# Three Lectures on:
# Control of Coupled Fast and Slow Dynamics

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# Control of Coupled Fast and Slow Dynamics

Zvi Artstein

**Plan:**

Modeling
Variational Limits
Classical Approach to slow-fast dynamics
What limits are appropriate? Young Measures
Modern Approach to slow-fast dynamics
Other chattering limits and averaging techniques
Control Invariant Measures
Stabilization
Optimal Control
Some special cases
Computations, error estimates
A Future Direction

# Lecture 2

**Plan:**

- √ Modeling
- √ Variational Limits
- √ Classical Approach to slow-fast dynamics
- √ What limits are appropriate? Young Measures
- √ Modern Approach to slow-fast dynamics
- **Other chattering limits and averaging techniques**
- Control Invariant Measures
- Stabilization
- Optimal Control
- Some special cases
- Computations, error estimates
- A Future Direction

## Recall: Singularly perturbed control systems:

$$\textit{minimize} \quad \boxed{\int_a^b c(x,y,u)\,dt}$$

$$\textit{subject to} \quad \boxed{\begin{aligned} \frac{dx}{dt} &= f(x,y,u) \\ \epsilon\frac{dy}{dt} &= g(x,y,u) \end{aligned}}$$

$$x(a) = x_0$$

$$y(a) = y_0$$

$$u \in U$$

Where: $x$ in $R^n$ the slow and $y$ in $R^m$ the fast, variables

Of interest: The behavior of the system as $\epsilon \rightarrow 0$

The general variational limit solution is of the form:

$$(x(t), \mu(x(t)))$$

Where: $x(t)$ solves the averaging equation

$$\frac{dx}{dt} = \int_{Y \times U} f(x, y, u)\mu(x)(dy \times du);$$

$\mu(x)(dy \times du)$ is a Young measure (parameterized by $x$)

and the limit cost is based on **averaging:**

$$\int_a^b \int_{Y \times U} c(x(t), y, u)\mu(x(t))(dy \times du) \; dt$$

**Notice,** the values of the Young measure are the **control variables**, (replacing the equilibrium points in the classical case

## A question:

Under what conditions is

$$\frac{dx}{dt} = \int_{Y\times U} f(x,y,u)\mu(dy \times du)$$

$$\mu \in IM(x)$$

a variational limit of

$$\frac{dx}{dt} = f(x,y,u)$$

$$\epsilon\frac{dy}{dt} = g(x,y,u)$$

## A Theorem:

The conditions are:

- Regularity (modest) of $f(x,y,u)$ and $g(x,y,u)$
- Uniform boundedness and controllability of solutions of

$$\epsilon \frac{dy}{dt} = g(x,y,u)$$

- The set-valued map

$$F(x) = \{ \int_{Y \times U} f(x,y,u) \; \mu(dy \times du) : \;\; \mu \in IM(x) \}$$

is Lipschitz

# Other chattering limits and averaging techniques

## The classical averaging:

Consider a smooth ordinary differential equation

$$\frac{dx}{dt} = f(x, \frac{t}{\epsilon}), \quad x(t_0) = x_0$$

where $f(x,t)$ periodic with period $T$.

The solutions converge as $\epsilon \to 0$ to the solution of

$$\frac{dx}{dt} = f_0(x), \quad x(t_0) = x_0$$

where $f_0(x) = \frac{1}{T}\int_0^T f(\tau, x)d\tau$

## More general averaging:

Consider an ordinary differential equation

$$\frac{dx}{dt} = f_\epsilon(x,t), \quad x(t_0) = x_0$$

If for every $x$ the functions $f_\epsilon(x,t)$ converge weakly to $f_0(x,t)$ then the solution converge to the solution of

$$\frac{dx}{dt} = f_0(x,t), \quad x(t_0) = x_0$$

## A different averaging (known to the Greeks):

Consider a **scalar** ordinary differential equation

$$\frac{dx}{dt} = f(\frac{x}{\epsilon}), \quad x(t_0) = x_0$$

Where $f(x)$ is periodic

say $\quad f(x) = 1 \;$ if $\quad 2k \leq x < 2k+1$

$$f(x) = 2 \qquad 2k+1 \leq x < 2k$$

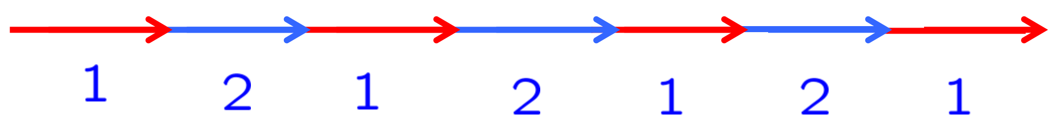
Consider a **scalar** ordinary differential equation

$$\frac{dx}{dt} = f\left(\frac{x}{\epsilon}\right), \quad x(t_0) = x_0$$

Where $f(x)$ is periodic

say $f(x) = 1$ if $2k \leq x < 2k+1$

$f(x) = 2 \qquad 2k+1 \leq x < 2k$



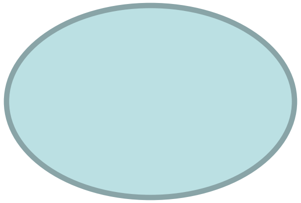

The correct average is the **harmonic average**

$$\frac{dx}{dt} = \frac{4}{3}, \quad \textit{in the example}$$

## Homogenization – An example:

## Heat Equation with a constant conductivity:

$$\frac{\partial u}{\partial t} = \alpha(\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} + \frac{\partial u}{\partial z^2})$$

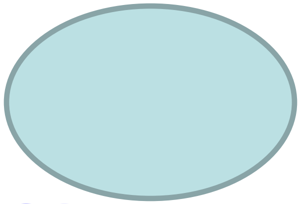


$$u = 0 \ \ on \ \ \partial\Omega$$

# Heat Equation with a varying conductivity:

$$\frac{\partial u}{\partial t} = div(\alpha(\omega)\ grad\ u)$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\alpha(\omega)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\alpha(\omega)\frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(\alpha(\omega)\frac{\partial u}{\partial z})$$



$$u = 0\ on\ \partial\Omega$$

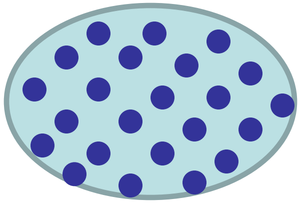
## Heat Equation in one dimension with a varying conductivity:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\alpha(x)\frac{\partial u}{\partial x})$$

$$u = 0 \ \ on \ \ \partial\Omega$$

# Heat Equation with a varying conductivity:

$$\frac{\partial u}{\partial t} = div(\alpha(\omega)\ grad\ u)$$



$$u = 0\ on\ \partial\Omega$$

$$u\ is\ continuous$$

## Heat Equation in one dimension with periodic conductivity:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\alpha(x)\frac{\partial u}{\partial x})$$

$u = 0 \ on \ \partial\Omega$

$u \ is \ continuous$



$\alpha \quad periodic$

Suppose $\alpha$ changes rapidly $\alpha = \alpha(\frac{x}{\epsilon})$ with small $\epsilon$
Can we average by taking the average of $\alpha$ ?

NO !!

The reason: Look at the equation with the small parameter

$$\frac{\partial u_\epsilon}{\partial t} = div(\alpha_\epsilon(\omega)\ grad\ u_\epsilon)$$

The functions $u$ converges strongly

The functions $\alpha_\epsilon$ and $grad\ u_\epsilon$ converge weakly but their product does not converge to the product of the weak limits.

An appropriate average is needed

# Heat Equation in one dimension with periodic conductivity:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\alpha(x)\frac{\partial u}{\partial x})$$

$u = 0 \; on \; \partial\Omega$

$u \; is \; continuous$

Suppose $\alpha$ changes rapidly $\alpha = \alpha(\frac{x}{\epsilon})$ with small $\epsilon$

The right average is the harmonic average of $\alpha$

In more dimensions and more complicated structures and other equations – consult homogenization theory

# Oscillatory solution of partial differential equations (Compensated compactness)

Luc Tartar

Main idea: Use the compactness to identify a limit then try to verify that the limit is a function (Compensated compactness)

# Oscillatory patterns in elasticity:

John M. Ball

Richard D. James

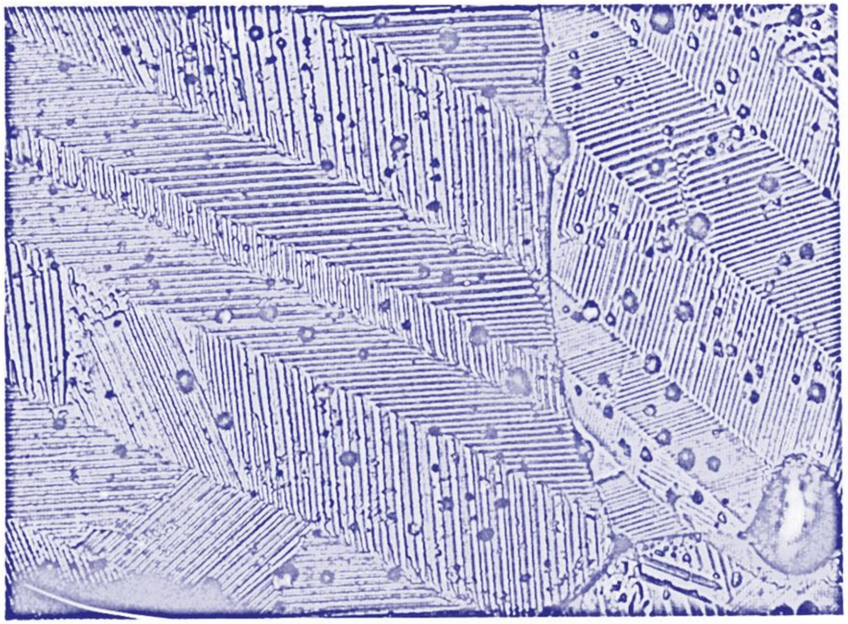
Wrote a variational problem for the arrangement of atoms in solids under stress. The optimal solution is a Young Measure, thus cannot be realized in reality?

**What does Nature do then?**

## Applied Platonism:

Nature is a very good approximation of Mathematics

# An earthy approximation of the ideal mathematics



John M. Ball and Richard D. James
1992

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- **Control Invariant Measures**
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# Recall: The variational limit for Singularly Perturbed ODE:

$$(x(t), \mu(x(t)))$$

where $\mu(x)(dy)$ is a Young measure

and $x(t)$ solves the averaging equation

$$\frac{dx}{dt} = \int_Y f(x, y)\mu(t)(dy)$$

## Recall: The variational limit for SP control systems:

$$(x(t), \mu(x(t)))$$

Where: $x(t)$ solves the averaging equation

$$\frac{dx}{dt} = \int_{Y\times U} f(x,y,u)\mu(x)(dy \times du);$$

$\mu(x)(dy \times du)$ is a Young measure (parameterized by $x$)

and the limit cost is based on **averaging:**

$$\int_a^b \int_{Y\times U} c(x(t),y,u)\mu(x(t)(dy \times du)\ dt$$

**Notice,** the values of the Young measure are the **control variables**, (replacing the equilibrium points in the classical case

## Recall the question:

Could any probability measure be a value for the Young Measure of the variational limit?

If not, how can the possible values be classified and identified?

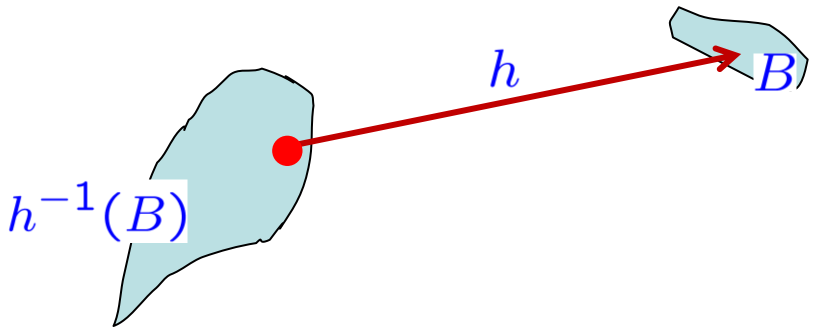
## Definition: An invariant measure of a mapping:

$$h(z) : Z \to Z$$

The probability measure $\mu(dz)$ on $Z$ is invariant with respect to $H$ if

$$\mu(h^{-1}(B)) = \mu(B)$$

for every measurable set $B$.



## An observation:

The invariant measures of a mapping form a convex set in the space of probability measures.

Follows from $\mu(h^{-1}(B)) = \mu(B)$

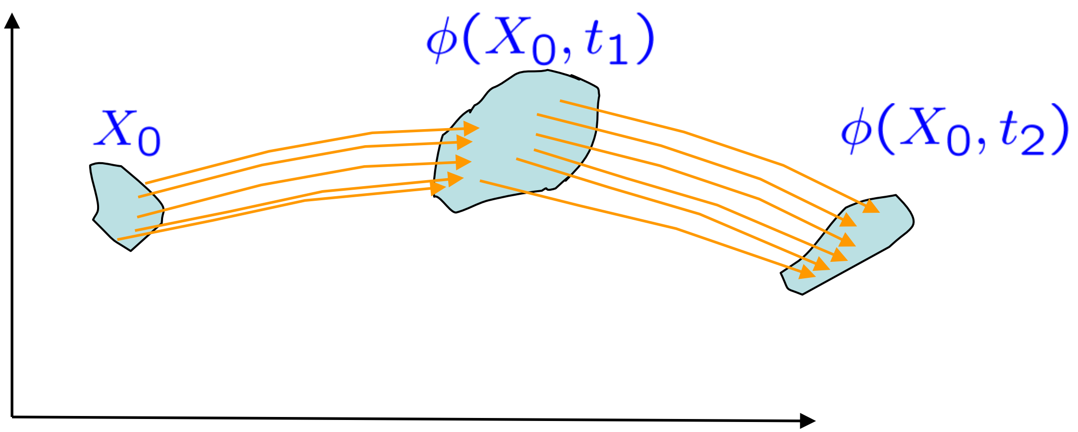
An invariant measure of an ordinary differential equation (with the uniqueness of solutions property):

$$\frac{dx}{dt} = f(x)$$

Is an invariant measure of the solution map

$$\phi(x_0, t)$$

for every time $t$.

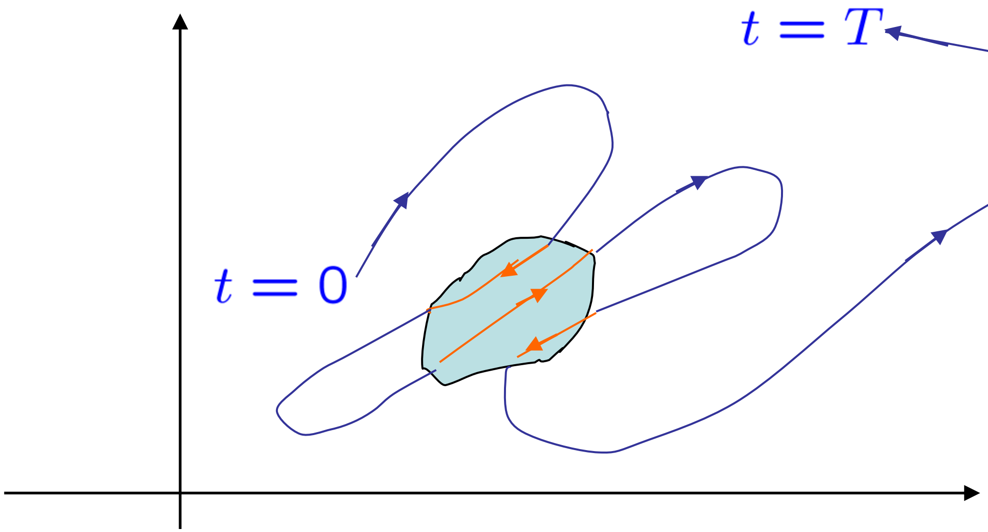


# An important property: convergence of occupational measures to an invariant measure

N.M. Kryloff
1879-1955

N. Bogoliuboff
1909-1992

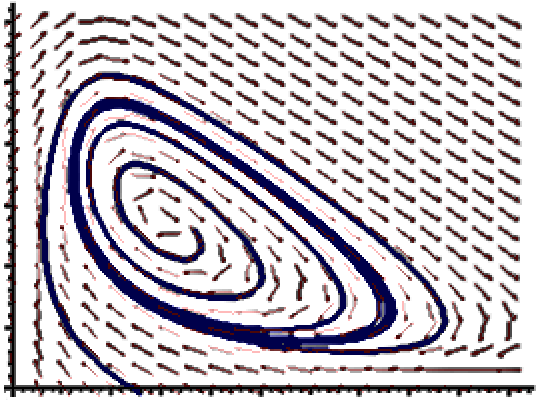
A probability measure determined by the relative time a trajectory spends in a given set, when the time is long, is almost an invariant measure of the differential equation. In the limit it converges to an invariant measure.



# The Poincare-Bendixson Theorem:

$$\frac{dx}{dt} = f(x)$$

In two dimensions, every bounded trajectory that stays away from rest points converges to a periodic orbit.



## A consquence:

$$\frac{dx}{dt} = f(x)$$

In two dimensions, every invariant measure is supported on periodic orbits and rest points

# Back to the variational limit for Singularly perturbed ODE:

$$\frac{dx}{dt} = f(x,y)$$

$$\epsilon\frac{dy}{dt} = g(x,y)$$

The limit is of the form $(x(t), \mu(x(t)))$ where $\mu(x)(dy)$ is a Young measure

Theorem: The values of this Young measure are invariant measures of

$$\frac{dy}{dt} = g(x,y)$$

## A consequence:

$$\frac{dx}{dt} = f(x, y)$$

$$\epsilon \frac{dy}{dt} = g(x, y)$$

If $y$ is two dimensional then the limit is composed of limit cycles and rest points of

$$\frac{dy}{dt} = g(x, y)$$

# Recall: A mathematical example capturing reality: An elastic structure in a rapidly flowing nearly invicid fluid

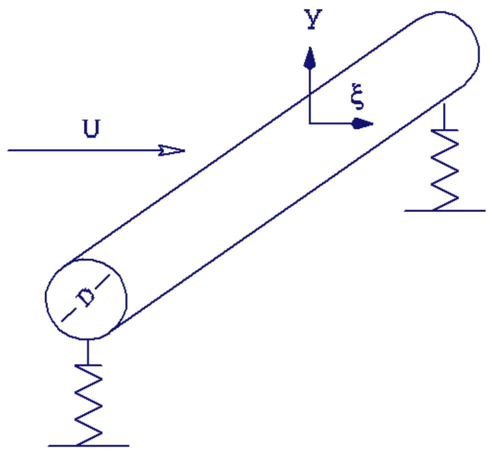



Figure 1

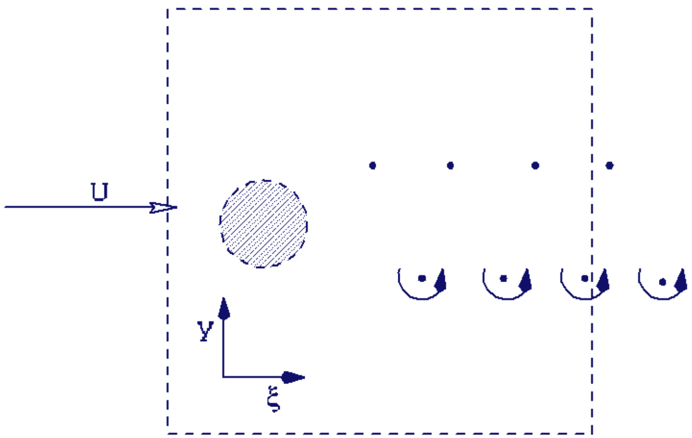



Figure 2

## To make the long story short:

Based on a model of Iwan/Belvins and Dowel/Ilgamov, the limit (after normalization) equations:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\alpha_1 x_1 - \alpha_2 x_2 \beta_3 \theta_1 + \beta_4 F(\gamma\theta_2)$$

$$\epsilon\frac{d\theta_1}{dt} = \theta_2$$

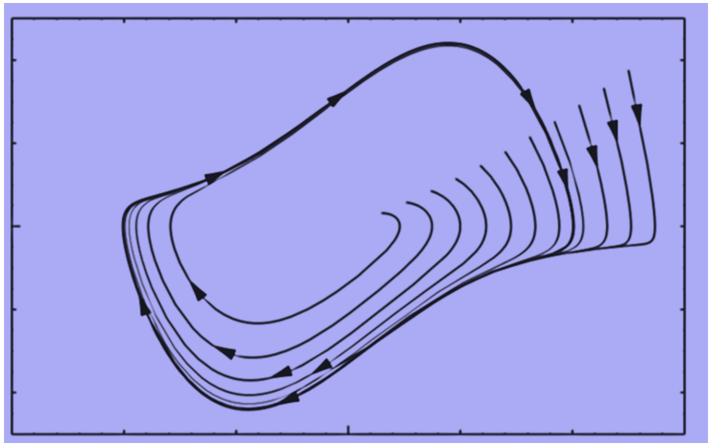
$$\epsilon\frac{d\theta_2}{dt} = -\beta_1\theta_1 + \beta_3 F(\gamma\theta_2) - \alpha_3 x_1 - \alpha_4 x_2$$

With $F(\theta)$ a generator of a van der Pol oscillator

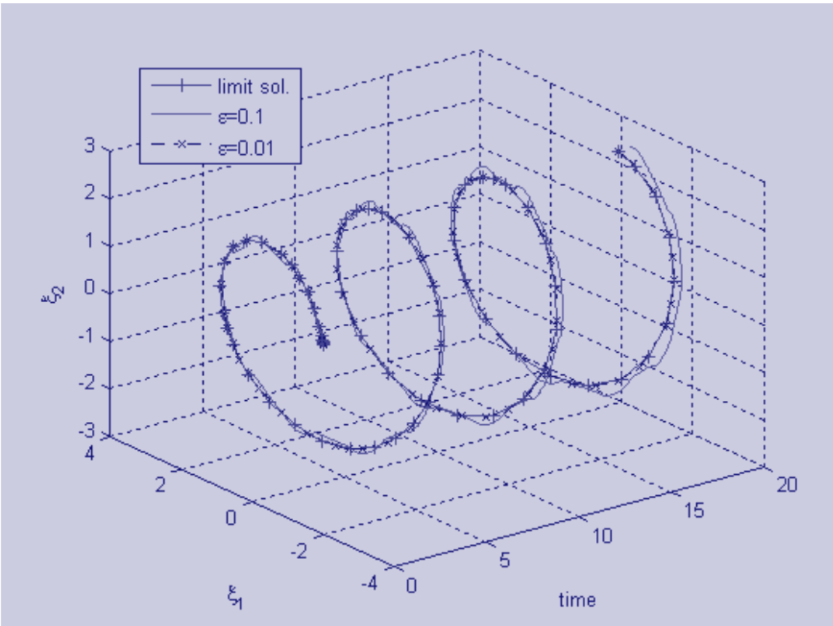
# The phase portrait of the van der pol oscillator:

$$\frac{d\theta_1}{dt} = \theta_2$$

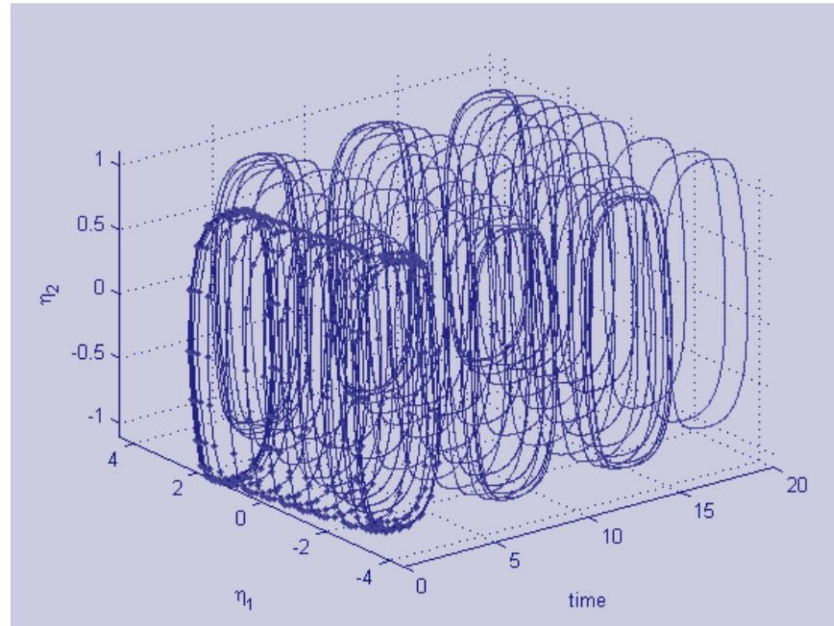
$$\frac{d\theta_2}{dt} = -\beta_1\theta_1 + \beta_3 F(\gamma\theta_2) - \alpha_3 x_1 - \alpha_4 x_2$$



# Numerical results:




The slow dynamics




The fast dynamics

**Now**: The variational limit for SP control systems:

$$\frac{dx}{dt} = f(x, y, u)$$
$$\epsilon \frac{dy}{dt} = g(x, y, u)$$

The limit is of the form $(x(t), \mu(x(t)))$ where $\mu(x)(dy \times du)$ is a Young measure

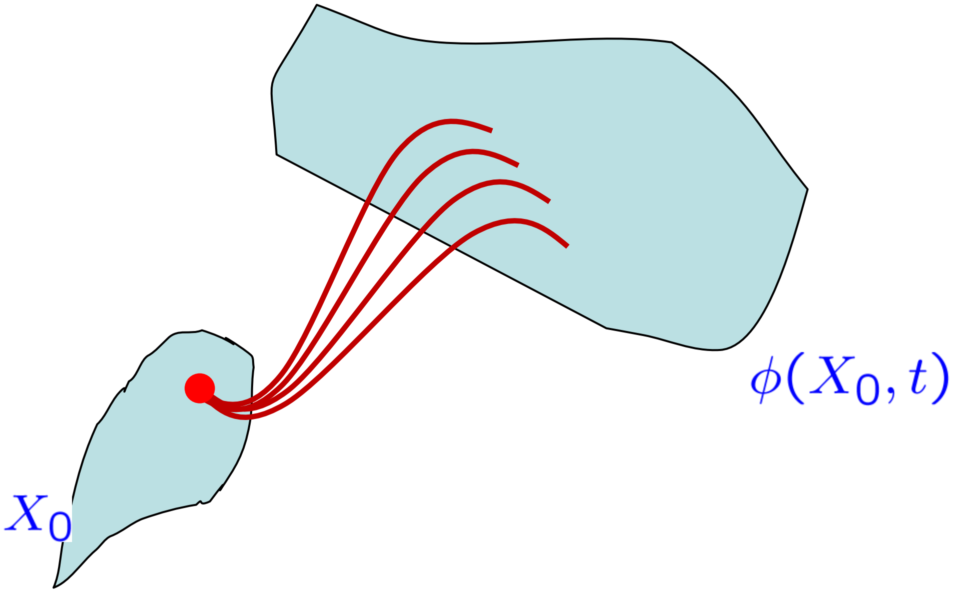
Question: Are the values of this Young measure invariant measures of

$$\frac{dy}{dt} = g(x, y, u)$$

**What** is an invariant measures of a multi-valued map?

# What is an invariant measures of a multi-valued map?

$$\frac{dx}{dt} \in F(x)$$

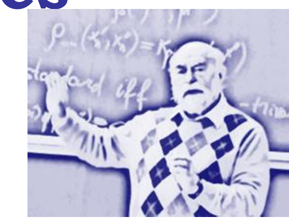


## We give four alternatives:


J.P. Aubin
H. Frankowaska
A. Lasota

1. **Sub-invariant measures**
2. **Markov-selectionable invariant measures**
3. **Projected invariant measures**
4. **Limit occupational measures**

Fritz Colonius



Anatoly Vershik

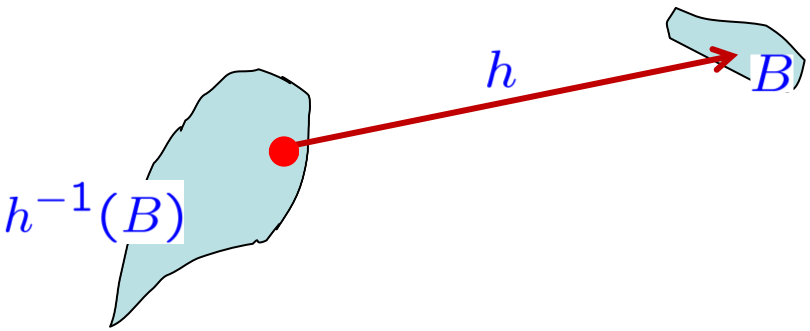
Vladimir Gaitsgory

Arie Leizarowirz 1953-2010

Wolfgang Kliemann

Recall: An invariant measure of a mapping:

$h(z) : Z \to Z$

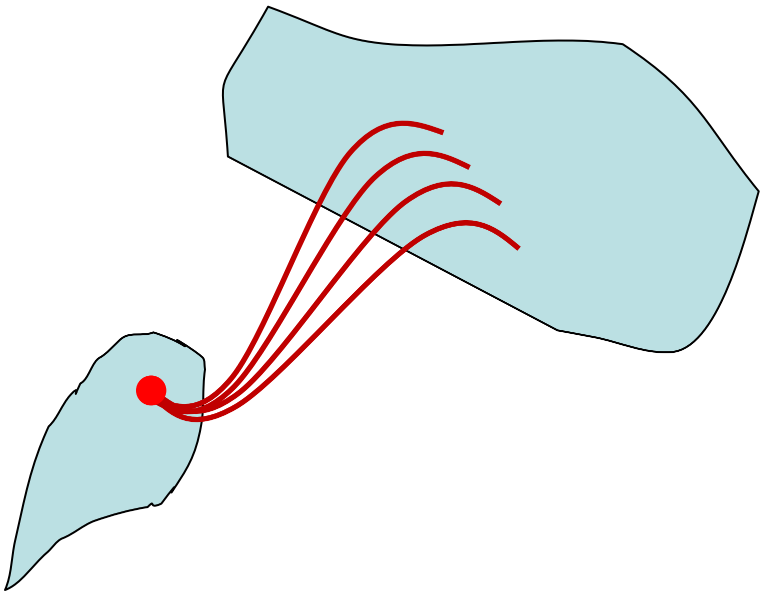



is determined by

$$\mu(h^{-1}(B)) = \mu(B)$$

**1. Define:** A measure $\mu(dz)$ is a **sub-invariant measure** of a multi-valued mapping:

$$H(z) : Z \rightrightarrows Z$$



**if** $\mu(h^{-1}(B)) \leq \mu(B)$ for every measurable set $B$

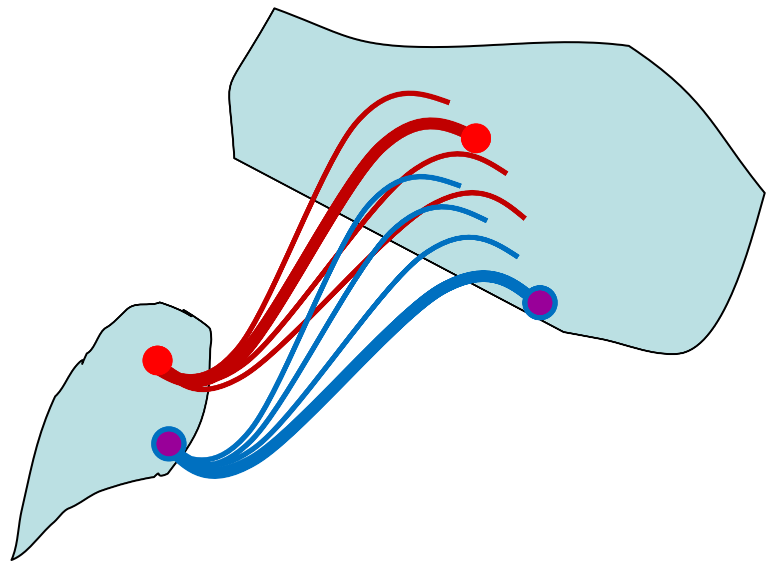
here $H^{-1}(B) = \{z : H(z) \cap B \neq \emptyset\}$

# So far

1. **A sub-invariant measure**
2. A Markov-selectionable invariant measure
3. A projected invariant measure
4. A limit occupational measure

**Try:** A measure $\mu(dz)$ is a selectionable invariant measure of a multi-valued mapping:

$$H(z) : Z \rightrightarrows Z$$

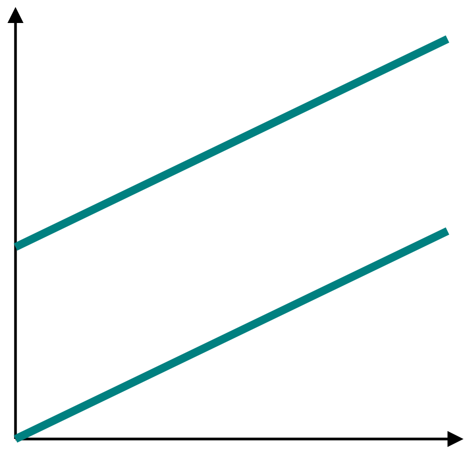


if there exists a point selection $h(z) \in H(z)$

Such that $\mu(dz)$ is invariant with respect to it

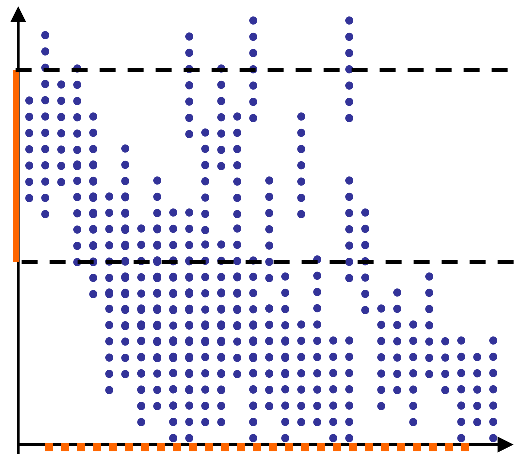
# A definition via point selection does not work!

An example showing a problem:



# Invariance with respect to a Markov transition

A Markov transition functions is a map



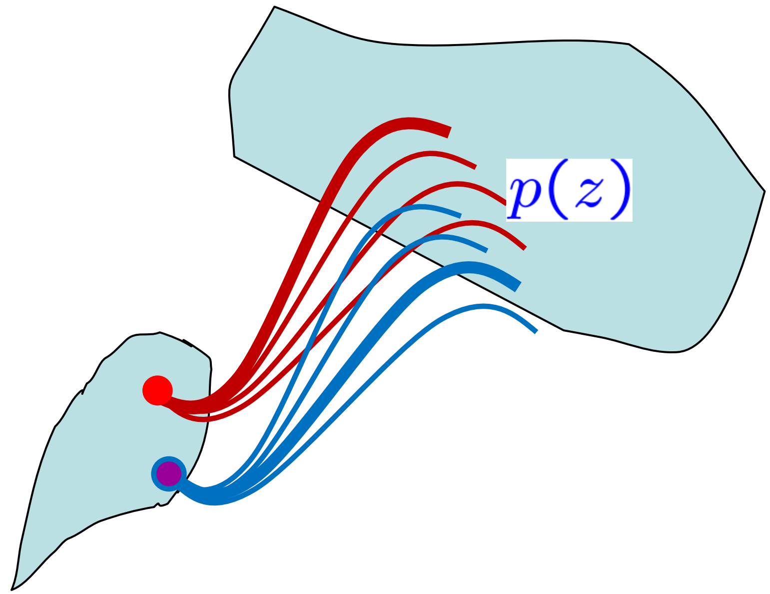
A probability measure $\mu(dz)$ is Markov-invariant with respect to $p(z)$ if

$$\mu(B) = \int p(z)(B)\mu(dz)$$

for every measurable set $B$.

**2. Definition:** A measure $\mu(dz)$ is a **Markov-selectionable invariant** of a multi-valued mapping:

$$H(z) : Z \rightrightarrows Z$$




if there exists a Markov transition $p(z)$ map pointwise supported on $H(z)$

such that $\mu(dz)$ is Markov invariant with respect to it

# So far

1. **A sub-invariant measure**
2. **A Markov-selectionable invariant measure**
3. A projected invariant measure
4. A limit occupational measure

# The lifted flow of a multi-valued mapping:

We associate with $H(z) : Z \rightrightarrows Z$

the family of sequences

$$\zeta = \{\cdot\ \cdot\ \cdot\ z_{-2},\ z_{-1},\ z_0,\ z_1,\ z_2,\ \cdot\ \cdot\ \cdot\ \}$$

where $z_{k+1} \in H(z_k)$

On this family the left shift determines a point-valued flow !

Thus, the notion of an invariant measure is well defined on the lifted space (but it will be a measure on sequences)

**3. Definition:** A measure $\mu(dz)$ is a **Projected invariant** measure of a multi-valued mapping:

$$H(z) : Z \rightrightarrows Z$$

if it is the projection on a coordinate of an invariant measure, say $p(d\zeta)$, of the lifted flow

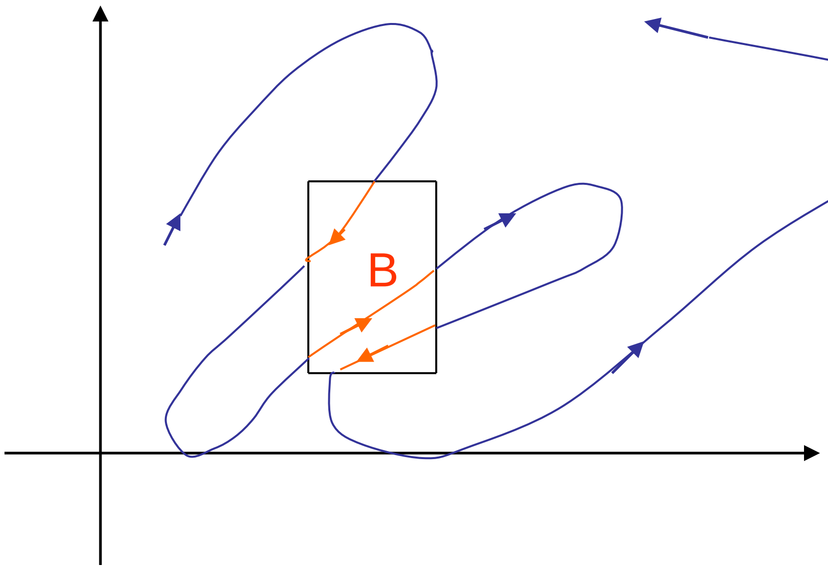
$$\mu(B) = p(\{\zeta : z_0 \in B\})$$

# So far

1. **A sub-invariant measure**
2. **A Markov-selectionable invariant measure**
3. **A projected invariant measure**
4. A limit occupational measure

## Empirical distribution, occupational measure:

The **empirical distribution** (= **occupational measure**) of a finite sequence $(z_0,\ z_1,\ z_2,\ \cdot\ \cdot\ \cdot\ z_k)$ (or a finite line) is the probability $\mu_{0,k}$ measures (on $Z$ )




$$\mu_{0,k}(B) = \#\{j : z_j \in B\}$$

**Terminology:** A measure $\mu(dz)$ is a **Limit weak Occupational measure** of a multi-valued mapping:

$$H(z) : Z \rightrightarrows Z$$

if it is the limit in the space of measures, of occupational measures of finite trajectories

$$(z_l,\ z_{l+1},\ z_{l+2},\ \cdot\ \cdot\ \cdot\ z_k)\ z_{k+1} \in H(z_k)$$

with $k = l \to \infty$

If all the finite trajectories are of one infinite trajectory then we get an **Extreme Limit Occupational measure**

**4. Definition:** A measure $\mu(dz)$ is a **Limit Occupational measure** of a multi-valued mapping:

$$H(z) : Z \rightrightarrows Z$$

if it is in the convex hull of the weak limit occupational measures

Theorem: If the graph of $H$ is closed and the space $Z$ is compact then the extreme limit occupational measures form the extreme points of the limit occupational measures.

We managed:

1. **A sub-invariant measure**
2. **A Markov-selectionable invariant measure**
3. **A projected invariant measure**
4. **A limit occupational measure**

THEOREM

$1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$

and

$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$

When the multi-dynamics has a closed graph and the space is locally compact

## Recall: A question:

Under what conditions is

$$\frac{dx}{dt} = \int_{Y \times U} f(x, y, u)\mu(t)(dy \times du)$$

$$\mu(t) \in IM(x(t))$$

a variational limit of

$$\frac{dx}{dt} = f(x, y, u)$$

$$\epsilon \frac{dy}{dt} = g(x, y, u)$$

## A Theorem:

The conditions are:

- Regularity (modest) of $f(x,y,u)$ and $g(x,y,u)$
- Uniform boundedness and controllability of solutions of

$$\epsilon \frac{dy}{dt} = g(x,y,u)$$

- The set-valued map

$$F(x) = \{\int_{Y\times U} f(x,y,u)\ \mu(x)(dy \times du) :\ \mu(x) \in IM(x)\}$$

is Lipschitz

# An important result

$$IM(x)$$

is the family of limit occupational measures of

$$\frac{dy}{dt} = g(x, y, u)$$

# A characterization of $IM(x)$

Vladimir Gaitsgory

$$\int_{R^m \times U} (\nabla \Phi) g(x,y,u) \mu(dy \times du) = 0$$

For every bounded and continuously differentiable $\Phi(y) : R^m \to R$ with a bounded gradient

## Another characterization of $IM(x)$

$$\int_{R^m \times U} b(y)\mu(dy \times du) = \int_{R^m \times U} b(g(x,y,u))\mu(dy \times du)$$

For every bounded and continuous

$$b(y) : R^m \to R$$

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# Singularly perturbed stabilization:

$$\frac{dx}{dt} = f(x, y, u)$$

$$\epsilon\frac{dy}{dt} = g(x, y, u)$$

$$u \in U$$

Where: $x$ in $R^n$ the slow and $y$ in $R^m$ the fast, variables

The goal: To stabilize the **slow dynamics** for small $\epsilon$

# The order reduction method

Stabilize:

$$\frac{dx}{dt} = f(x, y, u)$$

$$0 = g(x, y, u)$$

with a feedback $u = u(x)$ that solves the algebraic equation

It works in many systems (and in many practical applications) but may not be enough for many examples and applications

# The remedy: Young measures

Stabilize:

$$\frac{dx}{dt} = \int_{R^n} f(x, y, u)\ \mu(x)(dy, du)$$

with a feedback $\mu = \mu(x)$ that is an invariant measure of

$$\frac{dy}{dt} = g(x, y, u)$$

# Example:

$x \in R^2$

$y = (\rho, \theta)$

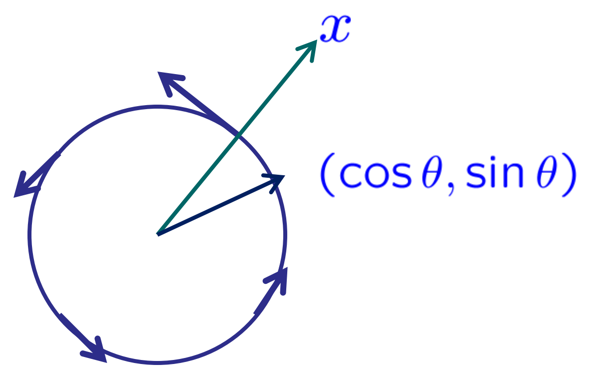
$v \in [-1, 1]$

$u \in [1, 2]$

$$\frac{dx}{dt} = (1 - \rho(x \cdot (\cos\theta, \sin\theta)))\ x$$

$$\epsilon \frac{d\rho}{dt} = v$$

$$\epsilon \frac{d\theta}{dt} = u$$



# Characterization

Continuous stabilizing feedback around $x_0$ exists if and only if there exists a smooth Liapunov function $V(x)$ such that for $x \neq x_0$

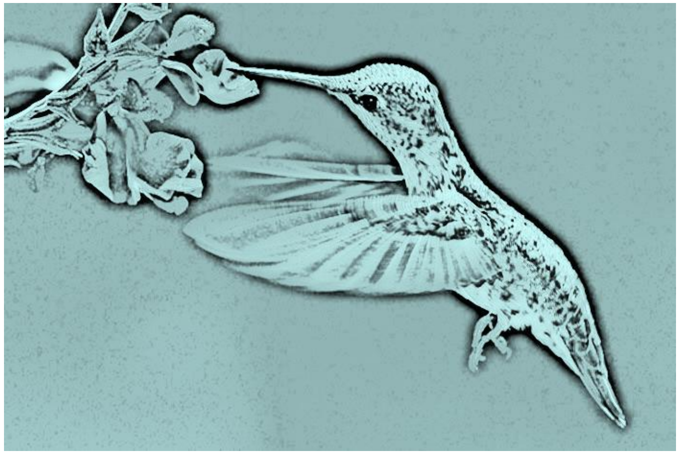
$$\nabla V(x) \int_{R^n} f(x,y,u)\ \mu(x)(dy,du) < 0$$

for some invariant measure $\mu = \mu(x)$

## Approximations in Nature and in Engineering:

Nature and Engineering are very good approximation of Mathematics

# Examples from real life:



The End

of Lecture 2

Thanks for the attention