Three Lectures on:
Control of Coupled Fast and Slow Dynamics

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Control of Coupled Fast and Slow Dynamics

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Plan:

Modeling

Variational Limits

Classical Approach to slow-fast dynamics

What limits are appropriate? Young Measures

Modern Approach to slow-fast dynamics

Other chattering limits and averaging techniques

Control Invariant Measures

Stabilization

Optimal Control

Some special cases

Computations, error estimates

A Future Direction

Lecture 3

Plan:

- √ Modeling
- √ Variational Limits
- √ Classical Approach to slow-fast dynamics
- √ What limits are appropriate? Young Measures
- √ Modern Approach to slow-fast dynamics
- √ Other chattering limits and averaging techniques
- √ Control Invariant Measures
- √ Stabilization
- **Optimal Control**
- Some special cases
- Computations, error estimates
- A Future Direction

The issues:

How to solve the limit optimal control problem?

How to relate the solution of the limit problem to the perturbed problem?

Are conditions (necessary, sufficient, …) for the limit problem, related to the perturbed problem?

Recall: An issue:

How to relate trajectories (say optimal solutions) of the limit problem to the perturbed problem?

The answer:

If $u_\epsilon(t)$ is designed such that $(y_\epsilon(t), u_\epsilon(t))$ approximates the limit Young Measure $\mu(x(t))$ (in the space of Young Measures), the outcome of the perturbed equation will be a good approximation of the limit (hence of the optimal solution to the perturbed equation. Under the conditions of the theorem his can be done !

The issues:

How to solve the limit optimal control problem?

√ How to relate the solution of the limit problem to the perturbed problem?

Are conditions (necessary, sufficient, …) for the limit problem, related to the perturbed problem?

Recall the limit optimal control problem:

minimize

$$\int_a^b \int_{Y \times U} c(x(t), y, u)\mu(t)(dy \times du)\ dt$$

subject to

$$\frac{dx}{dt} = \int_{Y \times U} f(x, y, u)\mu(t)(dy \times du)$$

$$\mu(t) \in IM(x(t))$$

$$x(a) = x_0, \quad y(a) = y_0$$

a optimal solution has the form:

$$(x(t), \mu(t))$$

Remark:

Under controllability and boundedness of the fast dynamics the fast initial and terminal conditions do not play a role, thus:

$$\textit{minimize} \quad \boxed{C(x(b))}$$

$$\textit{subject to} \quad \boxed{\frac{dx}{dt} = \int_{Y\times U} f(x,y,u)\mu(t)(dy \times du)}$$

$$\mu(t) \in IM(x(t))$$

$$x(a) = x_0, \ \textit{and possibly} \ \ x(b) \in C$$

The optimal control problem of the equivalent differential inclusion:

$$\textit{minimize} \quad C(x(b))$$

$$\textit{subject to} \quad \frac{dx}{dt} \in F(x)$$

$$x(a) = x_0, \ \textit{and possibly} \ \ x(b) \in C$$

where

$$F(x) = \{\int_{Y \times U} f(x, y, u) \ \mu(dy \times du) : \mu \in IM(x)\}$$

Ponryagin (et al.) necessary condition:

Lev Semenovich Pontryagin

1908-1988

Pontryagin maximum principle for:

$$\textit{minimize} \quad \boxed{C(x(b))}$$

$$\textit{subject to} \quad \boxed{\frac{dx}{dt} \in F(x)}$$

$$x(a) = x_0$$

(notice the lack of a terminal condition)

Pontryagin maximum principle:

If the trajectory $x^*(t)$ is optimal then there exists a function $p^*(t)$ satisfying

$$\frac{d}{dt}p^*(t) \in -\partial_x H(x^*(t), p^*(t))$$

$$\frac{d}{dt}x^*(t) \in \partial_p H(x^*(t), p^*(t))$$

plus a transversality condition at the end point

where

$$H(x,p) = max\{pv : v \in F(x)\}$$

How to identify the optimal invariant measure?

The key is

$$H(x^*(t), p^*(t)) = max\{p^*(t)v : v \in F(x^*(t))\}$$

where

$$F(x) = \{\int_{Y \times U} f(x, y, u)\ \mu(dy \times du) : \mu \in IM(x)\}$$

Hence: the optimal $\mu(t)$ in the limit occupational measure of $(y(t), u(t))$ that solves the infinite horizon problem:

$$\textit{maximize} \quad \lim_{S \to \infty} \frac{1}{S} \int_0^S p^*(t) f(x^*(t), y, u) ds$$

$$\textit{subject to} \quad \frac{dy}{dt} = g(x^*(t), y, u)$$

How to solve an infinite horizon optimal control problems?

$$maximize \quad \lim_{S\to\infty} \frac{1}{S}\int_0^S p^*(t) f(x^*(t), y, u) ds$$

$$subject\ to \quad \frac{dy}{dt} = g(x^*(t), y, u)$$

There is a vast literature. Of interest works by Vladimir Gaitsgory et al., utilizing the linearity of the problem in the invariant measure, thus reducing the problem to linear programming.

The issues:

√ How to solve the limit optimal control problem?

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Are conditions (necessary, sufficient, …) for the limit problem related to the perturbed problem?

The Pontryagin principle for the limit system in a special case

Consider the limit system:

$$\text{minimize} \quad C(x(b))$$

$$\text{subject to} \quad \frac{dx}{dt} = f(x, \mu)$$

$$x(a) = x_0$$

$$\mu \in M$$

Notice the assumption:

$M(x) = M$ does not depend on x

The Pontryagin principle

Given an optimal solution $(x^*(t), \mu^*(t))$

There exists "support" function $p(t)$ satisfying

$$\frac{dp}{dt} = -p(t) D_x f(x^*(t), \mu^*(t))$$

and for every fixed t

$$p(t) f(x^*(t), \mu) \le p(t) f(x^*(t), \mu^*(t))$$

whenever $\mu \in M$

and a transversality condition at the end point

The Pontryagin principle for the perturbed system:

Given an optimal solution $(x_\epsilon^*(t), y_\epsilon^*(t), u_\epsilon^*(t))$

There exists "adjoint" function $(p(t), q(t))$ satisfying:

$$\frac{dp}{dt} = -p_\epsilon(t) D_x f(x_\epsilon^*(t), y_\epsilon^*(T), u_\epsilon^*(t)) - q_\epsilon(t) \frac{1}{\epsilon} D_x g(x_\epsilon^*(t), y_\epsilon^*(T), u_\epsilon^*(t))$$

$$\frac{dq}{dt} = -p_\epsilon(t) D_y f(x_\epsilon^*(t), y_\epsilon^*(T), u_\epsilon^*(t)) - q_\epsilon(t) \frac{1}{\epsilon} D_y g(x_\epsilon^*(t), y_\epsilon^*(T), u_\epsilon^*(t))$$

and for every fixed t the control $u_\epsilon^*(t)$ maximizes

$$p_\epsilon(t) f(x_\epsilon^*(t), y_\epsilon^*(t), u) + q_\epsilon(t) \frac{1}{\epsilon} g(x_\epsilon^*(t), y_\epsilon^*(t), u)$$

Results:

There is no reason to expect that $q_\epsilon(t)$ converges

If $q_\epsilon(t)\frac{1}{\epsilon}D_x g(x_\epsilon^*(t), y_\epsilon^*(T), u_\epsilon^*(t))$ converge weakly to zero, and $p_\epsilon(b) = p(b)$ then $p_\epsilon(t)$ converges uniformly to the adjoint $p_0(t)$ of the limit equation

An example where $q_\epsilon(t)\frac{1}{\epsilon}D_x g(x_\epsilon^*(t), y_\epsilon^*(T), u_\epsilon^*(t))$ converge weakly to zero is when $g = g(y, u)$

A modified "cat" example

$$\text{minimize} \quad \int_0^1 (x^2(t) - |y_1(t) - 2y_2(t)|)dt$$

$$\text{subject to} \quad \frac{dx}{dt} = u$$

$$\epsilon \frac{dy_1}{dt} = -y_1 + u$$

$$\epsilon \frac{dy_2}{dt} = -2y_2 + u$$

$$x(0) = x_0$$

$$u \in [-1, 1]$$

The modified example - augmented

$$\textit{minimize} \qquad x_2$$

$$\textit{subject to} \qquad \frac{dx_1}{dt} = u$$

$$\frac{dx_2}{dt} = x_1^2(t) - |y_1(t) - 2y_2(t)|$$

$$\epsilon \frac{dy_1}{dt} = -y_1 + u$$

$$\epsilon \frac{dy_2}{dt} = -2y_2 + u$$

$$x_1(0) = x_0 \quad , x_2(1) = 0,$$

$$u \in [-1, 1]$$

The Pontryagin equations

Given an optimal solution $(x_1^*(t), \mu^*(t))$

The support vector is $(p(t), -1)$ where

$$\frac{dp}{dt} = -2x_1^*(t), \quad p(1) = 0$$

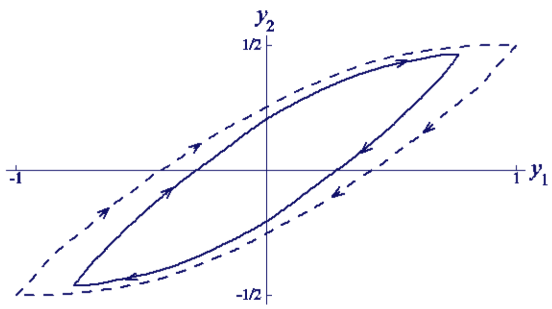
and $\mu^*(t)$ is generated by a solution to

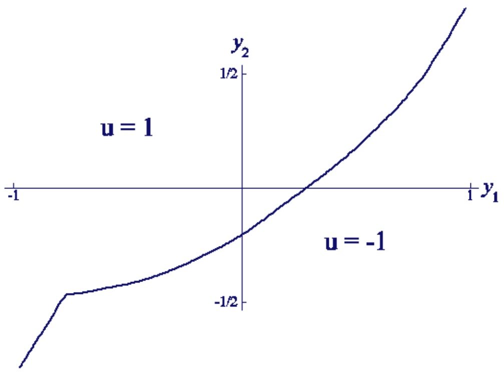
$$\text{maximize} \quad \lim_{S\to\infty} \frac{1}{S} \int_0^S (p(t)u(s) + |z_1(s) - 2z_2(s)|)ds$$

subject to the fast equation.

The solution (simplified a bit):

When $|p(t)| < 0.29129$:





When $|p(t)| \geq 0.29129$: $u = syg\ p(t)$

$p(t)$ can be found from the Pontryagin equations

Hamilton-Jacobi-Bellman (HJB) sufficient condition

William Rowan Hamilton

1805-1865

Carl Gustav Jacobi

1801-1851

Richard E. Bellman

1920-1984

Hamilton-Jacobi-Bellman equation for:

$$\textit{minimize} \quad C(x(b))$$

$$\textit{subject to} \quad \frac{dx}{dt} = f(x,\mu)$$

$$\mu \in IM(x)$$

Let $V(x,t)$ be the value function (when starting at (x,t))
Then it solves the partial differential equation

$$\frac{\partial V}{\partial t} + H(x, D_x V) = 0, \qquad V(x,b) = C(x)$$

where $H(x,p) = \max_{\mu \in IM(x)} \{-p \cdot f(x,\mu)\}$

Two approaches:

1. Use the general theory to be sure that the HJB equation for the limit captures the limit of solutions. Solve it to get this limit.

2. Show that the HJB for the limit is the limit for the HJB of the perturbed systems, and via that verify the continuity of the solutions (and get error estimates for the value in some cases).

Theory by Olivier Alvarez, Martino Bardi et al.

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Recall: Singularly perturbed control systems:

$$\text{minimize} \quad \int_a^b c(x,y,u)\,dt$$

$$\text{subject to} \quad \frac{dx}{dt} = f(x,y,u)$$

$$\epsilon \frac{dy}{dt} = g(x,y,u)$$

$$u \in U$$

Where: x in R^n the slow and y in R^m the fast, variables

Of interest: The behavior of the system as $\epsilon \to 0$

The order reduction method (Petar Kokotovic et al.)

The limit as $\epsilon \to 0$ is depicted by $\epsilon = 0$ namely, by:

$$\text{minimize} \quad \int_a^b c(x,y,u)\,dt$$

$$\text{subject to} \quad \frac{dx}{dt} = f(x,y,u)$$

$$0 = g(x,y,u)$$

$$u \in U$$

We spent a lot of time showing that the order reduction method is not adequate

But: The order reduction method may be good:

Case 1: The problem is convex

Case 2: The “fast variable” y is one dimensional

Arie Leizarowirz

Arie Leizarowitz proved:

Once the dimension of the fast variable is greater or equal to two, the Order reduction is invalid

In fact: in “most” of the systems with a two dimensional fast flow, the order reduction is not valid.

Linear quadratic tracking of periodic target:

Can be solved explicitly (was carried out by Z.A. and Vladimir Gaitsgory)

An observation:

Numerical examples in two dimensional problems seem to produce periodic solutions.

Question: Is it true that in general, when the state is two dimensional, there exists a periodic solution?

A Theorem
The analog of the Poicare-Bendixson theorem holds:

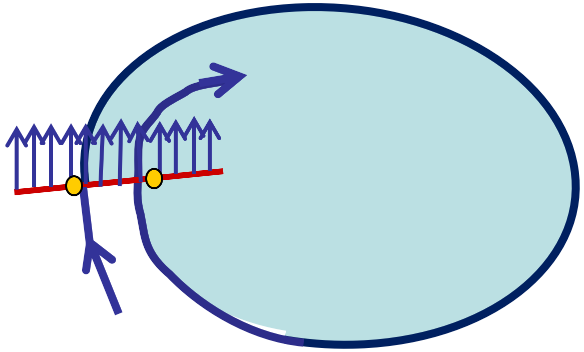
When the state variable is two-dimensional

It is enough to consider periodic trajectories (rest points, possibly) using, possibly, relaxed controls.

Ido Bright

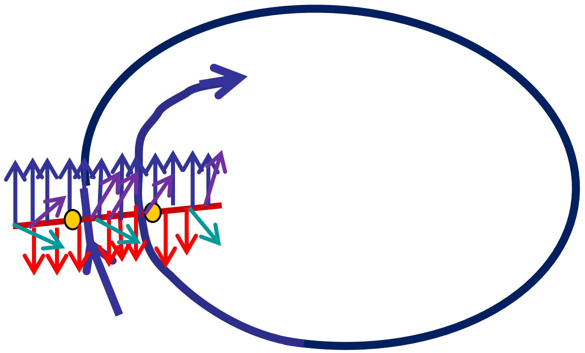
Recall: The Poincare-Bendixson Theorem:

A bounded solution of an ordinary differential equation in the plane either comes close to a rest point, or converges to a limit cycle.



This argument does not apply in the control setting

No notion of a transversal to a vector field!



Steps toward a solution: 1

Consider a sequence of periodic trajectories on longer and longer intervals that approximate, in the limit, the optimal cost.

Make sure each trajectory does not cross itself, by erasing loops (it can be done without decreasing the approximation property

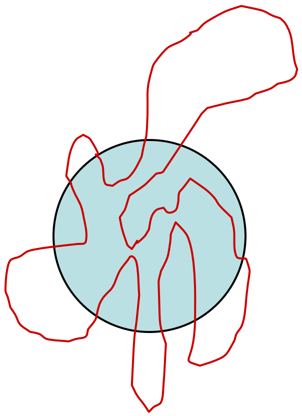
If the resulting sequence has a uniformly bounded period, take a limit periodic trajectory: **It is an optimal solution** (with, possibly, relaxed controls)

A Step toward the proof 2

An estimate. Let $x(t)$ be an absolutely continuous function with bound on the derivative, from an interval into R^2 with image forming a Jordan curve. Then for any Jordan curve B in R^2 with length L

$$\left| \int_{T_B} \frac{dx(t)}{dt} dt \right| \leq \frac{1}{2} L$$

where T_B is the part of the interval where the values of $x(t)$ are in B



Steps toward a solution: 3

If the resulting sequence has periods tending to infinity, employ the estimate to show that almost every point in the support of the limit measure is a stationary optimal solution.

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Some remarks on computations and error estimates:

Solving first the limit problem and using the solution in the perturbed one is efficient for small ϵ

But then it is uniformly efficient !!

Recall: A mathematical example capturing reality: An elastic structure in a rapidly flowing nearly invicid fluid (with Marshall Slemrod)

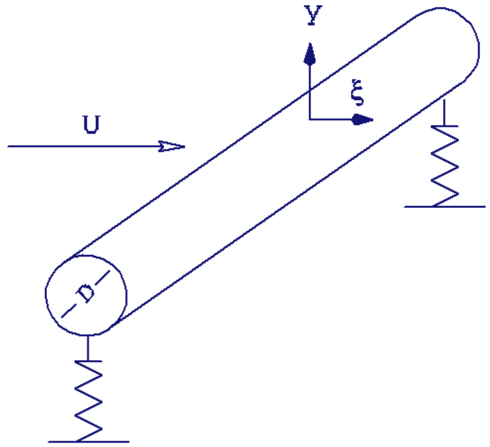


Figure 1

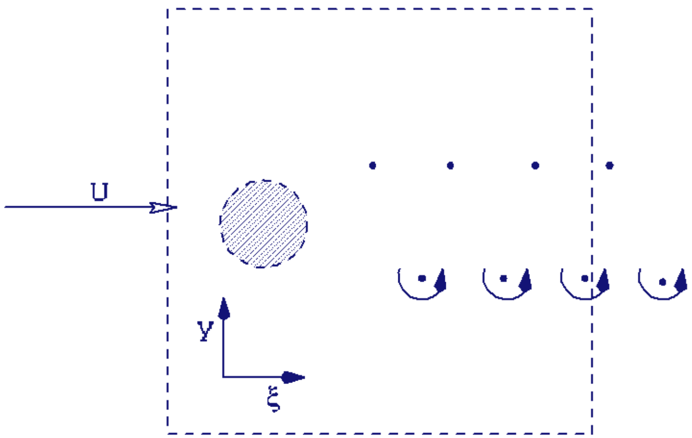
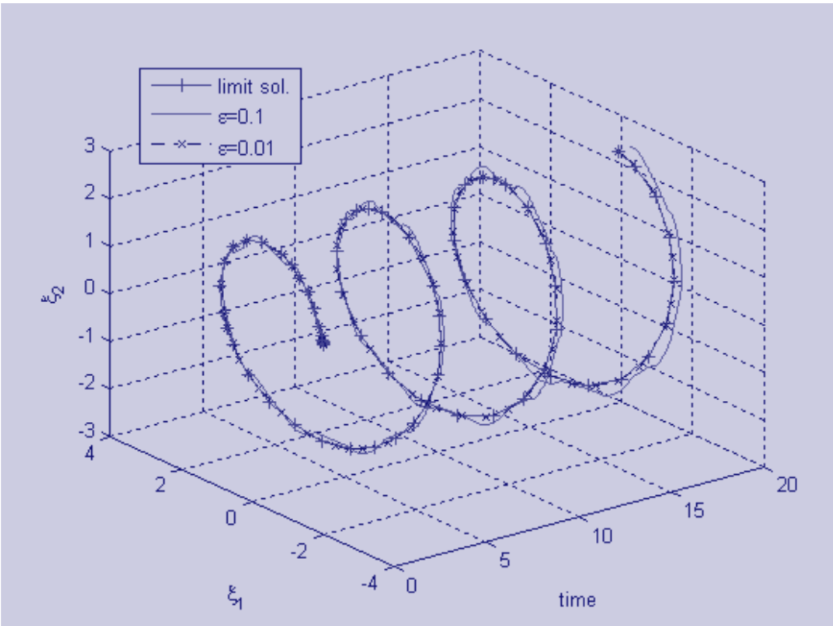
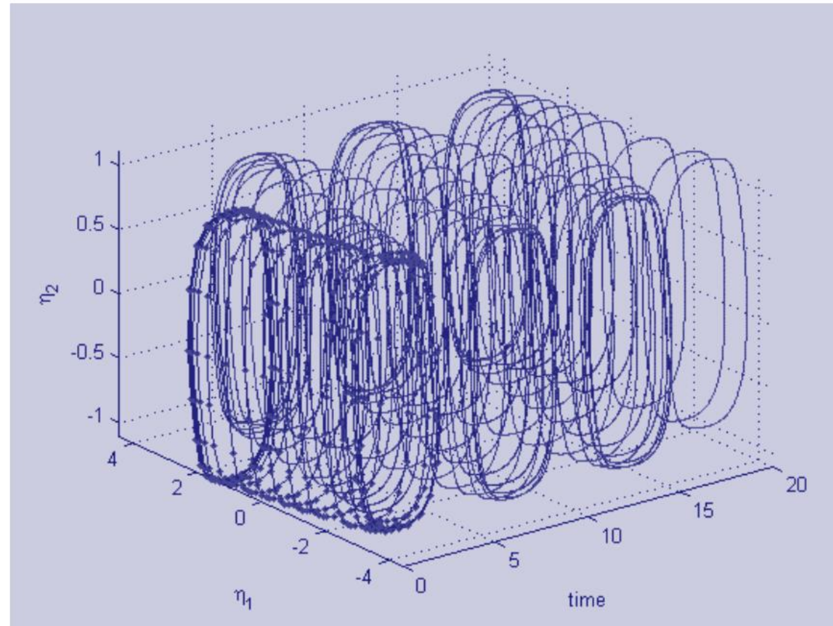


Figure 2

Numerical results:





The slow dynamics

The fast dynamics

Exact error estimates:

Are available for averaging problem:

Typical result:

For periodic systems the error is of order ϵ

For almost periodic systems the error is of order $\epsilon^{\frac{1}{2}}$

These results extend to averaging of control systems.

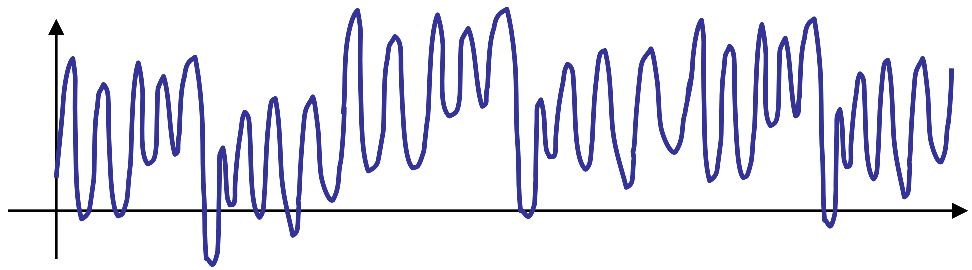
Exact error estimates:

Are available for approximation of relaxed controls

The question:

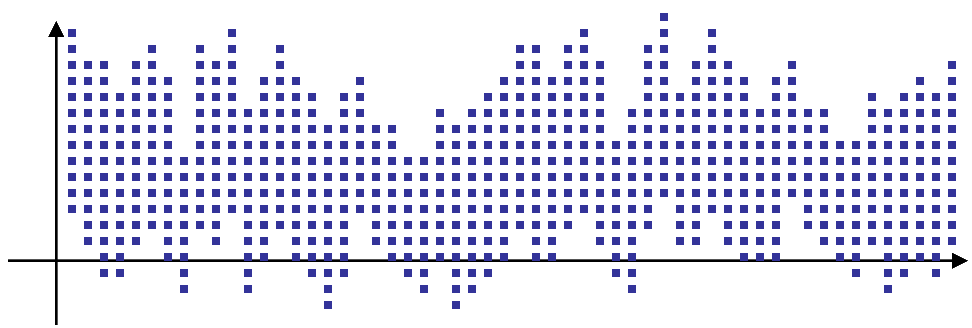
What is an appropriate distance between a point valued control function

$u_\epsilon(t)$



and a measure-valued relaxed control

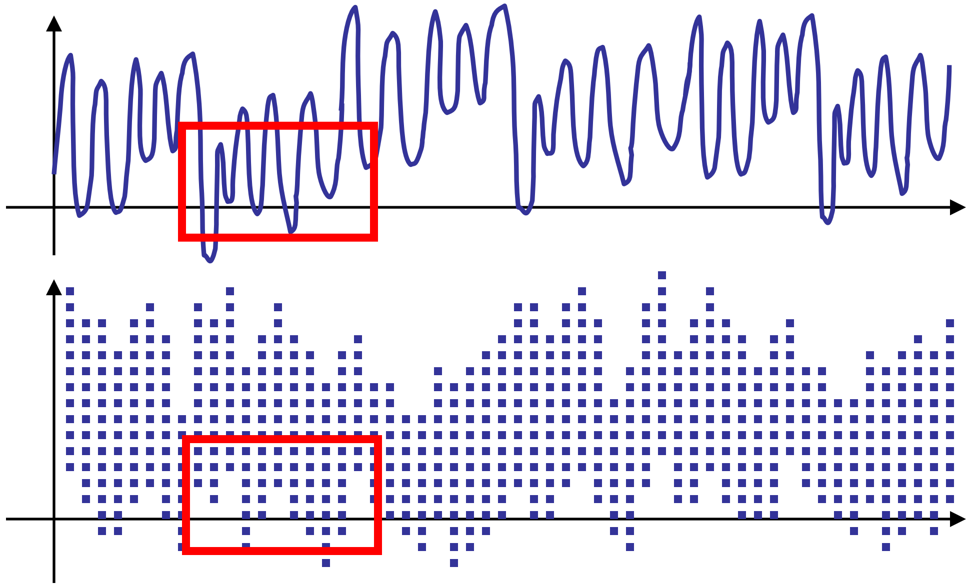
$\mu_0(t)$



A distance for quantitative estimates:

Prohorov metric $Proh(\nu, \mu)$ between measures μ and ν is the smallest η such that for every Borel set B

$$\mu(B) \leq \nu(B^{\eta}) + \eta \quad \text{and} \quad \nu(B) \leq \mu(B^{\eta}) + \eta$$

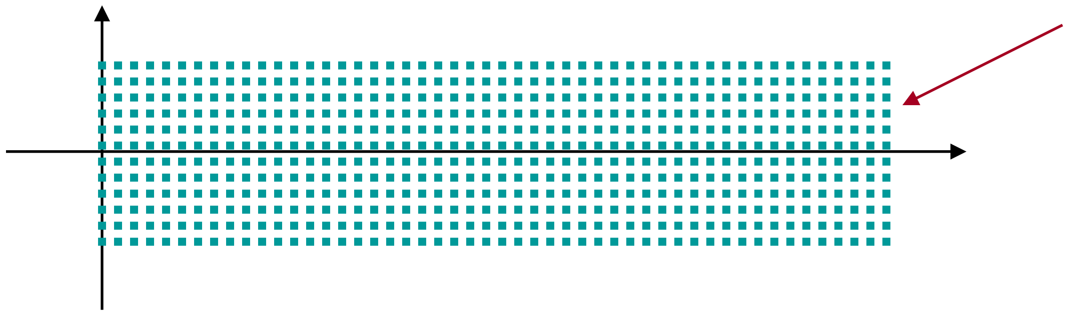


Exercise:

The Prohorov distance between

$$u_\epsilon(t) \;=\; sin(\epsilon^{-1}t)$$

and its Young measure limit on the unit interval



is between $\pi\epsilon$ and $2\pi\epsilon$

A result:

Consider

$$\text{minimize} \quad \int_0^1 c(x,u)dt$$

$$\text{subject to} \quad \frac{dx}{dt} = f(x,u)$$

Suppose that rather than the optimal relaxed control $\mu^*(t)$, an approximation $u(t)$ is used.

The resulting error? (when data are Lipschitz) is of the order of the Prohorov distance $Proh(\mu^*(\cdot), u(\cdot))$.

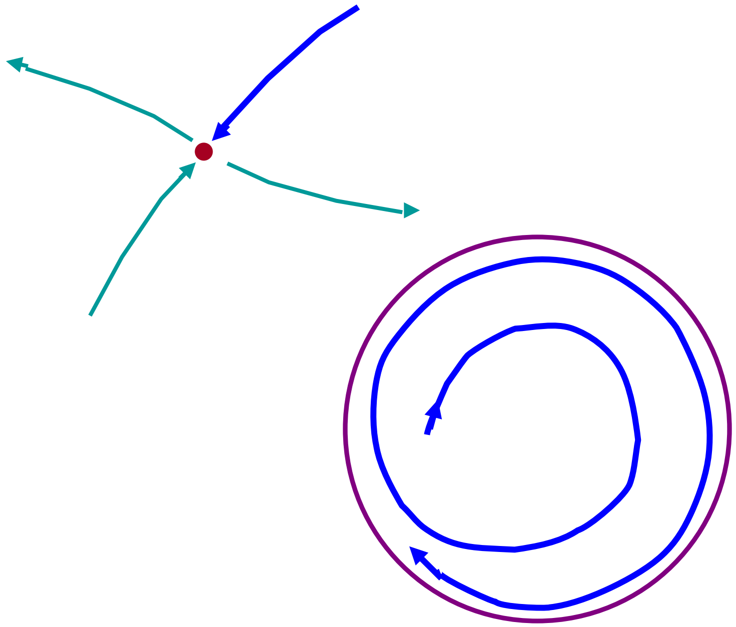
Exact error estimates:

Are available for some very particular cases of singularly perturbed ordinary differential equations

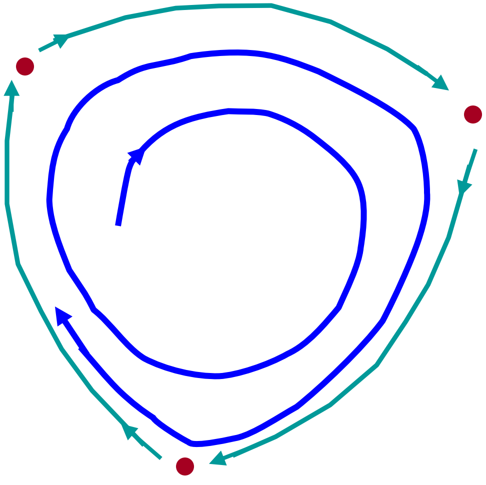
The generic case of two dimensional bounded dynamics:

The equation: $$\varepsilon \frac{dy}{dt} = g(y)$$

Convergence to a <u>hyperbolic fixed point</u> or to a <u>hyperbolic cycle</u>:



Convergence to a <u>heteroclinic cycle</u>:



The setting:

We consider the equation

$$\varepsilon \frac{dy}{dt} = g(y)$$

With fixed initial condition y_0 and on a time interval $[0, 1]$ we consider the distributional $y_\epsilon(\cdot)$ distance of the solution from the Young measure limit

Rates of convergence:

The cases of hyperbolic fixed point or an hyperbolic limit cycle: The rate of convergence is

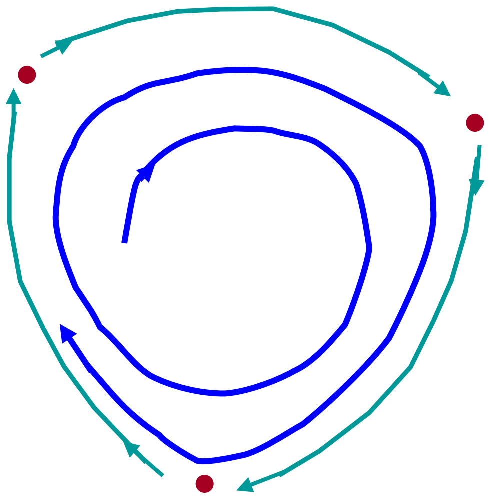
$$\epsilon \log\log\frac{1}{\epsilon} \prec d_0(\epsilon) \prec \epsilon\log\frac{1}{\epsilon}$$

Where $\prec$ denotes strictly smaller in the little oh sense

The heterocline cycle cases:

Denote α_i and β_i the eigenvalues of the incoming and outgoing trajectories at the fixed points and

$$\gamma = \frac{\alpha_1}{\beta_1} \cdots \frac{\alpha_m}{\beta_m}$$



Then $\gamma > 1$

is a sufficient condition for the existence of a converging trajectory while

$\gamma \geq 1$

is a necessary condition for the existence

Rates of convergence:

The case of heteroclinic cycle with $\gamma = 1$ and a contractive Poincare map: The rate of convergence is

$$\epsilon^{\frac{1}{2}} \log\log\frac{1}{\epsilon} \prec d_1(\epsilon) \prec \epsilon^{\frac{1}{2}} \log\frac{1}{\epsilon}$$

The case of heteroclinic cycle with $\gamma > 1$: The rate of convergence is

$$\epsilon \log\frac{1}{\epsilon} \prec d_2(\epsilon) \prec \epsilon(\log\frac{1}{\epsilon})^k$$

For some natural number k. In this case the convergence is to a family of Young measures.

The hierarchy:

$$\epsilon \log\log\frac{1}{\epsilon} \prec d_0(\epsilon) \prec \epsilon\log\frac{1}{\epsilon} \prec d_2(\epsilon) \prec \epsilon(\log\frac{1}{\epsilon})^k$$
$$\prec \epsilon^{\frac{1}{2}}\log\log\frac{1}{\epsilon} \prec d_1(\epsilon) \prec \epsilon^{\frac{1}{2}}\log\frac{1}{\epsilon}$$

Where $d_0(\epsilon)$ is the rate of convergence to a limit cycle, $d_2(\epsilon)$ is the rate of convergence to a heteroclinic cycle with $\gamma = 1$ and $d_1(\epsilon)$ is the rate of convergence to a heteroclinic cycle with $\gamma > 1$

Convergence rates for the slow dynamics:

The rates are maintained by the slow dynamics

Exact error estimates in general:

Except for some very particular cases exact error estimates **are not** available for the for approximation of singularly perturbed differential equations and optimal control problems

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Other coupled slow and fast dynamics

$$\frac{dx}{dt} = G(x) + \frac{1}{\epsilon}F(x)$$

$$\frac{dx}{dt} = G(x,u) + \frac{1}{\epsilon}F(x,u)$$

Again, we are interested in the **limit behavior** of the system as $\epsilon \to 0$

Singular perturbations without split to slow and fast coordinates:





C. William Gear

Yannis Kevrekidis

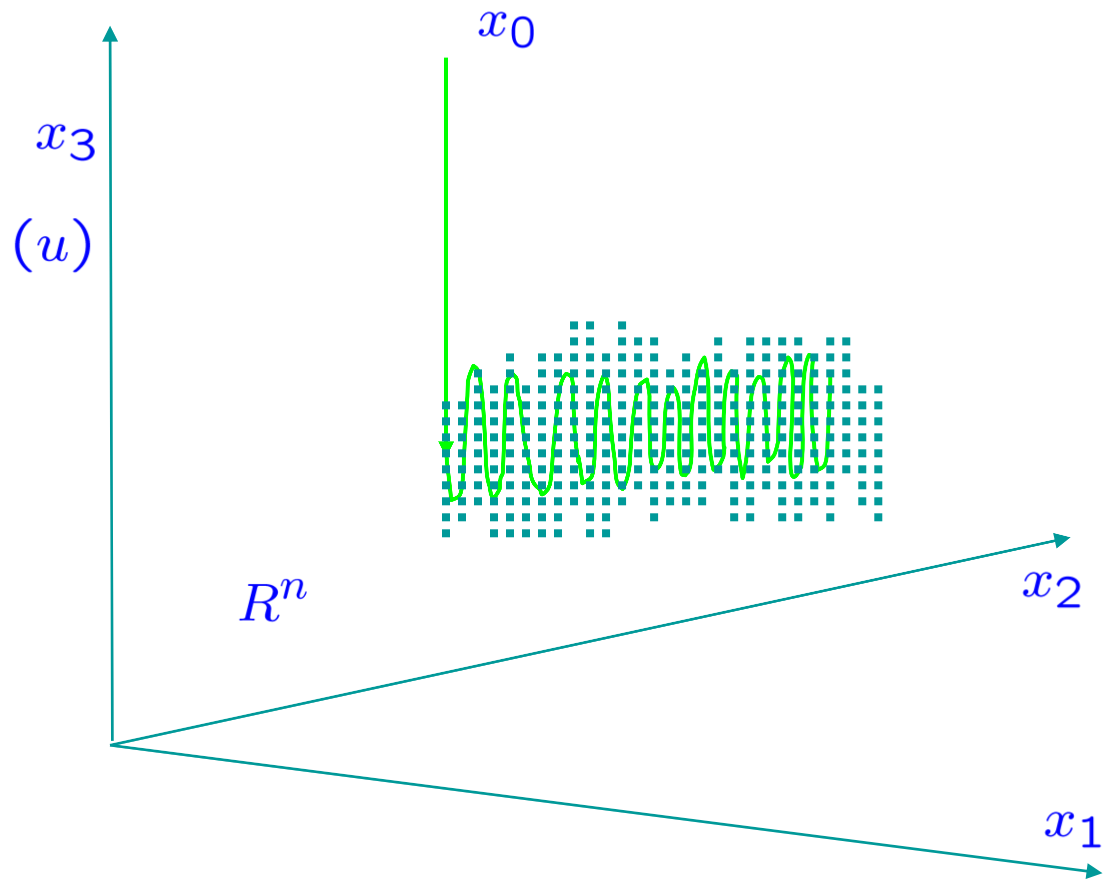
Marshall Slemrod

Edriss Titi

The perturbed system:

$$\frac{dx}{dt} = G(x) + \frac{1}{\epsilon} F(x)$$

The general situation:



The Young measure is defined on the time interval with values being probability measures on the x-space

Identifying slow and fast contributions :

Fast equation:

$$\frac{dx}{dt} = \frac{1}{\epsilon} F(x)$$

Equivalently:

$$\epsilon \frac{dx}{dt} = F(x)$$

The complete system:

$$\frac{dx}{dt} = G(x) + \frac{1}{\epsilon} F(x)$$

The limit solution:

As $\epsilon \to 0$ the limit (in the sense of Young measures) of the solution of the perturbed system:

$$\frac{dx}{dt} = G(x) + \frac{1}{\epsilon}F(x)$$

is an invariant measure of the fast equation:

$$\frac{dx}{dt} = \frac{1}{\epsilon}F(x)$$

drifted by the slow component.

How to track the evolution of the invariant measure?

The idea:
A probability measure is determined by its (generalized) moments, that is, integrals with respect to the measure of continuous real-valued functions

The trajectory of invariant measures:

The drift (change in time) of the measures $\mu_0(t)$ is determined by generalized moments, or observables, preferably first integrals of the fast equation:

$$v = v(t)$$

The dynamics of the observables satisfies:

$$\frac{dv}{dt} = \int_{R^n} (\nabla v)(x) \cdot G(x) \mu_0(t)(dx)$$

The novelty: The observables are not part of the state space.

The limit solution:

$$(v(t), \mu(v(t)))$$

where $\mu(v)(dx)$ is an invariant measure of the equation

$$\frac{dx}{ds} = F(x)$$

And $v(t)$ solves the "equation"

$$\frac{dv}{dt} = \int_{R^n} (\nabla v)(x) \cdot G(x) \mu_0(t)(dx)$$

This is the slow progress of fast dynamics

Research issues:

- Examples
- Singular cases: When is the differential relation a closed differential equation?
- When are there enough first integrals?
- How to determine (to characterize) the invariant measures?
- Numerical procedures

An example:

$$\frac{dU_k}{dt} + \frac{1}{2h\epsilon} U_k (U_{k+1} - U_{k-1}) = \frac{1}{h^2}(U_{k+1} - 2U_k + U_{k-1})$$

$$k = 1, 2, \ldots, 2n$$

With periodic boundary conditions. This is the Lax-Goodman discretization of the KdV-Burgers

$$u_t + u(u_x + \frac{h^2}{6} u_{xxx}) = \epsilon u_{xx}$$

$$0 \leq x \leq 2\pi$$

with periodic boundary conditions

The first integrals of the fast equation:

$$\frac{dU_k}{dt} + \frac{1}{2h\epsilon} U_k (U_{k+1} - U_{k-1}) = 0$$

$$k = 1, 2, \ldots, 2n$$

are the traces of the so called Lax pairs – these are computable even polynomials

Computing the dynamics of these time-varying polynomial enables the construction of the drift of the invariant measures

Computational results for an invariant measure:

$$\frac{dU_k}{dt} + \frac{1}{2h\epsilon} U_k (U_{k+1} - U_{k-1}) = 0$$

$$k = 1, 2, \ldots, 6$$

for the limit as $\epsilon \to 0$

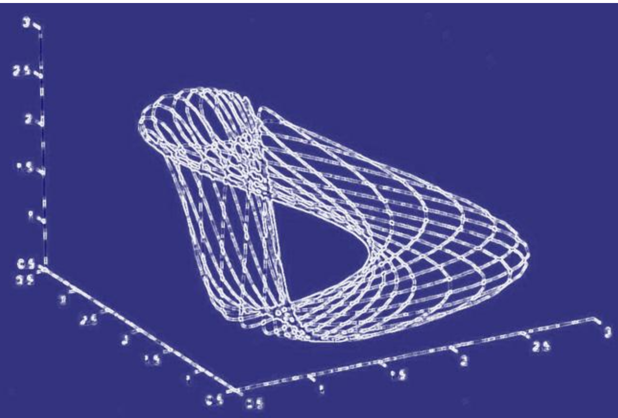


FIGURE 1. Torus for the case $N = 6$ of system (3.1). Initial values were [1 1 1 3 2 1].

Computational results for the drifted measure:

$$\frac{dU_k}{dt} + \frac{1}{2h\epsilon} U_k (U_{k+1} - U_{k-1}) = \frac{1}{h^2} (U_{k+1} - 2U_k + U_{k-1})$$

$$k = 1, 2, \ldots, 6$$

for the limit as $\epsilon \to 0$

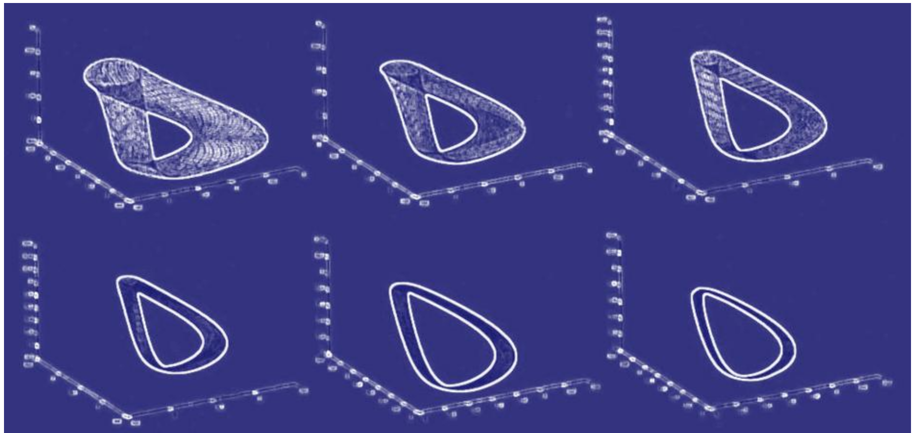


Figure 7. Evolution of Tori of system (6.1) for initial conditions [1 1 1 1 6 1]

Very little has been done for control case
What is lacking:

- **Examples !!!**
- When are there enough first integrals?
- Singular cases: When is the differential relation a closed differential equation?
- Is the dynamics regular enough for a necessary conditions?
- In general: How to treat dynamics in the space of probability measures?

Plan:

- √ Modeling
- √ Variational Limits
- √ Classical Approach to slow-fast dynamics
- √ What limits are appropriate? Young Measures
- √ Modern Approach to slow-fast dynamics
- √ Other chattering limits and averaging techniques
- √ Control Invariant Measures
- √ Stabilization
- √ Optimal Control
- √ Some special cases
- √ Computations, error estimates
- √ A Future Direction

The End

of the series of letures

Thanks for the attention

And all the best