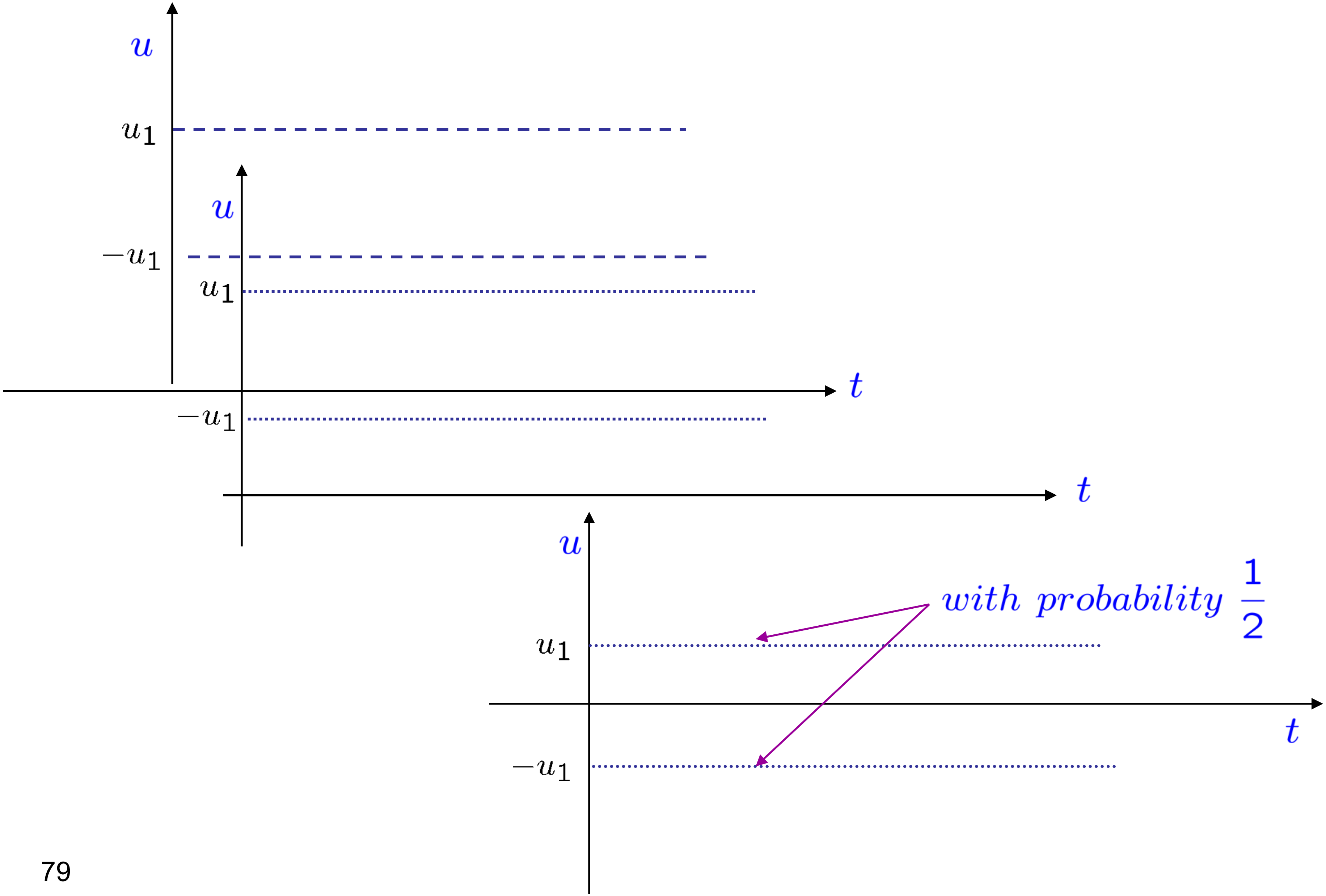
$$\text{minimize} \quad \int_0^1 \int_{R^n} L(x(t), y, t)\mu(t)(dy)\ dt$$

Recall the illustration of a control problem:



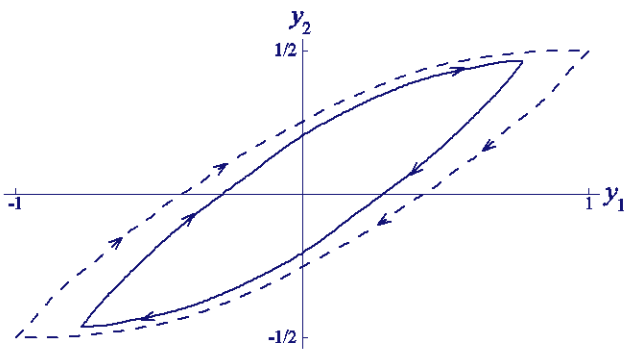
<u>The questions</u>: when should the switch be made? How should this be carried out when the speed is very fast?

In the limit: A distribution arises

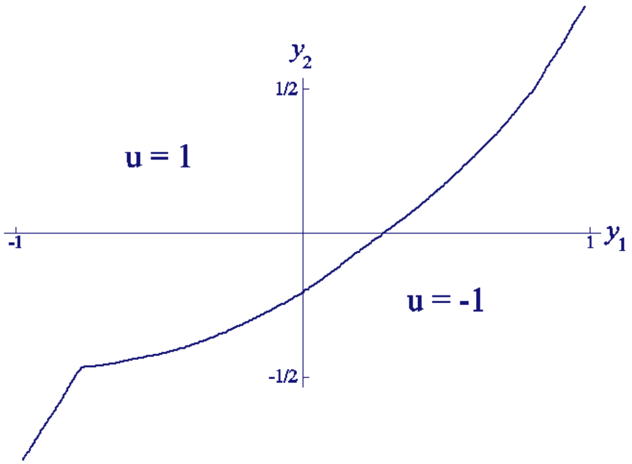


The limit solution:

The limit strategy as $\epsilon \to 0$ can be expressed as a bang-bang feedback $u(y_1, y_2)$ resulting in:



Limit measure



The bang-bang feedback

Recall: A mathematical example capturing reality: An elastic structure in a rapidly flowing nearly invicid fluid

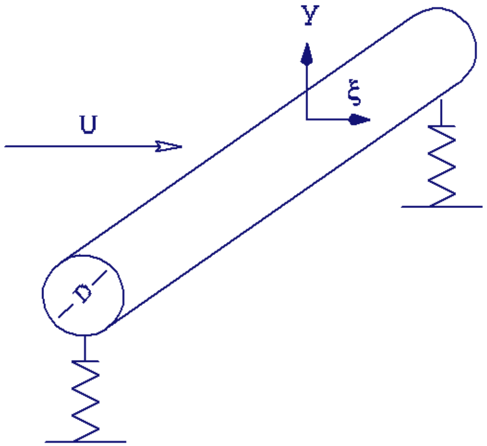


Figure 1

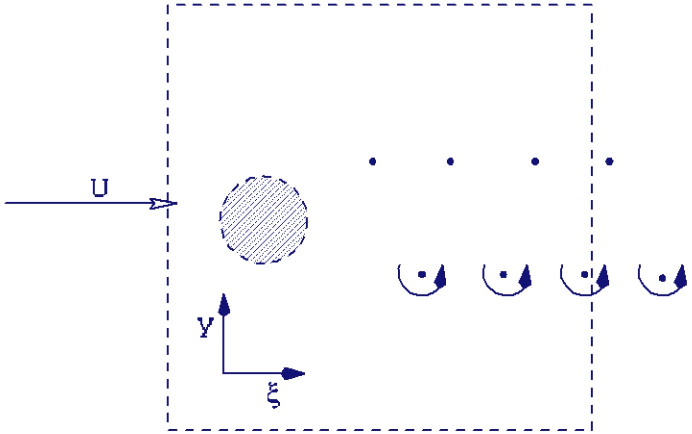
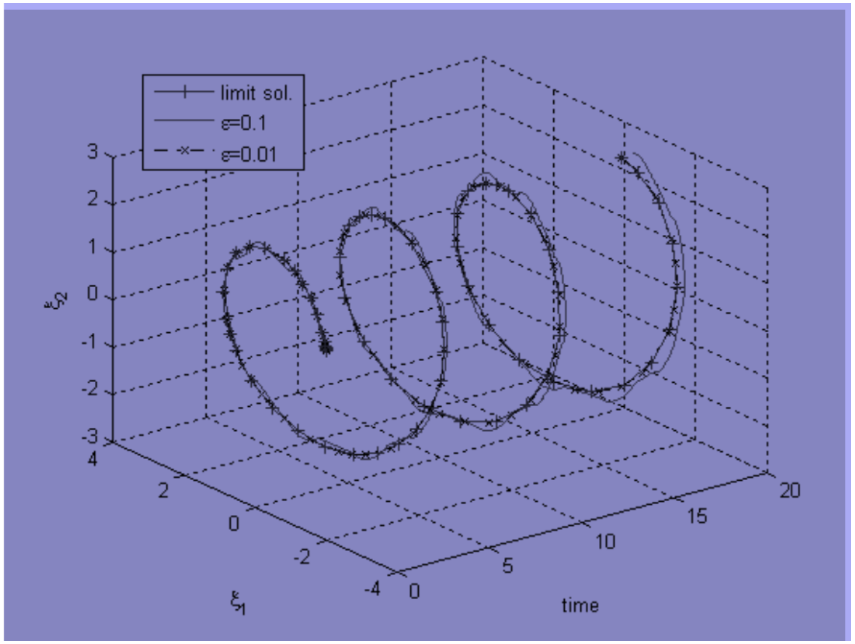
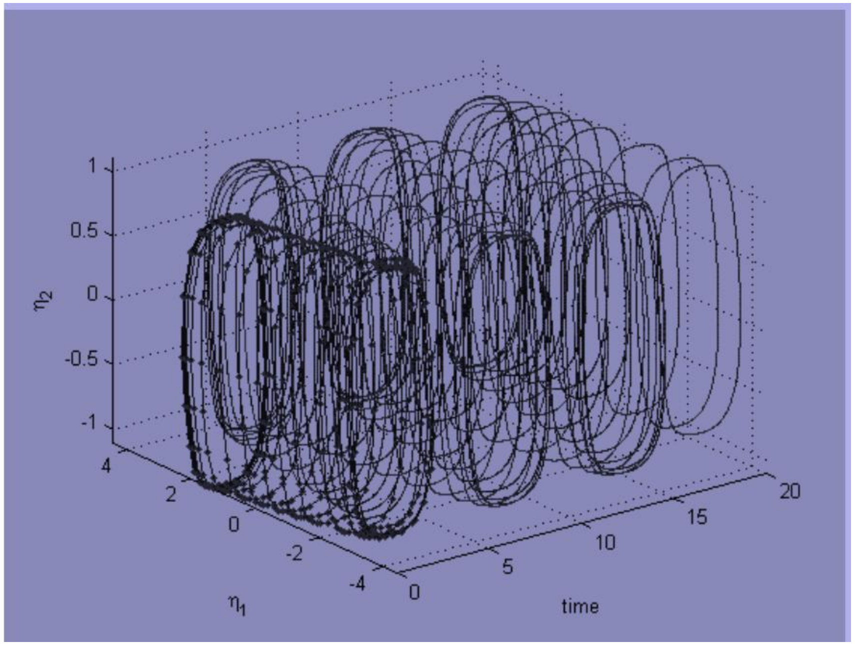


Figure 2

Numerical results:





The slow dynamics

The fast dynamics

Recall: Singular perturbations as a model of coupled slow and fast motions:

The perturbed system:

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = \frac{1}{\varepsilon}\, g(x, y)$$

Equivalently:

$$\epsilon\frac{dy}{dt} = g(x, y)$$

We are interested in the **limit behavior** of the system as $\epsilon \to 0$

The classical Tikhonov approach

Write the perturbed system as:

$$\frac{dx}{dt} = f(x,y)$$

$$\epsilon\frac{dy}{dt} = g(x,y)$$

The limit behavior as $\epsilon \to 0$ is captured by the system:

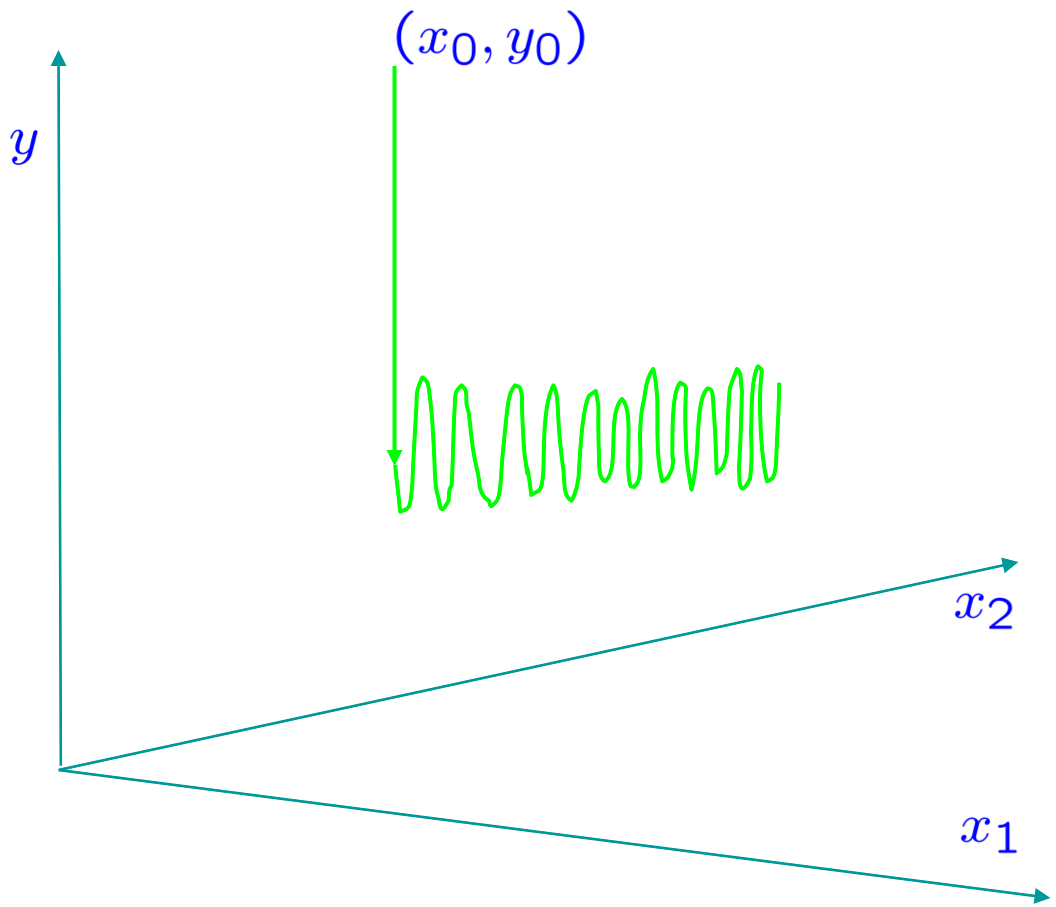
$$\frac{dx}{dt} = f(x,y)$$

$$0 = g(x,y)$$

This type of variational limit does not capture the general situation

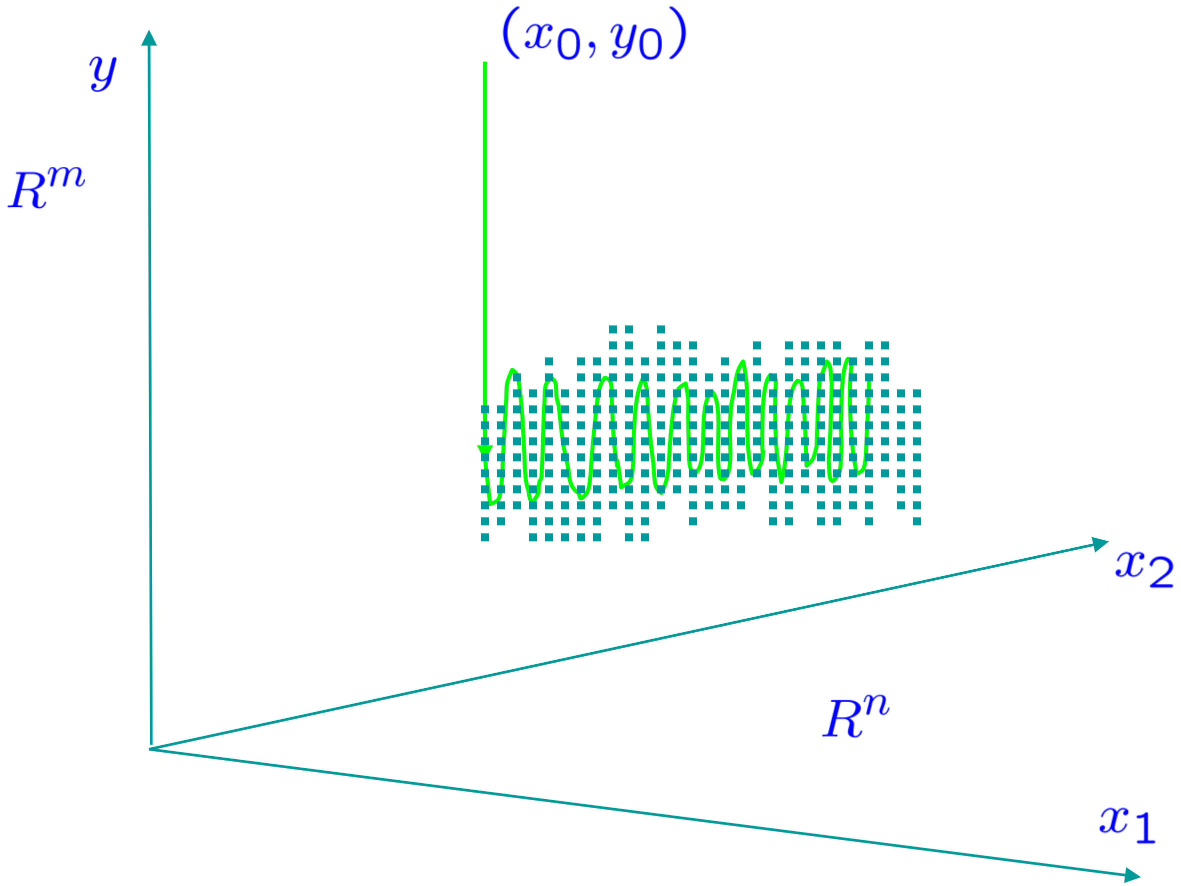
A program to exploit Young Measures this started by Zvi Artstein and Alexander Vigodner, 1996.

The general situation:



The solution oscillates. The limit (as $\epsilon \to 0$) may be described as a Young measure

The general situation:



The Young measure is defined on the x-space with values being probability measures on the y-space

The variational limit solution in the new formulation :

$$(x(t), \mu(x(t)))$$

where $\mu(x)(dy)$ is a Young measure

and $x(t)$ solves the averaging equation

$$\frac{dx}{dt} = \int_Y f(x,y)\mu(x)(dy)$$

Now to control systems

Recall: Singularly perturbed control systems:

$$\textit{minimize} \quad \int_a^b c(x,y,u)dt$$

$$\textit{subject to} \quad \frac{dx}{dt} = f(x,y,u)$$

$$\epsilon\frac{dy}{dt} = g(x,y,u)$$

$$x(a) = x_0$$

$$y(a) = y_0$$

$$u \in U$$

Where: x in R^n the slow and y in R^m the fast, variables

Of interest: The behavior of the system as $\epsilon \to 0$

The order reduction method (Petar Kokotovic et al.)

The limit as $\epsilon \to 0$ is depicted by $\epsilon = 0$ namely, by:

$$\textit{minimize} \quad \int_a^b c(x,y,u)\,dt$$

$$\textit{subject to} \quad \frac{dx}{dt} = f(x,y,u)$$

$$0 = g(x,y,u)$$

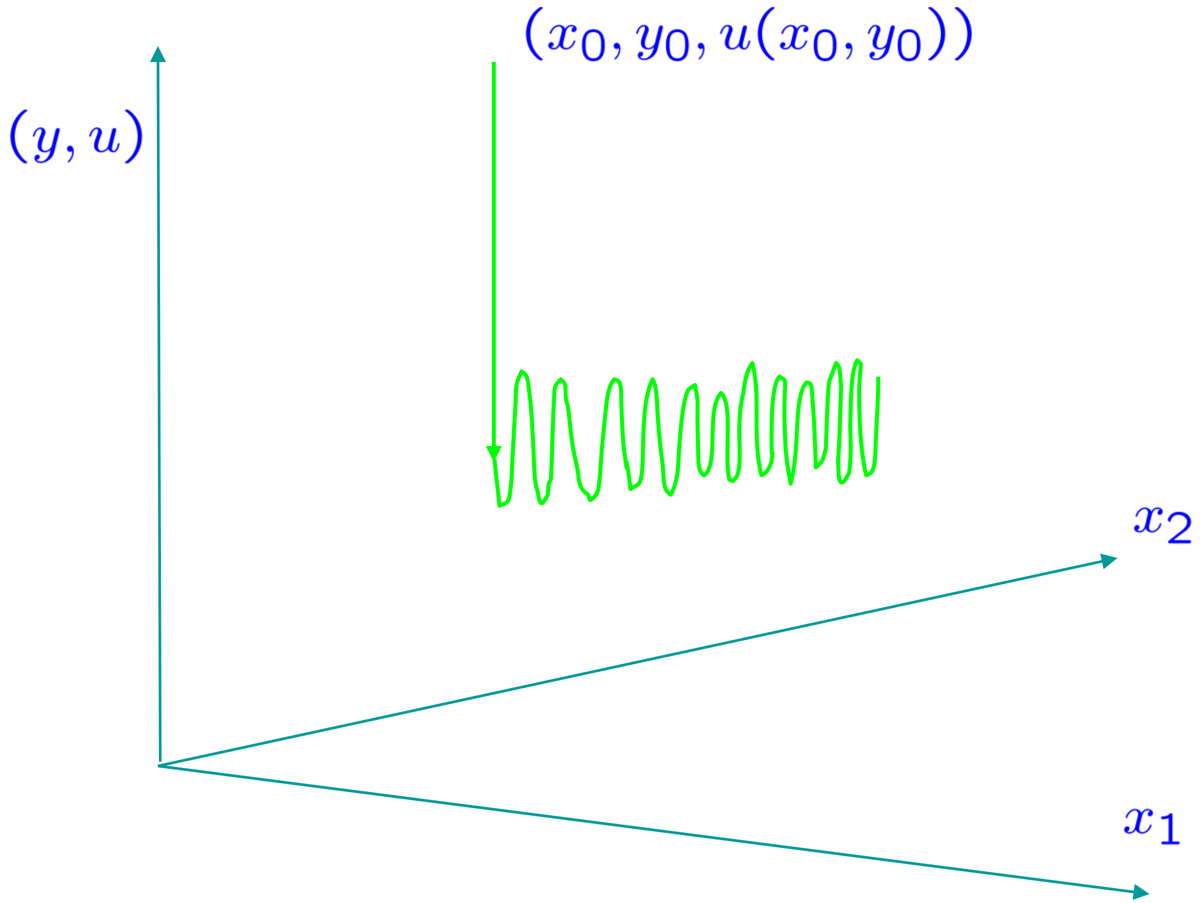
$$x(a) = x_0$$

$$y(a) = y_0$$

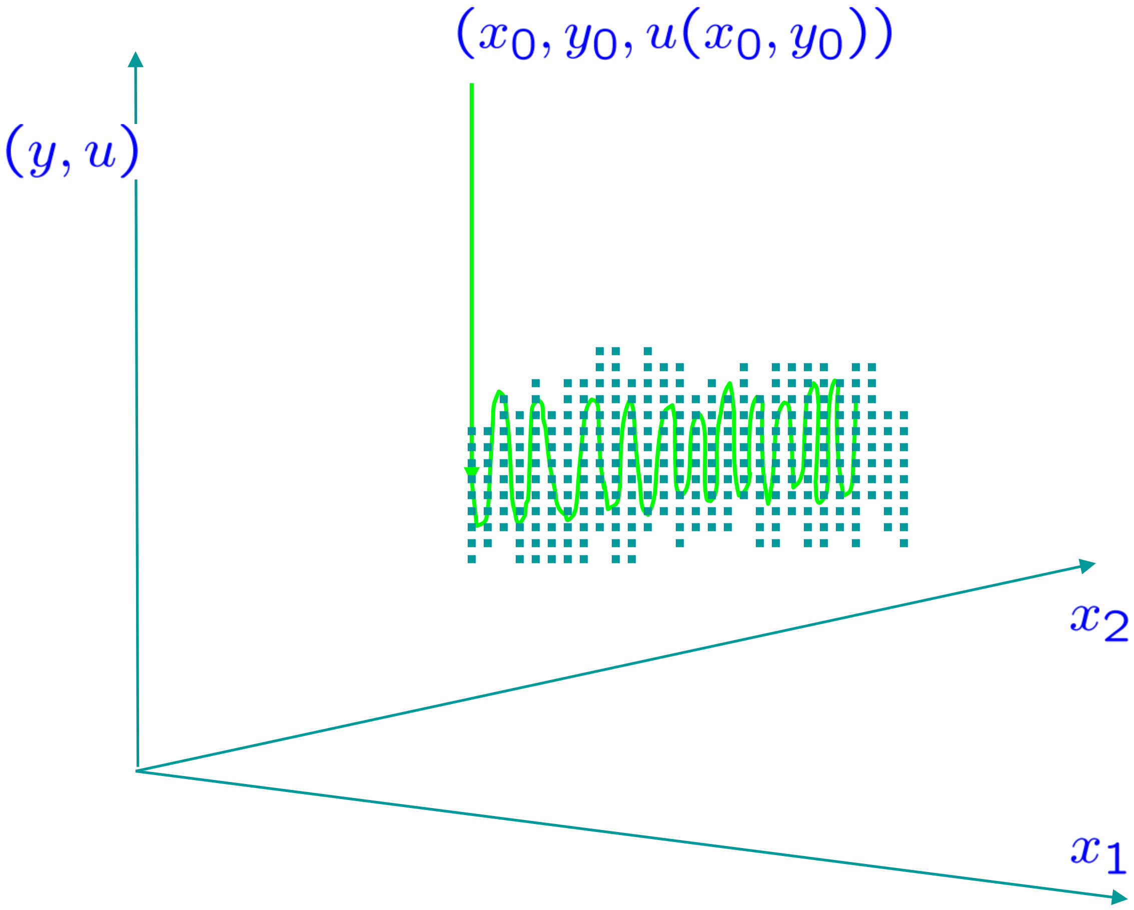
$$u \in U$$

The general situation:

There is no reason why the optimal fast solution will converge and not, say, oscillate!



The general situation:



The values of the Young measure are:

measures of the (fast state, control) dynamics !

The general variational limit solution is of the form:

$$(x(t), \mu(x(t)))$$

Where: $x(t)$ solves the averaging equation

$$\frac{dx}{dt} = \int_{Y \times U} f(x, y, u)\mu(x)(dy \times du);$$

$\mu(x)(dy \times du)$ is a Young measure (parameterized by x)

and the limit cost is based on **averaging:**

$$\int_a^b \int_{Y \times U} c(x(t), y, u)\mu(x(t))(dy \times du)\ dt$$

Notice, the values of the Young measure are the **control variables**, (replacing the equilibrium points in the classical case

The “equivalent” differential inclusion:

$x(t)$ solves the differential inclusion

$$\frac{dx}{dt} \in F(x)$$

where

$$F(x) = \{ \int_{Y \times U} f(x, y, u) \; \mu(x)(dy \times du) \}$$

$\mu(x)(dy \times du)$ is a Young measure (parameterized by x)

Notice, the values of the Young measure are the **control variables**, here they determine the velocity of the slow variable

A question:

Could any probability measure be a value for the Young Measure of the variational limit?

If not, how can the possible values be classified and identified?

A promise:

We shall soon give a characterization of the probability measures that may appear as values in the variational limit.

We denote this family by $IM(x)$

Recall: A variational limit

What do we want from a variational limit?

1. Convergence of the values
2. Convergence of trajectories
3. Convergence of optimal controls

and

4. Possibility to construct near optimal solution for the perturbed system given an optimal solution to the variational limit.

A question:

Under what conditions is

$$\frac{dx}{dt} = \int_{Y \times U} f(x, y, u) \mu(dy \times du)$$

$$\mu \in IM(x)$$

a variational limit of

$$\frac{dx}{dt} = f(x, y, u)$$

$$\epsilon \frac{dy}{dt} = g(x, y, u)$$

A Theorem:

The conditions are:

- Regularity (modest) of $f(x,y,u)$ and $g(x,y,u)$
- Uniform boundedness and controllability of solutions of

$$\epsilon \frac{dy}{dt} = g(x,y,u)$$

- The set-valued map

$$F(x) = \{ \int_{Y\times U} f(x,y,u)\ \mu(dy \times du) : \ \mu \in IM(x) \}$$

is Lipschitz

The Lipschitz condition cannot be dropped:

Example (Olivier Alvarez, Martino Bardi):

$$\textit{minimize} \quad x(1)$$

$$\textit{subject to} \quad \frac{dx}{dt} = min(|\theta|, |\theta - 2\pi|)$$

$$\epsilon \frac{d\theta}{dt} = x + u$$

$$u \in [0, 1]$$

θ is a polar coordinate

An issue:

How to relate trajectories (say optimal solutions) of the limit problem to the perturbed problem?

The answer:

If $u_\epsilon(t)$ is designed such that $(y_\epsilon(t), u_\epsilon(t))$ approximates the limit Young Measure $\mu(x(t))$ (in the space of Young Measures), the outcome of the perturbed equation will be a good approximation of the limit (hence of the optimal solution to the perturbed equation. Under the conditions of the theorem his can be done !

The End

of lecure 1

Thanks for the attention

See you tomorrow