Optimal control, Hamilton-Jacobi equations and singularities in euclidean and riemaniann spaces

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Università di Roma “Tor Vergata”

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Outline

1. Introduction to semiconcave functions, generalized differentials, and singularities
   - Semiconcave functions
   - Semiconcavity of value functions
   - Generalized differentials
   - Optimal synthesis
   - Singular sets, rectifiability

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Oleinik 1957 One-sided Lipschitz estimate as a uniqueness criterion for scalar hyperbolic conservation laws

Kruzhkov 1960, Douglis 1961 Semiconcavity as a uniqueness criterion for Hamilton-Jacobi equations

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Reference: Cannarsa–S.:"Semiconcave functions, Hamilton-Jacobi equations and optimal control" (Birkhäuser, 2004)

Semiconcave functions have been widely applied in other fields, e.g. nonlinear second order PDEs (Krylov), geometry of Alexandrov spaces (Perelman, Petrunin), etc.

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Semiconcave functions

Definition

A function $u \in C(A)$, with $A \subset \mathbb{R}^n$ is called semiconcave *in A (with a linear modulus) if there exists $C \geq 0$ such that*

$$u(x+h) + u(x-h) - 2u(x) \leq C|h|^2,$$

for all $x, h \in \mathbb{R}^n$ such that $[x-h, x+h] \subset A$.

C is called a semiconcavity constant *for u in A.*

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Equivalent formulations

Proposition

The following properties are equivalent:

- u *is semiconcave with constant* C;
- *the function* $x \to u(x) - \frac{C}{2}|x|^2$ *is concave in* A;
- $u = u_1 + u_2$, *with* u_1 *concave and* $u_2 \in C^2(A)$ *such that* $||D^2 u_2||_\infty \leq C$;
- *for any* $\nu \in \mathbb{R}^n$ *such that* $|\nu| = 1$ *we have* $\frac{\partial^2 u}{\partial \nu^2} \leq C$ *in* A *weakly;*
- $u(x) = \inf_{i \in \mathcal{I}} u_i(x)$, *where* $\{u_i\}_{i \in \mathcal{I}} \subset C^2(A)$ *such that* $||D^2 u_i||_\infty \leq C$ *for all* $i \in \mathcal{I}$.
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Generalizations

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A function $u : A \to \mathbb{R}$ *is called* semiconcave with modulus $\omega(\cdot)$, *where* $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ *is nondecreasing and satisfies* $\lim_{\rho \to 0^+} \omega(\rho) = 0$, *if*

$$\begin{aligned} & \lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \\ \leq \quad & \lambda(1-\lambda)|x-y|\omega(|x-y|) \end{aligned}$$

for any pair $x, y \in A$, *such that* $[x, y] \subset S$ *is contained in* S *and for any* $\lambda \in [0, 1]$.

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Generalizations (II)

- u semiconcave with modulus ω iff $u = \inf u_i$, with $u_i \in C^1$ and Du_i has a uniform modulus of continuity $\omega(\cdot)$, for every i.
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The distance function

Given any $S \subset \mathbb{R}^n$ closed, define

$$d_S(x) = \min_{y \in S} |y - x|, \qquad x \in \mathbb{R}^n,$$

distance function from the set S.

It is a special case of the *minimum time function*, corresponding to

$$y' = a(t) \in A = B_1 \text{ (unit ball).}$$

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Semiconcavity of the distance function

Proposition

- *The squared distance function d_S^2 is semiconcave in $\mathbb{R}^n$ with semiconcavity constant 2.*
- *d_S is locally semiconcave in $\mathbb{R}^n \setminus S$. More precisely, given Ω such that $\operatorname{dist}(S, \Omega) > 0$, d_S is semiconcave in Ω with semiconcavity constant equal to $\operatorname{dist}(S, \Omega)^{-1}$.*

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Semiconcavity of the distance function (II)

Proof of the semiconcavity of d_S^2

For any $x \in \mathbb{R}^n$ we have

$$d_S^2(x) - |x|^2 = \min_{y \in S} |x-y|^2 - |x|^2 = \min_{y \in S} \left(|y|^2 - 2\langle x, y \rangle \right) .$$

$\Longrightarrow$ $d_S^2(x) - |x|^2$ is concave (infimum of linear functions)

$\Longrightarrow$ $d_S^2(\cdot)$ semiconcave with constant 2. $\square$

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Semiconcavity of the distance function (III)

Proof of the local semiconcavity of d_S

Take $z, h \in \mathbb{R}^n$, $z \neq 0$. By a direct computation

$$|z+h| + |z-h| - 2|z| \leq \frac{|h|^2}{|z|}.$$

Let now Ω be a set with positive distance from S. For any x, h such that $[x-h, x+h] \subset \Omega$, let $\bar{x} \in S$ be a projection of x onto S. Then

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Interior sphere property

We say that $S \subset \mathbb{R}^n$ satisfies the *interior sphere property* for some $r > 0$ if, for any $x \in S$ there exists y such that $x \in \overline{B_r(y)} \subset S$.

Proposition

If S satisfies the interior sphere property for some $r > 0$, then d_S is semiconcave in $\overline{\mathbb{R}^n \setminus S}$ with constant equal to r^{-1}.

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The Mayer problem with fixed horizon

- (f, A) control process in $\mathbb{R}^n$, $T > 0$
- given (t, x) and a control $\alpha : [t, T] \to A$

$$y(\cdot; t, x, \alpha) \quad \text{solution of} \quad \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in [t, T] \\ y(t) = x \end{cases}$$

- $\psi : \mathbb{R}^n \to \mathbb{R}$ final cost

Problem (Mayer with fixed horizon)

minimize $\psi(y(T; t, x, \alpha))$ *over all* $\alpha \in L^1(t, T; A)$

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Semiconcavity of the value function

Value function

$$V(t,x) = \min_{\alpha} \psi(y(T;t,x,\alpha)), \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Theorem

(Cannarsa-Frankowska, 1991) *Suppose that*

– *The control set A is compact.*
– *$f(x,a)$ is differentiable w.r.t. x.*
– *$f(x,a)$ and $f_x(x,a)$ are Lipschitz continuous w.r.t. x, uniformly in a.*
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Then the value function V is semiconcave in $[0,T] \times \mathbb{R}^n$ (jointly in (t,x)).

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Proof of the semiconcavity (I)

Estimates on trajectories starting at different points but following the same control.

Lemma

There exists $c > 0$ such that

$$|y(T;t,x_0,\alpha) - y(T;t,x_1,\alpha)| \leq c|x_0 - x_1|,$$

$$\left| y(T;t,x_0,\alpha) + y(T;t,x_1,\alpha) - 2y\left(T;t,\frac{x_0+x_1}{2},\alpha\right)\right| \leq c|x_0 - x_1|^2$$

for all $\alpha : [t,T] \to U$ and $x_0, x_1 \in \mathbb{R}^n$.

Regularity of f, Gronwall Lemma.

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Proof of the semiconcavity (II)

For simplicity, we only prove semiconcavity w.r.t. x.

Consider $x-h, x, x+h \in \mathbb{R}^n$ and $t \in [0,T)$. Let $\alpha : [t,T] \to A$ be an optimal control for the middle point (t,x).

Let us set for simplicity

$$y(\cdot) = y(\cdot; t, x, \alpha), \quad y_-(\cdot) = y(\cdot; t, x-h, \alpha), \quad y_+(\cdot) = y(\cdot; t, x+h, \alpha).$$

By the previous lemma

$$|y_+(T) - y_-(T)| \le c|h|, \qquad |y_+(T) + y_-(T) - 2y(T)| \le c|h|^2.$$

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Proof of the semiconcavity (III)

It follows,

$$
\begin{aligned}
& V(t, x+h) + V(t, x-h) - 2V(t,x) \\
& \quad \le \psi(y_+(T)) + \psi(y_-(T)) - 2\psi(y(T)) \\
& \quad = \psi(y_+(T)) + \psi(y_-(T)) - 2\psi\left(\frac{y_+(T) + y_-(T)}{2}\right) \\
& \qquad +2\psi\left(\frac{y_+(T) + y_-(T)}{2}\right) - 2\psi(y(T)) \\
& \quad \le C_\psi |y_+(T) - y_-(T)|^2 + L_\psi |y_+(T) + y_-(T) - 2y(T)| \\
& \quad \le (C_\psi c^2 + L_\psi c)|h|^2,
\end{aligned}
$$

which proves the semiconcavity w.r.t. x. □

Minimum time function

- (f, A) control process in $\mathbb{R}^n$,
- $f(x, a)$ Lipschitz w.r.t. x, A compact;
- given $\alpha : [0, \infty) \to A$ control,

$$y(\cdot; x, \alpha) \quad \text{solution of} \quad \begin{cases} \dot{y}(t) = f(y(t), \alpha(t)) & (t \geqslant 0) \\ y(0) = x \end{cases}$$

- target $S \subset \mathbb{R}^n$ nonempty closed set
- transition time $\quad \tau(x, \alpha) = \inf \{ t \geqslant 0 \mid y(t; x, \alpha) \in S \}$
- controllable set $\quad \mathcal{C} = \{ x \in \mathbb{R}^n \mid \exists \alpha \ : \ \tau(x, \alpha) < \infty \}$
- minimum time function $\quad T(x) = \inf_\alpha \tau(x, \alpha) \quad x \in \mathcal{C}$

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Petrov condition

Definition

Given $y \in \partial S$, a vector $\nu \in \mathbb{R}^n$ is called a proximal normal *to S at y if*

$$\mathrm{proj}_S(y + \varepsilon\nu) = \{y\}$$

for $\varepsilon > 0$ small enough.

Definition

We say that (f, A) satisfies the Petrov condition *on S if there exists $\mu > 0$ such that*

$$\min_{a \in A} f(x, a) \cdot \nu \leq -\mu|\nu|$$

for any $x \in \partial S$, ν proximal normal to S at x.

Local controllability

Theorem

(Petrov 1970, Bardi-Falcone 1990, . . .) Let the Petrov condition hold. Then

- $\mathcal{C}$ *is an open neighbourhood of* S;
- *there exist* $k, \delta > 0$ *such that*

$$T(x) \leq k d_S(x), \qquad \forall x \text{ s.t. } d_S(x) \leq \delta$$

- T *is locally Lipschitz continuous on* $\mathcal{C}$.

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Semiconcavity of T

Theorem

Let the Petrov condition hold and let $f(x, a)$ be $C^{1,1}$ w.r.t. x.

- *If S satisfies an interior sphere property, then T is locally semiconcave in $\overline{\mathcal{C} \setminus S}$. (Cannarsa-S., 1995)*
- *If $f(x, A)$ is convex and satisfies an interior sphere property for x near S, then T is locally semiconcave in $\mathcal{C} \setminus S$. (Cannarsa-Frankowska, S., 2004)*
- *If $f(x, a) = Ax + a$ for some matrix A and S is convex, then T is locally semiconvex in $\overline{\mathcal{C} \setminus S}$. (Cannarsa-S., 1995)*

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Outline

1. Introduction to semiconcave functions, generalized differentials, and singularities
   - Semiconcave functions
   - Semiconcavity of value functions
   - Generalized differentials
   - Optimal synthesis
   - Singular sets, rectifiability

Lipschitz continuity

Proposition

If $u : A \to \mathbb{R}$ is semiconcave (with a general modulus), it is locally Lipschitz continuous in the interior of A.

Corollary

Semiconcave functions are differentiable almost everywhere (Rademacher's theorem).

Theorem

(Alexandroff) Semiconcave functions with linear modulus is twice differentiable almost everywhere.

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Fréchet differentials

Let $u : A \to \mathbb{R}$, with $A \subset \mathbb{R}^n$ open.

Definition

Given $x \in A$, the sets

$$D^- u(x) = \left\{ p \in \mathbb{R}^n \ : \ \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\},$$

$$D^+ u(x) = \left\{ p \in \mathbb{R}^n \ : \ \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}$$

are called, respectively, the (Fréchet) subdifferential *and* superdifferential *of u at x.*

Reachable gradients

Definition

Given $u : A \to \mathbb{R}$ and $x \in A$, we say that p is a reachable gradient of u at x if there exists $\{x_n\} \subset A$ such that u is differentiable at x_n and

$$x = \lim_{n\to\infty} x_n \qquad p = \lim_{n\to\infty} Du(x_n).$$

We denote by $D^*u(x)$ the set of reachable gradients.

If $u \in Lip_{loc}(A)$, then $D^*u(x) \neq \emptyset$ for any $x \in A$.

If $u \in Lip_{loc}(A)$, the convex hull of $D^*u(x)$ coincides with *Clarke's generalized gradient*.

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Differential properties

Proposition

Let $u : A \to \mathbb{R}$ be semiconcave (with general modulus). Then

- $D^+u(x) = co(D^*u(x))$.
- $D^+u(x) \neq \emptyset$.
- $D^*u(x) \subset \partial D^+u(x)$.
- If $x_k \to x$ and if $p_k \in D^+u(x_k)$ satisfy $p_k \to p$, then $p \in D^+u(x)$ (upper semicontinuity of D^+u).
- If $D^+u(x)$ is a singleton, then u is differentiable at x.

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Differential properties

For simplicity, linear modulus of semiconcavity, A open convex.

Proposition

Let $u : A \to \mathbb{R}$ be semiconcave with constant C. Then

- $p \in D^+u(x)$ *if and only if*

$$u(y) \leq u(x) + \langle p, y - x \rangle + \frac{C}{2}|x - y|^2$$

for all $y \in A$;

- *given* x, y *and* $p \in D^+u(x), q \in D^+u(y)$, *we have*

$$\langle q - p, y - x \rangle \leq C|x - y|^2 \qquad \text{(monotonicity of } D^+u\text{)}.$$

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An approximation lemma

Proposition

Let $u : A \to \mathbb{R}$ be semiconcave, $x_0 \in A$ and V an open set such that $x_0 \in V \subset \overline{V} \subset A$. Then, for any $p \in D^+u(x_0)$ there is a sequence $u_k \in C^\infty(V)$ such that

- $u_k \to u$ *uniformly in* V
- $Du_k(x_0) \to p$
- $||u_k||_\infty \le M, ||Du_k||_\infty \le L, ||D^2u_k||_\infty \le C$, *for all* k, *where* M, L *and* C *are respectively the supremum, the Lipschitz constant and the semiconcavity constant of* u *on* A.

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Semiconcavity and viscosity

Consider the Hamilton-Jacobi equation

$$(HJ) \qquad H(x, u, Du) = 0, \qquad x \in \Omega \subset \mathbb{R}^n.$$

with H a continuous function.

$u \in C(\Omega)$ is a *viscosity solution* of (HJ) if it satisfies, for any $x \in \Omega$,

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Semiconcavity and viscosity

Consider the Hamilton-Jacobi equation

$$(HJ) \qquad H(x, u, Du) = 0, \qquad x \in \Omega \subset \mathbb{R}^n.$$

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Semiconcavity and viscosity (II)

Proposition

Suppose that $H(x, u, p)$ is convex w.r.t. p. Let $u : \Omega \to \mathbb{R}$ be a semiconcave function which satisfies (HJ) at all points of differentiability. Then u is a viscosity solution of (HJ).

Proof — At the points x where u is differentiable — trivial.

If u is not differentiable at x, then $D^- u(x) = \emptyset$, while $D^+ u(x) = \text{co}(D^* u(x))$.

By continuity, $H(x, u(x), p) = 0$ for all $p \in D^* u(x)$.

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Marginal functions

Marginal functions: infimum of smooth functions

($\longleftrightarrow$ semiconcave functions.)

$A \subset \mathbb{R}^n$ open, $S \subset \mathbb{R}^m$ compact.
$F = F(s, x)$ continuous in $S \times A$ together with $D_x F$.

Define $u(x) = \min_{s \in S} F(s, x)$. Then u is semiconcave.

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Marginal functions (II)

Theorem

Let $u(x) = \min_{s \in S} F(s,x)$ as above. Given $x \in A$, define

$$M(x) = \{s \in S : u(x) = F(s,x)\},$$

$$Y(x) = \{D_x F(s,x) : s \in M(x)\}.$$

Then, for any $x \in A$

$$D^+ u(x) = \mathrm{co}\, Y(x).$$

In particular, u is differentiable at x if and only if $Y(x)$ is a singleton.

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Generalized gradients of the distance

Corollary

Let S be a nonempty closed subset of $\mathbb{R}^n$. Then

- *d_S is differentiable at $x \notin S$ if and only if $\mathrm{proj}_S(x)$ is a singleton and in this case*

$$Dd_S(x) = \frac{x-y}{|x-y|}$$

  where y is the unique element of $\mathrm{proj}_S(x)$.

- *If $\mathrm{proj}_S(x)$ is not a singleton then we have*

$$D^+ d_S(x) = \mathrm{co}\left\{ \frac{x-y}{|x-y|} \;:\; y \in \mathrm{proj}_S(x) \right\},$$

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Back to the Mayer problem

- (f, A) control process in $\mathbb{R}^n$, $T > 0$
- given (t, x) and a control $\alpha : [t, T] \to A$

$$y(\cdot; t, x, \alpha) \quad \text{solution of} \quad \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in [t, T] \\ y(t) = x \end{cases}$$

- $\psi : \mathbb{R}^n \to \mathbb{R}$ final cost

Mayer problem: minimize $\psi(y(T; t, x, \alpha))$ over all $\alpha \in L^1(t, T; A)$

Value function $V(t, x) = \min_\alpha \psi(y(T; t, x, \alpha))$.

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We assume in the following

- A compact
- $f(x,a)$ of class $C^{1,1}$ w.r.t. x
- $\psi : \mathbb{R}^n \to \mathbb{R}$ of class C^1 and semiconcave

Pontryagin's maximum principle

Theorem

- $\alpha^* \in L^1(0, T; A)$ *and* $y^*(\cdot) := y(\cdot; x, \alpha^*)$ *optimal pair*

$$\psi(y^*(T)) = \min_{\alpha \in L^1(0,T;A)} \psi(y(T; x, \alpha))$$

- *let* p^* *be the solution of the adjoint problem*

$$\begin{cases} \dot{p}(s) = -f_x\big(y^*(s), \alpha^*(s)\big)^{tr} p(s) & (s \in [0, T]) \\ p(T) = D\psi(y^*(T)) \end{cases}$$

then

$$p^*(s) \cdot f\big(y^*(s), \alpha^*(s)\big) = \min_{a \in A} p^*(s) \cdot f\big(y^*(s), a\big) \quad (s \in [0, T] \; a.e.)$$

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Dual arc inclusion for the Mayer problem

Denote by $\nabla^+ V(t,x)$, $\nabla^- V(t,x)$ the super- and subdifferential of V at (t,x) with respect to the x variable alone.

Theorem

(Clarke-Vinter 1987, Cannarsa-Frankowska, 1991) Under the previous assumptions, we have that

$$p(s) \in \nabla^+ V(s, y(s)), \quad \forall s \in [t, T].$$

If in addition $p(t) \in \nabla^- V(t, y(t))$, then we also have

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Hamiltonian form of PMP

Assume that $f(x, A)$ is a (n-dimensional) uniformly convex set for all x.

This implies that $H(x, p) = \max_{a \in A} -p \cdot f(x, a)$ is smooth for $p \neq 0$.

Theorem

Let (α, y) be an optimal pair for the point $(t, x) \in [0, T] \times \mathbb{R}^n$ and let $p : [t, T] \to \mathbb{R}^n$ be a dual arc associated with (α, y) such that $p(\bar{s}) \neq 0$ for some $\bar{s} \in [t, T]$. Then $p(s) \neq 0$ for all $s \in [t, T]$ and (y, p) solves the system

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Given a point $(t,x) \in [0,T[\times\mathbb{R}^n$ and a vector $\bar{p} = (\bar{p}_t, \bar{p}_x) \in D^*V(t,x)$ such that $\bar{p} \neq 0$, let us associate with $\bar{p}$ the pair $(y(\cdot), p(\cdot))$ which solves the hamiltonian system with initial conditions $y(t) = x$, $p(t) = \bar{p}_x$.

Then $y(\cdot)$ is an optimal trajectory for (t,x) and $p(\cdot)$ is a dual arc associated with $y(\cdot)$.

The map from $D^*V(t,x)$ to the set of optimal trajectories from (t,x) defined in this way is injective, and it is one-to-one if $0 \notin D^*V(t,x)$.

Corollary

If $0 \notin D^*V(t,x)$, then the optimal trajectory at (t,x) is unique if and only if V is differentiable at (t,x).

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   - Semiconcave functions
   - Semiconcavity of value functions
   - Generalized differentials
   - Optimal synthesis
   - Singular sets, rectifiability

The singular set

Given $u : A \to \mathbb{R}$ semiconcave, the *singular set* of u is

$$\begin{aligned}\Sigma(u) &= \{x \in A \ : \ u \text{ is not differentiable at } x\} \\ &= \{x \in A \ : \ D^+u(x) \text{ is not a singleton}\}.\end{aligned}$$

We know: Σ has measure zero.

Much sharper results can be given in terms of *rectifiability* properties.

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rectifiable sets

Let $k \in \{0, 1, \ldots, n\}$ and let $C \subset \mathbb{R}^n$.

- C is called a k–*rectifiable* set if there exists a Lipschitz continuous function $f : \mathbb{R}^k \to \mathbb{R}^n$ such that $C \subset f(\mathbb{R}^k)$.
- C is called a *countably* k–*rectifiable* set if it is the union of a countable family of k–rectifiable sets.
- C is called a *countably* $\mathcal{H}^k$–*rectifiable set* if there exists a countably k–rectifiable set $E \subset \mathbb{R}^n$ such that $\mathcal{H}^k(C \setminus E) = 0$. Here $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

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- C is called a *countably* k–*rectifiable* set if it is the union of a countable family of k–rectifiable sets.
- C is called a *countably* $\mathcal{H}^k$–*rectifiable set* if there exists a countably k–rectifiable set $E \subset \mathbb{R}^n$ such that $\mathcal{H}^k(C \setminus E) = 0$. Here $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

rectifiability results

It is easy to see that, if u is semiconcave with a linear modulus, then Du is a function of bounded variation.

The singular set $\Sigma(u)$ coincides with the *jump set* of Du in the theory of BV functions.

Standard results about BV functions then imply that $\Sigma(u)$ is *countably* $\mathcal{H}^{n-1}$*–rectifiable*.

More precise results can be obtained by a direct approach.

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rectifiability results (II)

$D^+u(x)$ is a convex set $\Longrightarrow$ it has an integer dimension.

For $k = 1, \ldots, n$, we define

$$\Sigma^k(u) = \{x \in \Sigma \ : \ \dim(D^+u(x)) = k\}.$$

Theorem

If $u : \Omega \to \mathbb{R}$ is semiconcave (with a general modulus) then the set $\Sigma^k(u)$ is countably $(n-k)$-rectifiable for any $k = 1, \ldots, n$.

Zajíček (1978), Veselý (1986), Alberti-Ambrosio-Cannarsa (1992).

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example

Let $u(x,y) = -|x| - |y|$, concave on $\mathbb{R}^2$.

Then $\Sigma(u) = \{(x,y) \ : \ x = 0 \text{ or } y = 0\}$.

If $x = 0$ and $y > 0$, then $D^+u(x,y) = [-1,1] \times \{-1\}$. Similarly, any point with $x = 0, y \neq 0$, or with $x \neq 0, y = 0$ belongs to $\Sigma^1(u)$.

Finally, $D^+u(0,0) = [-1,1] \times [-1,1]$, and $\Sigma^2(u) = \{(0,0)\}$.

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sketch of the proof

Definition

Let $S \subset \mathbb{R}^n$ and $x \in \overline{S}$ be given. The contingent cone *(or* Bouligand's tangent cone*) to S at x is the set*

$$T(x,S) = \left\{ \lim_{i\to\infty} \frac{x_i - x}{t_i} \; : \; x_i \in S, x_i \to x, t_i \in \mathbb{R}_+, t_i \downarrow 0 \right\}.$$

The vector space generated by $T(x,S)$ is called tangent space *to S at x and is denoted by $\mathrm{Tan}(x,S)$.*

sketch of the proof (II)

Theorem

Let $S \subset \mathbb{R}^n$ be a set such that $\dim \mathrm{Tan}(x, S) \leq k$, for all $x \in S$, for a given integer $k \in [0, n]$. Then S is countably k–rectifiable.

Given $\rho > 0$, we denote by $\Sigma^k_\rho(u)$ the set of all $x \in \Sigma^k(u)$ such that $D^+u(x)$ contains a k–dimensional sphere of radius ρ.

Theorem

If u is semiconcave in Ω, then

$$\mathrm{Tan}(x, \Sigma^k_\rho(u)) \subset [D^+u(x)]^\perp, \quad \forall x \in \Sigma^k_\rho(u).$$

The rectifiability theorem follows. □

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Thank you for your attention!