Optimal control, Hamilton-Jacobi equations and singularities in euclidean and riemaniann spaces

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Università di Roma "Tor Vergata"

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NEW TRENDS IN OPTIMAL CONTROL

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Outline

1. Propagation of singularities in euclidean space
   - Semiconcave functions
   - Solutions of HJ equations
   - Weak KAM theory

Outline

propagation of singularities

$u : \Omega \to \mathbb{R}$ semiconcave $x_0 \in \Sigma(u)$

Definition

singularity at x_0 propagates: $\exists \delta > 0$ *and* $\{x_k\}_k \subset \Sigma(u) \setminus \{x_0\}$ *such that*

$$x_k \to x_0 \quad \& \quad diam\left(D^+u(x_k)\right) > \delta$$

want to give conditions for a given singularity to propagate

- C – Soner 1987, 1989
- Ambrosio – C – Soner 1993
- Albano – C 1999, 2000, 2002; Albano 2002
- C – Sinestrari 2004
- Yu 2006, 2007; C – Yu 2009
- Albano – C – Nguyen, Sinestrari 2012
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magnitude of a singular point

Definition

magnitude of $x_0 \in \Sigma(u)$: $\kappa(x_0) = \dim\left(D^+ u(x_0)\right)$

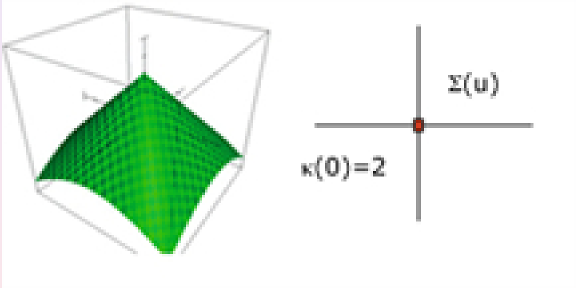


Figure: no propagation for $u(x,y) = 3 - \sqrt{x^2 + y^2}$

do singularities of lower magnitude propagate?

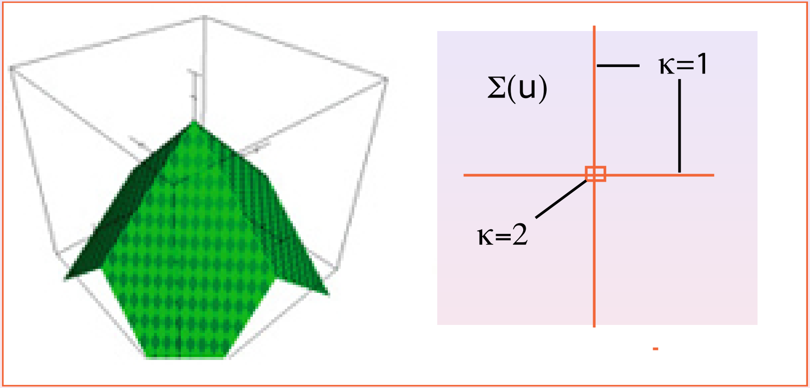


Figure: magnitude 1 singularities of $u(x,y) = 3 - |x| - |y|$ do propagate along straight lines

NO

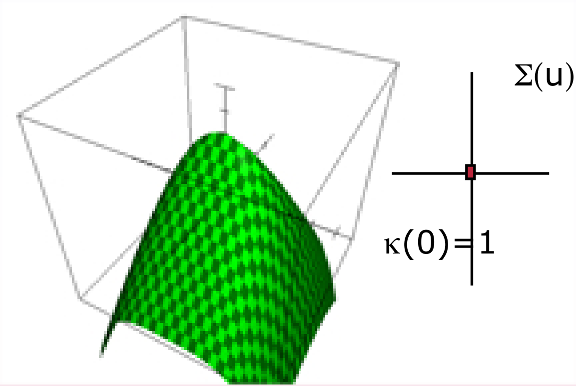


Figure: an isolated singularity of magnitude 1 at the origin

$$u(x,y) = 3 - \sqrt{\left(\frac{3x}{2}\right)^2 + \left(\frac{2y}{3}\right)^4}$$

“pattern recognition”

a closer look at D^+u (recall $D^*u \subset \partial D^+u$)

Example

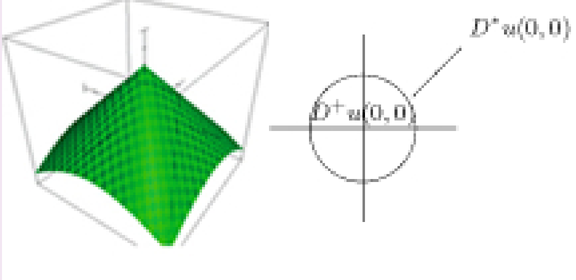


Figure: in example 1 $D^*u(0,0) = \partial D^+u(0,0)$

example 2

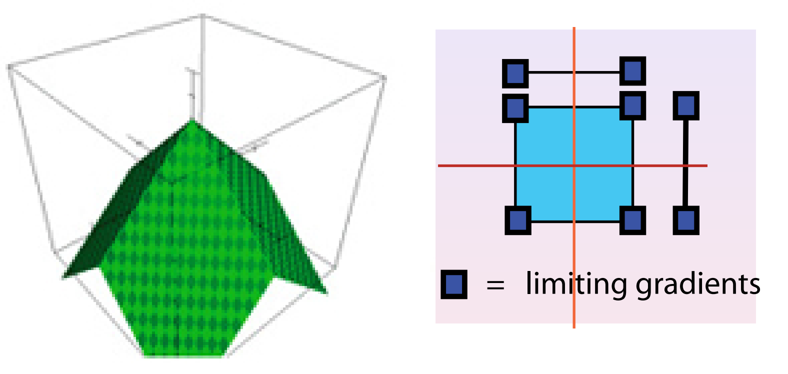
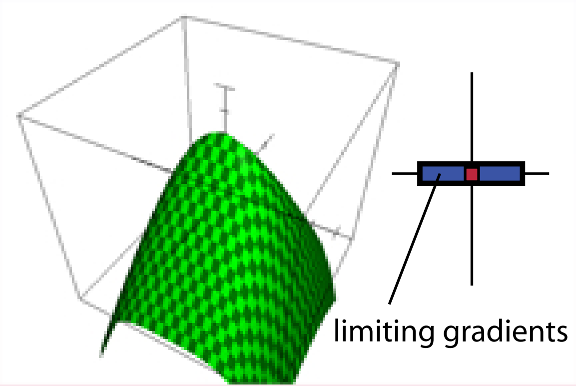


Figure: here

$D^*u(0,0) \subsetneq \partial D^+u(0,0)$

example 3



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$$D^* u(0,0) = D^+ u(0,0)$$

propagation principle

Theorem (Albano – C 1999; C – Yu 2009)

$u : \Omega \to \mathbb{R}$ *semiconcave* $x_0 \in \Sigma(u)$

suppose

$$\varnothing \neq \partial D^+ u(x_0) \setminus D^* u(x_0) \ni p_0$$

Let $q \in \mathbb{R}^N \setminus \{0\}$ *be such that* $q \cdot (p - p_0) \geqslant 0 \quad \forall p \in D^+ u(x_0)$

Then $\exists\, x(\cdot) : [0, \tau) \to \Sigma(u)$ *Lipschitz*

- $x(0) = x_0 \quad \& \quad \dot{x}^+(0) = q$
- $\dot{x}^+$ *continuous from the right*
- $\dot{x}(t) \in q - p_0 + D^+ u(x(t)) \quad (t \in [0, \tau)$ *a.e.*$)$
- $\inf_{t \in [0,\tau]} \operatorname{diam} D^+ u(x(t)) > 0$

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back to example 2

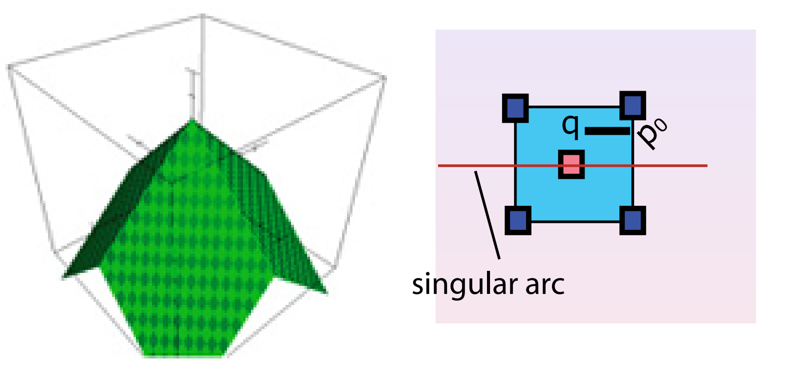

Figure: the propagation principle at work

construction of singular arc

- for $t > 0$ small let $\boxed{x(t)}$ be the unique max point of the strictly concave function

$$\phi_t(x) = u(x) - u(x_0) - (p_0 - q) \cdot (x - x_0) - \frac{1}{2t}|x - x_0|^2 \quad (x \in \overline{B}_r(x_0))$$

- then $\quad 0 < |x(t) - x_0| < \dfrac{2|q|t}{1 - Ct}$ indeed

$$\begin{cases} \phi_t(x(t)) > 0 \quad (\forall t > 0 \text{ small}) & \text{because } p_0 - q \notin D^+u(x_0) \\ \phi_t(x) \leqslant q \cdot (x - x_0) + \left(\frac{C}{2} - \frac{1}{2t}\right)|x - x_0|^2 & \text{by semiconcavity} \end{cases}$$

- by Fermat's rule

$$0 \in D^+\phi_t(x(t)) = D^+u(x(t)) - \underbrace{\left(p_0 - q + \frac{x(t) - x_0}{t}\right)}_{p(t)}$$

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application to distance function

$\varnothing \neq S \subset \mathbb{R}^n$ closed

$$d_S(x) = \min_{y \in S} |x - y| \qquad (x \in \mathbb{R}^n)$$

Corollary

$\boxed{x \in \Sigma(d_S) \setminus S}$ *the following properties are equivalent*

1. *x isolated singularity of d_S*
2. $\partial D^+ d_S(x) = D^* d_S(x) (= \partial B_1)$
3. $\mathrm{proj}_S(x) = \partial B_r(x)$ *with $r = d_S(x)$*

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application to distance function

$\emptyset \neq S \subset \mathbb{R}^n$ closed

$$d_S(x) = \min_{y \in S} |x - y| \qquad (x \in \mathbb{R}^n)$$

Corollary

$\boxed{x \in \Sigma(d_S) \setminus S}$ *the following properties are equivalent*

1. *x isolated singularity of d_S*
2. $\partial D^+ d_S(x) = D^* d_S(x) (= \partial B_1)$
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$$p_0 \in D^+u(x_0)\,, \quad N_{p_0} = \Big\{ q \in \mathbb{R}^n \;\Big|\; |q| = 1\,,\; q\cdot(p-p_0) \geq 0\,,\; \forall p \in D^+u(x_0) \Big\}$$

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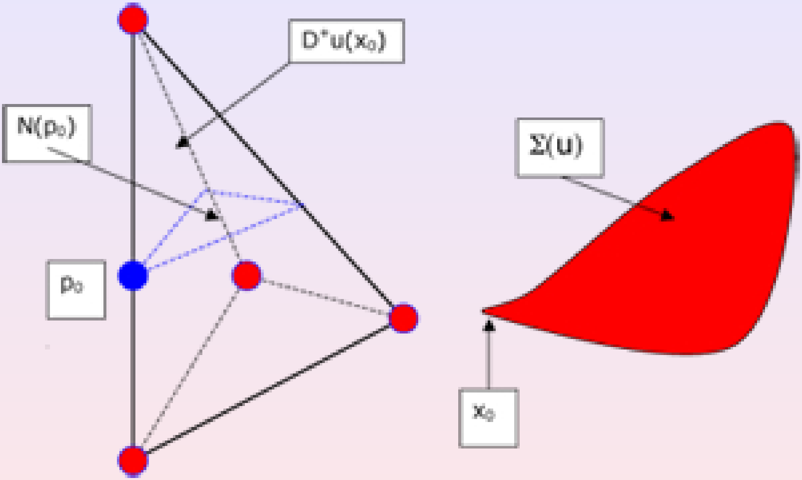
D*u(x0)
N(p0)
p0
Σ(u)
x0

Outline

1. Propagation of singularities in euclidean space
   - Semiconcave functions
   - Solutions of HJ equations
   - Weak KAM theory

semiconcave solutions of HJB equations

$u : \Omega \to \mathbb{R}$ semiconcave $H = H(x, u, p)$ continuous

$$H(x, u(x), \nabla u(x)) = 0 \qquad x \in \Omega \subset \mathbb{R}^N \quad \text{a.e.} \qquad (H)$$

with

- $H(x, u, \cdot)$ convex with strictly convex level sets $\forall (x, u) \in \Omega \times \mathbb{R}$

Proposition

$u : \Omega \to \mathbb{R}$ *semiconcave solution* (H) *then*

- u *viscosity solution*
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generalized characteristics

$u : \Omega \to \mathbb{R}$ semiconcave $H(x, u, \cdot) \in C^1(\mathbb{R}^n)$

Definition

Lipschitz arc $\xi(\cdot) : [0, \tau) \to \Omega$ *generalized characteristic for* (u, H)

$$\dot\xi(t) \in \mathrm{co}\, \partial_p H\Big(\xi(t), u(\xi(t)), D^+u(\xi(t))\Big) \quad \text{for a.e. } t \in [0, \tau)$$

Dafermos 1977, Albano – C 2000, C – Yu 2009

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recall approximation lemma

Lemma

let $u : \Omega \to \mathbb{R}$ be semiconcave, $x_0 \in \Omega$ and V an open set such that

$$x_0 \in V \subset \overline{V} \subset \Omega$$

then, for any $p^0 \in D^+u(x_0)$ there is a sequence $u_k \in C^\infty(V)$ such that

- $u_k \to u$ uniformly in V
- $\nabla u_k(x_0) \to p^0$
- $||u_k||_\infty \leqslant M, ||\nabla u_k||_\infty \leqslant L, ||D^2 u_k||_\infty \leqslant C$, for all k,

where M, L and C are respectively the supremum, the Lipschitz constant and the semiconcavity constant of u on Ω

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proof of approximation lemma

let

$$p^0 = \sum_{i=1}^{n+1} \lambda_i p^i\,, \quad p^i \in D^* u(x_0)\,, \quad \sum_{i=1}^{n+1} \lambda_i = 1\,,\ \lambda_i \geqslant 0$$

then $\forall\, k \geqslant 1\,,\ \forall i = 1, \dots, n+1\,,\ \exists\, x_k^i \in V$ such that

- u differentiable at $x_k^i \in B_{1/k}(x_0)$
- $\left|\nabla u(x_k^i) - p^i\right| \leq \frac{1}{k}$ and $\left|\sum_{i=1}^{n+1} \lambda_i \nabla u(x_k^i) - p^0\right| \leq \frac{1}{k}$
- moreover $\exists\, \epsilon_k \in (0, 1/k)$ such that

$$\sup_{x \in \overline{B}_{\epsilon_k}(x_k^i)} d_{D^+ u(x_k^i)}(\nabla u(x_k^i)) \leqslant \frac{1}{k} \qquad (i = 1, \dots, n+1)$$

fix $\eta \in C_0^\infty(B_1(0))$ be a cut-off function with $\eta \geqslant 0\,,\ \int_{\mathbb{R}^n} \eta = 1\,,$ and define

$$\eta_k(x) = \sum_{i=1}^{n+1} \frac{\lambda_i}{\epsilon_k^n} \eta\Big(\frac{x_0 - x_k^1 - x}{\epsilon_k}\Big)$$

then the required approximation is

$$u_k(x) := \int_{\mathbb{R}^n} u(x-y)\, \eta_k(y)\, dy$$

proof of approximation lemma

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$$p^0 = \sum_{i=1}^{n+1} \lambda_i p^i \,, \quad p^i \in D^* u(x_0) \,, \quad \sum_{i=1}^{n+1} \lambda_i = 1 \,, \ \lambda_i \geqslant 0$$

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  so that ξ turns out to be a generalized characteristic for (u, H)
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$$\frac{d}{dt} H(\xi_k, u_k(\xi_k), \nabla u_k(\xi_k)) \leqslant C|\partial_p H|^2 + |\partial_p H||\nabla u_k||\partial_u H| + |\partial_x H||\partial_p H| \leqslant C$$

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- conclude $\lim_{k\to\infty,\ t\to 0^+} \sup_{s\in[0,t]} |\nabla u_k(\xi_k(s)) - p_0| = 0$

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propagation along generalized characteristics

generalized characteristics may well be constant however...

Theorem

- $u : \Omega \to \mathbb{R}$ *semiconcave solution* $H(x, u, \nabla u) = 0$ *a.e. in* Ω
- $x_0 \in \Sigma(u)$ *such that* $0 \notin \partial_p H(x_0, u(x_0), D^+u(x_0))$

then $\exists \xi : [0, \tau) \to \Omega$ *generalized characteristic for* (u, H) *with*

- $\xi(0) = x_0$
- $\dot{\xi}^+(0) \neq 0$
- $\xi(t) \in \Sigma(u)$ *for all* $t \in [0, \tau)$

Albano – C 2000, Yu 2006, C – Yu 2009

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proof of theorem

- arguing as before $\forall t \in [0,\tau]\ \exists p_t \in D^+u(\xi(t))$ such that

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nonuniqueness of generalized chacteristics

Example

- *let*

$$H(p_1,p_2)=\frac{1}{4}(p_1^4+p_2^4)-\frac{p_1}{2}-\frac{p_2}{2} \quad \textit{and} \quad u(x)=\min\{x_1,x_2\}$$

- *then* $D^+u(0,0)=\{(t,1-t) \mid 0\le t\le 1\}$
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find ξ_1 [*answer:* $\xi_1(t)=-\left(\frac{3}{8},\frac{3}{8}\right)t$]

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Exercise

find ξ_1 [*answer:* $\xi_1(t)=-\left(\frac{3}{8},\frac{3}{8}\right)t$]

nonuniqueness of generalized chacteristics

Example

- *let*

$$H(p_1, p_2) = \frac{1}{4}(p_1^4 + p_2^4) - \frac{p_1}{2} - \frac{p_2}{2} \quad \textit{and} \quad u(x) = \min\{x_1, x_2\}$$

- *then* $D^+u(0,0) = \{(t, 1-t) |\ 0 \le t \le 1\}$
- *since* $(0,0) \in \mathrm{co}\, \partial_p H(D^+u(0,0))$

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uniqueness yields regularity

the following corollary follows from theorem by translation invariance of generalized characteristic flow

Corollary

suppose

$$\boxed{\forall x \in \Omega\ \exists! \ \text{generalized characteristic for } (u, H) \text{ starting from } x}$$

then $\forall\, \xi : [0, \tau] \to \Omega$ *generalized characteristic for* (u, H)

- $\exists \dot{\xi}^+(t)$ *for all* $t \in [0, \tau)$ *and*

$$\dot{\xi}^+(t) = \partial_p H\Big(\xi(s), u(\xi(s)), p(s)\Big)$$

  where $p(s)$ *is the unique point of* $D^+u(\xi(s))$ *such that*

$$H\Big(\xi(s), u(\xi(s)), p(s)\Big) = \min_{p \in D^+u(\xi(s))} H\Big(\xi(s), u(\xi(s)), p\Big)$$

- $\dot{\xi}^+$ *is right-continuous*

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generalized gradient flow

Example

for $u : \Omega \to \mathbb{R}$ *semiconcave and* $\boxed{H(x,p) = \frac{1}{2}\left(|p|^2 - 1\right)}$ *we have*

$$\forall x \in \Omega\ \exists! \ \text{ generalized characteristic for } (u, H) \text{ starting from } x$$

indeed if ξ_1 *and* ξ_2 *are two generalized characteristics from the same point*

$$\dot{\xi}_i(s) \in D^+u(\xi_i(s)) \quad (s \in [0, \sigma] \ a.e.).$$

so, by monotonicity of D^+u,

$$\frac{1}{2}\frac{d}{ds}\left|\xi_2(s) - \xi_1(s)\right|^2 = \left(\dot{\xi}_2(s) - \dot{\xi}_1(s)\right) \cdot \left(\xi_2(s) - \xi_1(s)\right) \leq C\left|\xi_2(s) - \xi_1(s)\right|^2$$

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show uniqueness of generalized characteristic with same initial datum for

$$H(x,p) = p_{n+1} + |p'|^2 \quad \text{where} \quad p := (p', p_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$$

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Outline

1. Propagation of singularities in euclidean space
   - Semiconcave functions
   - Solutions of HJ equations
   - Weak KAM theory

preliminaries from weak KAM theory

- $\mathbb{T}^N = N-$dimensional flat torus
- $V(\cdot)$ smooth $\mathbb{T}^N-$periodic function

Theorem

for each $P \in \mathbb{R}^N$ there is a unique real number $\overline{H}(P)$ such that the (cell) problem

$$\frac{1}{2}|P + \nabla u(x)|^2 + V(x) = \overline{H}(P)$$

has a $\mathbb{T}^N-$periodic solution v
moreover

- *v is semiconcave in $\mathbb{R}^n$*
- *$\overline{H}$ is convex and* $\min_{\mathbb{R}^N} \overline{H} = \overline{H}(0) = \max_{\mathbb{T}^n} V$ (Mañé's critical value)

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global propagation in weak KAM theory

Theorem (C – Yu, 2009)

fix $P \in \mathbb{R}^N$ *and let*

- v *be a* $\mathbb{T}^N$*–periodic solution of*

$$\frac{1}{2}|P + \nabla u(x)|^2 + V(x) = \overline{H}(P)$$

- *the (energy) condition* $\overline{H}(P) > \max_{\mathbb{T}^n} V$ *be satisfied*
- $x_0 \in \Omega \subset \mathbb{R}^N$ *bounded with* $\partial\Omega \sim S^{N-1}$

then

$$x_0 \in \Sigma(v) \implies \partial\Omega \cap \Sigma(v) \neq \emptyset$$

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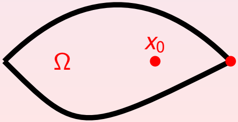

preliminaries of the proof

- argue by contradiction and suppose v differentiable on $\partial\Omega$
- let $u(x) := P \cdot x + v(x)$ observe u semiconcave, $\Sigma(u) = \Sigma(v)$, and

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hence $\forall x \in \mathbb{R}^n \ : \ \exists \nabla u(x)\ \exists! \xi_x \in C^1\big((\infty, 0]; \mathbb{R}^n\big)$ such that

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moreover ξ_x minimizes the action functional

$$J_T[\xi] = \int_T^0 \Big[\frac{1}{2}|\dot{\xi}|^2 - V(\xi)\Big]\, dt \qquad (\forall T < 0)$$

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core of the proof

$\forall\ t < 0$ define

- $\gamma_t : \partial\Omega \to \mathbb{R}^n$ by $\gamma_t(x) = \xi_x(t)$
- $\Gamma_t := \gamma_t(\partial\Omega)$

observe

$$\gamma_t \begin{cases} \text{continuous} \\ \text{one-to-one} \end{cases} \Longrightarrow \gamma_t \text{ homeomorphism} \Longrightarrow \Gamma_t \text{ homeomorphic to } S^{n-1}$$

moreover

- $t \mapsto \Gamma_t$ is continuous with respect to $d_{\mathcal{H}}$
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$$P \cdot (x - \xi_x(t)) + v(x) - v(\xi_x(t)) = u(x) - u(\xi_x(t)) = \int_t^0 \nabla u(\xi_x(s)) \cdot \dot{\xi}_x(s)\, ds$$

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$$\gamma_t \begin{cases} \text{continuous} \\ \text{one-to-one} \end{cases} \Longrightarrow \gamma_t \text{ homeomorphism} \Longrightarrow \Gamma_t \text{ homeomorphic to } S^{n-1}$$

moreover

- $t \mapsto \Gamma_t$ is continuous with respect to $d_{\mathcal{H}}$
- $\Gamma_T \subset \{ y \in \mathbb{R}^n \mid P \cdot y \leqslant P \cdot x_0 - 1 \}$ for $T << 0$

$$P \cdot (x - \xi_x(t)) + v(x) - v(\xi_x(t)) = u(x) - u(\xi_x(t)) = \int_t^0 \nabla u(\xi_x(s)) \cdot \dot{\xi}_x(s)\, ds$$

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$$\Longrightarrow \quad P \cdot \xi_x(t) \leqslant P \cdot x_0 + |p| \operatorname{diam}(\Omega) + 2\|v\|_\infty + 2\mu\, t \qquad (x \in \partial\Omega\,,\ t \leqslant 0)$$

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$$\mathbb{R}^n \setminus \Gamma_t = \Omega_t \cup \overline{\Omega}_t^c \quad \text{with} \quad \Omega_t \quad \text{bounded} \qquad (\forall\, t \leqslant 0)$$

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$$\bar{d}_{\Gamma_t}(x) = d_{\mathbb{R}^n \setminus \Omega_t}(x) - d_{\overline{\Omega}_t}(x) = \begin{cases} d_{\Gamma_t}(x) & \text{if } x \in \overline{\Omega}_t \\ -d_{\Gamma_t}(x) & \text{if } x \in \overline{\Omega}_t^c \end{cases}$$

turns out to be continuous in both x and t

- the continuous function $h(t) := \bar{d}_{\Gamma_t}(x_0)\ (t \leqslant 0)$ satisfies
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local propagation in weak KAM theory

Theorem (C – Cheng – Zhang, 2012)

fix $P \in \mathbb{R}^N$ *and let*

- v *be a* $\mathbb{T}^N$*–periodic solution of*

$$\frac{1}{2}|P + \nabla u(x)|^2 + V(x) = \overline{H}(P)$$

- *the (energy) condition* $\quad \overline{H}(P) > \max_{\mathbb{T}^n} V \quad$ *be satisfied*
- $x_0 \in \Sigma(u)$

then there exists $\xi : [0, \tau) \to \Sigma(u)$ *Lipschitz such that* $\xi(0) = x_0$ *and* $\dot{\xi}^+(0) \neq 0$

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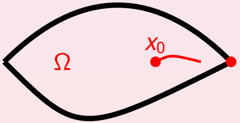

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