Optimal control, Hamilton-Jacobi equations and singularities in euclidean and riemaniann spaces

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Outline

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1. Generalized gradient flow of the distance function
   - Generalized gradient flow
   - Invariance of the singular set
   - Homotopy results
   - Riemannian manifolds
   - Semiconcave functions on riemannian manifolds
   - Propagation of singularities

The distance function

Let $\Omega \subset \mathbb{R}^n$ open and bounded (no regularity is assumed). For $x \in \Omega$, we set

$$d_{\partial\Omega}(x) = d(x) = \min_{y \in \partial\Omega} |y - x|$$

distance function (from the boundary of Ω).

We set

$$\mathrm{proj}(x) = \{y \in \partial\Omega \ : \ d(x) = |y - x|\}$$

projections of x onto $\partial\Omega$.

Properties of the distance function

- d is Lipschitz continuous with constant 1.
- d^2 is semiconcave with constant 2.
- d is differentiable at $x \in \Omega$ if and only if $\mathrm{proj}(x)$ is a singleton and in this case

$$Dd(x) = \frac{x - y}{|x - y|}$$

  where y is the unique element of $\mathrm{proj}(x)$.
- If $\mathrm{proj}(x)$ is not a singleton then we have

$$D^+d(x) = \mathrm{co}\left\{\frac{x - y}{|x - y|} \ : \ y \in \mathrm{proj}(x)\right\}.$$

Singular set of the distance function

d is not everywhere differentiable in Ω (Ω bounded).

Singular set of the distance function:

$$\begin{aligned}\Sigma &:= \{x \in \Omega \;:\; d \text{ is not differentiable at } x\,\} \\ &= \{x \in \Omega \;:\; \mathrm{proj}(x) \text{ is not a singleton}\,\} \\ &= \{x \in \Omega \;:\; \min_{p \in D^+ d(x)} |p| < 1\,\}.\end{aligned}$$

$\overline{\Sigma}$ is called cut-locus

Σ is $\mathcal{H}^{n-1}$–rectifiable.

If Ω is smooth, then $\overline{\Sigma}$ is also $\mathcal{H}^{n-1}$–rectifiable (Mantegazza-Mennucci, 2003)

Smooth gradient flow

Given $x_0 \in \Omega \setminus \overline{\Sigma}$, let $\gamma(\cdot)$ be the solution of

$$\begin{cases} \dot{\gamma}(t) = Dd(\gamma(t)) \\ \gamma(0) = x_0 \end{cases}$$

gradient flow of the distance function.

$\gamma(\cdot)$ is a segment with direction $Dd(x_0)$; that is, with direction $x_0 - \mathrm{proj}(x_0)$, going from x_0 to $\overline{\Sigma}$.

It is interesting to define a generalized gradient flow when the trajectory reaches $\overline{\Sigma}$

Generalized gradient flow

Differential equation $\longrightarrow$ differential inclusion
$\dot\gamma(t) = Dd(\gamma(t)) \longrightarrow \dot\gamma(t) \in D^+d(\gamma(t))$ a.e.

Some related works:

Dafermos (1977) generalized characteristics for hyperbolic conservation laws: study of the regularity and of the long-time behaviour of solutions

Grove-Shiohama (1977), Shafarudtinov (1977), Gromov (1981), Perelman, Petrunin ('90) gradient flow of the distance function on riemannian manifolds and Alexandrov spaces: topological applications

Cannarsa-Soner (1987), Albano-Cannarsa (2002), Cannarsa-Yu (2007) propagation of singularities for semiconcave functions and for solutions to Hamilton-Jacobi equations, see Lecture 3

Ambrosio-Gigli-Savarè (2005) gradient flow of the distance function for general classes of metric spaces

Generalized gradient flow (II)

Theorem (Albano,Cannarsa,Yu)

1. *for any* $x_0 \in \Omega$ *there is a unique Lipschitz curve* $\gamma : [0, \infty) \to \Omega$ *s. t.*

$$\begin{cases} \dot\gamma(t) \in D^+ d(\gamma(t)) & t \in [0,\infty) \ a.e. \\ \gamma(0) = x_0 \end{cases}$$

2. $|\dot\gamma(t)| = \min \{ \, |p| \ : \ p \in D^+ d(\gamma(t)) \, \}$
3. $\gamma(t) \in \Sigma \iff |\dot\gamma(t)| < 1$
4. $\dfrac{d}{dt} d(\gamma(t)) = |\dot\gamma(t)|^2$ *a.e.*
5. $x_0 \in \Sigma \implies \exists \tau = \tau(x_0) > 0 \ : \ \gamma(t) \in \Sigma \,, \quad \forall t \in [0, \tau)$

   (propagation of singularities)

Outline

Invariance of the singular set

The previous theorem does not exclude that $\gamma(t) \in \Sigma$ for $t \in [0, \tau)$ but $\gamma(\tau) \notin \Sigma$ and $\mathrm{diam}(D^+ d(\gamma(t))) \to 0$ as $t \uparrow \tau$.

We show now that this cannot occur.

Theorem (Albano – Cannarsa – Khai T. Nguyen – S., (2012))

If $\gamma(t_0) \in \Sigma$ for some $t_0 \geq 0$, then $\gamma(t) \in \Sigma$ for every $t \in [t_0, +\infty)$.

Sketch of the proof

for simplicity: $\delta(t) := d(\gamma(t))$, $\quad p(t) := \dot\gamma(t) \quad \Rightarrow \quad p(t) \in D^+ d(\gamma(t))$.

- d^2 semiconcave $\Rightarrow$ monotonicity of $D^+ d^2$,

  $\langle 2d(x_2)p_2 - 2d(x_1)p_1, x_2 - x_1\rangle \leq C|x_2 - x_1|^2, \qquad \forall\, p_i \in D^+ d(x_i)$
- given $s < t$, we apply this with

  $$x_2 = \gamma(t), \quad x_1 = \gamma(s), \quad p_2 = p(t), \quad p_1 = p(s)$$

  to obtain

  $$2\,\langle\, \delta(t)p(t) - \delta(s)p(s)\,,\, \gamma(t) - \gamma(s)\,\rangle \leq C|\gamma(t) - \gamma(s)|^2,$$

  that can be rewritten as

  $$\begin{aligned} & 2\delta(t)\langle p(t) - p(s), \gamma(t) - \gamma(s)\rangle \\ \leq\ & C|\gamma(t) - \gamma(s)|^2 - 2(\delta(t) - \delta(s))\langle p(s), \gamma(t) - \gamma(s)\rangle \end{aligned}$$

Sketch of the proof (II)

$$\begin{aligned} & 2\delta(t)\langle p(t)-p(s),\gamma(t)-\gamma(s)\rangle \\ \leq \; & C|\gamma(t)-\gamma(s)|^2 - 2(\delta(t)-\delta(s))\langle p(s),\gamma(t)-\gamma(s)\rangle \end{aligned}$$

Now we have, for a.e. s,

$$\lim_{t\to s}\frac{\gamma(t)-\gamma(s)}{t-s}=\dot\gamma(s)=p(s), \qquad \lim_{t\to s}\frac{\delta(t)-\delta(s)}{t-s}=|p(s)|^2.$$

Therefore, dividing by $(t-s)^2$ and taking $\lim_{t\to s}$

$$2\delta(s)\langle \dot p(s),p(s)\rangle \leq C|p(s)|^2 - 2|p(s)|^2\langle p(s),p(s)\rangle$$

that can be rewritten as

$$\frac{d}{ds}|p(s)|^2 \leq \frac{2}{\delta(s)}\,|p(s)|^2\left(\frac{C}{2}-|p(s)|^2\right).$$

Sketch of the proof (III)

- Now since $C = 2$ this becomes

$$\frac{d}{ds}|p(s)|^2 \leq \frac{2}{\delta(s)}\,|p(s)|^2\left(1 - |p(s)|^2\right)$$

- by comparison arguments for ODE we obtain

$$|p(s)| < 1 \quad \Longrightarrow \quad |p(t)| < 1\,, \forall t \geq s$$

- since $\gamma(t) \in \Sigma(d)$ iff $|p(t)| < 1$ we obtain

$$\gamma(s) \in \Sigma \quad \Longrightarrow \quad \gamma(t) \in \Sigma\,, \forall t \geq s$$

- actually $p(\cdot)$ is only in L^∞, but the argument can be made rigorous

Outline

Homotopic equivalence

X, Y topological spaces. There is a definition of *homotopic equivalence* between X and Y which implies that X, Y have the same homotopy groups (e.g. X is simply connected iff Y is).

When $Y \subset X$ a *sufficient condition* for homotopic equivalence is the existence of a continuous map $\mathbb{H} : X \times [0,1] \to X$ s.t.

1. $\mathbb{H}(x,0) = x \qquad \forall x \in X$
2. $\mathbb{H}(x,1) \in Y \qquad \forall x \in X$
3. $\mathbb{H}(x,t) \in Y \qquad \forall (x,t) \in Y \times [0,1]$.

If in addition $H(\cdot,t)$ is the identity on Y for every t, then Y is called a *strong deformation retract* of X

Homotopy equivalence of Ω and $\Sigma(d)$

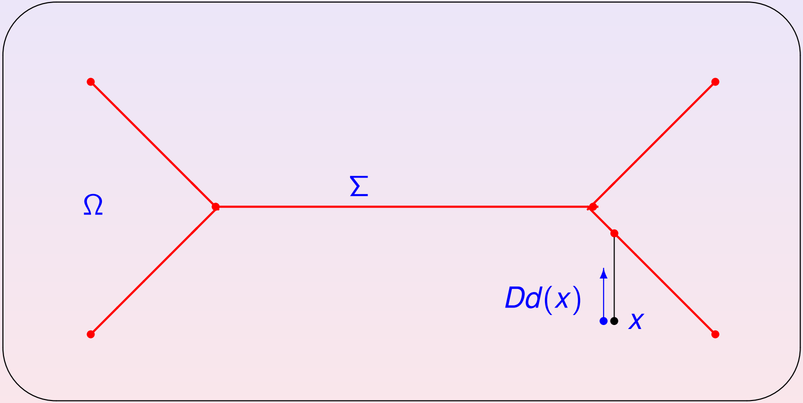

Figure: is Σ a strong deformation retract of Ω?

Homotopy equivalence of Ω and $\Sigma(d)$

Wolter (1993): Σ is a *deformation retract* of Ω if

- $\Omega \subset \mathbb{R}^n$ and $\partial\Omega \in \mathcal{C}^2$
- $\Omega \subset \mathbb{R}^2$ and $\partial\Omega$ is piecewise $\mathcal{C}^2$

Use smooth gradient flow of the distance. It works only under the regularity assumptions above.

Nevertheless, we have in general

Theorem

Ω is homotopically equivalent to Σ

- Lieutier (2004) (elementary, but technically involved)
- easy corollary of the invariance of Σ under the generalized gradient flow

proof

- it suffices to find a continuous map $\mathbb{H} : \Omega \times [0,1] \to \Omega$ s.t.
  1. $\mathbb{H}(x,0) = x \qquad \forall x \in \Omega$
  2. $\mathbb{H}(x,1) \in \Sigma \qquad \forall x \in \Omega$
  3. $\mathbb{H}(x,t) \in \Sigma \qquad \forall (x,t) \in \Sigma \times [0,1]$
- $\gamma(t,x)$ generalized gradient flow

$$\begin{cases} \dot\gamma(t) \in D^+ d(\gamma(t)) & t \in [0,\infty) \ a.e. \\ \gamma(0) = x \end{cases}$$

  locally Lipschitz on $[0,\infty) \times \Omega$
- easy to see: $\exists T > 0 \ : \ \gamma(T,x) \in \Sigma \quad \forall x \in \Omega$
- we choose $\mathbb{H}(x,t) = \gamma(tT,x)$ $\square$

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Riemannian geometry

We give now an *informal* review of the basic notions of riemannian geometry.

M. Do Carmo, Riemannian geometry. Birkhäuser, (1992).

S. Gallot, D. Hulin, J. Lafontaine. Riemannian Geometry. Springer, 2004.

P. Petersen, Riemannian geometry. Springer (2006)

Immersed manifolds

n-dimensional immersed manifolds: image of subsets of $\mathbb{R}^n$ by smooth maps to $\mathbb{R}^N$, with $N > n$.

Definition

An n-dimensional immersed manifold in $\mathbb{R}^N$ is a subset $M \subset \mathbb{R}^N$ such that, for any $p \in M$, there exist opens sets $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^N$ with $p \in B$, and a map $F : A \to \mathbb{R}^N$ such that

$$F(A) = B \cap M, \qquad rk\, DF(x) \equiv n$$

F associates to any $q \in M \cap B$ a set of coordinates $(x_1, \ldots, x_n)$
F is called a *local coordinate chart*.

We consider for simplicity smooth manifolds (e.g. $F \in C^\infty$)

immersed manifolds

For any $p \in M$, $p = F(x)$, the subspace generated by

$$e_1 := \frac{\partial F}{\partial x_1}(x) \, , \, \ldots \, , \, e_n = \frac{\partial F}{\partial x_n}(x)$$

is called *tangent space* to M at p and is denoted by T_pM.

The dual of T_pM is called *cotangent space* and is denoted by T_p^*M.

abstract manifolds

Abstract manifold: a topological space M, such that every point $p \in M$ has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$.

The charts induce a differentiable structure. There is an abstract definition of tangent space T_pM. Given a coordinate chart, there is a basis $\{e_1, \dots, e_n\} \in T_pM$ associated with the coordinates.

contrast with euclidean space

There is no canonical way of comparing tangent vectors at different points.

Expressions like:

$$q - p, \quad p, q \in M, \qquad \text{or: } p + v, \quad p \in M, v \in T_pM$$

are not well defined on a manifold.

The differential of a *function* $f : M \to \mathbb{R}$ and the tangent to a *curve* $\gamma : [a, b] \to M$ are well defined, but there is no canonical way of differentiating *vector fields*.

Riemannian manifolds

Manifolds with a *metric structure*.

A manifold M is called *riemannian* if there is a scalar product defined on T_pM for every $p \in M$.

An immersed manifold has a natural riemannian structure by considering the restriction on each T_pM of the scalar product of the ambient space $\mathbb{R}^N$.

metric and coordinates

Given $x_1, \ldots, x_n$ local coordinates, $e_1(p), \ldots, e_n(p)$ basis of T_pM associated with the coordinates set $g_{ij}(p) = \langle e_i(p), e_j(p) \rangle$.

It is possible to choose coordinates such that $\{e_i\}$ is orthonormal *at a given point*, but in general not *at every point*.

Cotangent space T_p^*M isomorphic to T_pM via the scalar product. Let $e^1(p), \ldots, e^n(p)$ basis of T_p^*M dual to $e_1, \ldots, e_n$ i.e. such that $\langle e^i, e_j \rangle = \delta_j^i$.
Then $\langle e^i(p), e^j(p) \rangle = g^{ij}(p)$, where g^{ij} is the inverse matrix to g_{ij}.

metric and distance

Curve $\gamma : [a, b] \to M$. Length of γ:

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt.$$

Distance between two points $p, q \in M$:

$$\text{dist}(p, q) = \min\{L(\gamma) \ : \ \gamma \text{ curve joining } p \text{ and } q \ \}$$

covariant derivative

On a riemannian manifold, there is a canonical definition of the derivative of a vector field, called *covariant derivative*.

Case of an immersed manifold. Let γ be a curve on M and let X be a vector field defined on the points of γ.

The covariant derivative of X along γ is the tangential component of $\frac{d}{dt}X(\gamma(t))$ computed in the ambient space $\mathbb{R}^N$.
It is denoted by $\nabla_{\dot\gamma}X$.

parallel transport

$\gamma : [a, b] \to M$ curve, X vector field along γ.

We say that X is *parallel* along γ if $\nabla_{\dot{\gamma}} X \equiv 0$.

Given a vector $X_a \in T_{\gamma(a)}M$, there is a unique vector field $X(t) \in T_{\gamma(t)}M$ parallel along γ such that $X(a) = X_a$ ($\exists!$ for ODE). $X(t)$ is called the *parallel transport* of X_a along γ.

Parallel transport is an *isometry*. It allows to compare tangent vector at different points. However, such a correspondence depends on the choice of the curve joining the points.

geodesics

A curve $\gamma : [a, b] \to M$ is called a *geodesic* if its speed $\dot\gamma$ is parallel along γ itself ($\dot\gamma$ "constant").

Equivalent property: stationary point of the energy functional

$$\int_a^b |\dot\gamma(t)|^2 \, dt$$

with fixed endpoints.

Any distance-minimizing curve is a geodesic. Conversely, a geodesic is distance-minimizing on small intervals, but in general not on arbitrarily large intervals.

Example: on a sphere, the geodesics are the maximal circles.

exponential map

Given $p \in M$ and $v \in T_pM$, there exists a unique geodesic $\gamma(t)$ satisfying $\gamma(0) = x$ and $\dot\gamma(0) = v$ ($\exists!$ for ODE).

We set

$$\exp_p(v) = \gamma(1)$$

The map $\exp_p : T_pM \to M$ is called *exponential map*.
It is a diffeomorphism in a neighbourhood of 0.

For simplicity, from now on we make the following abuse of notation. Given $x, y \in M$ close enough, let $\gamma : [0, 1] \to M$ the minimizing geodesic from x to y.

We write $y - x$ to mean the vector $v \in T_x M$ such that $\dot{\gamma}(0) = v$ (so that $y = \exp_x v$).

If $v \in T_x M$, $w \in T_y M$, we write $w - v$ to mean the vector $\Pi w - v \in T_p M$, where $\Pi : T_y M \to T_x M$ is the isomorphism induced by the parallel transport along γ.

a taste of curvature

Positive curvature: Sphere S^2 immersed in $\mathbb{R}^3$.

- Two geodesics always meet (no parallels).
- Sum of angles of a triangle greater than π.

Negative curvature: Hyperbolic plane H^2

Upper half-plane model: $H^2 = \mathbb{R} \times \mathbb{R}_+$ with the metric $g_{ij} = \frac{1}{y^2}\delta_{ij}$

- Infinitely many parallels to a geodesic through a given point
- Sum of angles of a triangle less than π.

Both have n-dimensional generalizations.

Outline

semiconcave functions

Reference: C. Villani, Optimal transport, old and new. (2009).

Definition

A function $u : A \to \mathbb{R}$*, with* $A \subset M$*, is called* semiconcave *in* A *with constant* C *if, for every geodesic* $\gamma : [0,1] \to U$ *and* $t \in [0,1]$*, we have*

$$u(\gamma(0)) + u(\gamma(1)) - 2u\,(\gamma\,(1/2)) \leq \frac{C}{4}\, dist(\gamma(0), \gamma(1))^2\,.$$

Superdifferential

Definition

Given $u : M \to \mathbb{R}$, *we say that* $p \in T_x^* M$ *belongs to* $d^+ u(x)$, *the* superdifferential *of* u *at* x, *if*

$$u(\exp_x(v)) - u(x) \le \langle p, v \rangle + o(|v|), \qquad v \in T_x M, \quad v \to 0.$$

Proposition

If $u : U \to \mathbb{R}$ *is semiconcave, then* $d^+ u(x) \neq \emptyset$ *for all* $x \in U$. *In addition,* $p \in d^+ u(x)$ *if and only if*

$$u(\exp_x(v)) - u(x) \le \langle p, v \rangle + C \frac{|v|^2}{2}$$

for all $v \in T_x M$ *such that* $\exp_x(v) \in U$.

Superdifferential (II)

By our abuse of notation, this can be rewritten as

$$u(y) - u(x) \leq \langle p, y - x \rangle + C \, \frac{\mathrm{dist}(x, y)^2}{2}.$$

If $q \in d^+u(y)$, summing up the two inequalities we obtain

$$\langle q - p, y - x \rangle \leq C \, \mathrm{dist}(x, y)^2,$$

recovering the monotonicity of d^+u as in the euclidean case.

sectional curvature

Given $x \in M$ and $v, w \in T_pM$ linearly independent, one can define the *sectional curvature* $K_p(v, w)$.

Manifolds with constant sectional curvature (space forms):

- $K > 0$ sphere
- $K = 0$ euclidean space
- $K < 0$ hyperbolic space

Roughly speaking: given any 2-dimensional submanifold of M through p tangent to v, w, it “behaves locally” like the 2-dimensional space form with curvature $K_p(v, w)$.

Semiconcavity of the distance function

Theorem

Let M be a manifold with constant curvature K. Consider the distance function $d_S(\cdot)$ from a closed $S \subset \mathbb{R}^n$. Given $\Omega \subset M$, let

$$M_- = \inf_\Omega d_S, \qquad M_+ = \sup_\Omega d_S.$$

Then

- *If $K = \alpha^2 > 0$ then the function $-\cos(\alpha d_S(\cdot))$ is semiconcave on Ω with constant $\alpha^2 \cos(\alpha M_-)$.*
- *If $K = 0$ then $d_S^2(\cdot)$ is semiconcave on Ω with constant 2.*
- *If $K = -\alpha^2 < 0$ and Ω is bounded, then $\cosh(\alpha d_S(\cdot))$ is semiconcave on Ω with constant $\alpha^2 \cosh(\alpha M_+)$.*

Semiconcavity of the distance function (II)

Equivalently stated: define

$$\phi_K(r) = \begin{cases} -\cos(\alpha r) & \text{if } K = \alpha^2 > 0 \\ r^2 & \text{if } K = 0 \\ \cosh(\alpha r) & \text{if } K = -\alpha^2 < 0 \end{cases}$$

Then we have

Theorem

Let M be a space with constant curvature K and let $S \subset M$ be a closed set. Then $\phi_K(d_S)$ is semiconcave on any bounded open set $A \subset M$, with constant

$$C := \sup_{x \in A} \phi_K''(d_S(x))$$

Comparison results

Classical *comparison theorems* in riemannian geometry (Rauch, Toponogov,...), imply that the same statement is true on a manifold M where the sectional curvatures are *greater than* K everywhere.

Therefore the distance function has a *stronger semiconcavity* if the manifold is *more positively curved*.

The squared distance function d^2 is semiconcave with constant $C = 2$ only on manifolds with nonnegative curvature.

Comparison results (III)

On general manifolds, we have the following.

Corollary

Let M be any riemannian manifold, $\Omega \subset M$ open bounded. Then, for a suitably large $\alpha > 0$, if we set $\phi_\alpha(r) = \cosh(\alpha r)$, we have that $\phi_\alpha(d_{\partial\Omega}(\cdot))$ is semiconcave on any $A \subset \Omega$ with semiconcavity constant $C = \sup_{x \in A} \phi''_\alpha(d_{\partial\Omega}(x))$

Just set $\alpha = \sqrt{-K}$, with $K < 0$ a lower bound for the sectional curvatures of M in Ω.

Outline

local propagation result

Theorem

Let us write $u(x) = d_{\partial\Omega}(x)$. *For every* $x_0 \in \Omega$ *there exists a unique lipschitz continuous arc* $\gamma : [0, +\infty[\to \Omega$ *such that*

$$\dot{\gamma}(t) \in d^+ u(\gamma(t)) \quad t \in [0, +\infty[\text{ a.e.} \qquad \gamma(0) = x_0.$$

$\dot{\gamma}(t)$ *is the element of* $d^+ u(\gamma(t))$ *with minimal norm; hence* $\gamma(t) \in \Sigma(u)$ *if and only if* $|\dot{\gamma}(t)| < 1$.

The arc γ *satisfies* $\frac{d}{dt} u(\gamma(t)) = |\dot{\gamma}(t)|^2$ *a.e.*

Moreover, for any $t_0 \geq 0$ *such that* $\gamma(t_0) \in \Sigma(u)$ *there exists* $\sigma = \sigma(t_0) > 0$ *such that* $\gamma(t) \in \Sigma(u)$ *for all* $t \in [t_0, t_0 + \sigma[$.

It follows from the results in euclidean spaces. In local coordinates, u solves

$$\langle G^{-1}(x)Du(x), Du(x)\rangle = 1,$$

with $G(x)$ the matrix of the scalar product.

Then the results by Albano, Cannarsa, Yu on solutions to Hamilton-Jacobi equations can be applied.

global propagation result

Theorem (Albano – Cannarsa – Khai T. Nguyen – S., (2012))

Let M be a smooth complete riemannian manifold, let $\Omega \subset M$ any bounded open set and let $\gamma(\cdot)$ be a trajectory of the generalized gradient flow of $d_{\partial\Omega}$ as in the previous theorem.

If $\gamma(t_0) \in \Sigma(d_{\partial\Omega})$ for some $t_0 \geq 0$ then $\gamma(t) \in \Sigma(d_{\partial\Omega})$ for all $t \in [t_0, +\infty[$.

Corollary

Any bounded open set $\Omega \subset M$ is homotopically equivalent to the singular set Σ of its distance function.

proof of the propagation

Sketch of the proof

Set

$$\delta(t) = d_{\partial\Omega}(\gamma(t)), \qquad p(t) = \dot{\gamma}(t).$$

Let $\phi(r) = \cosh(\alpha r)$ with $\alpha > 0$ large enough.

Fix any s such that $\gamma(s) \in \Sigma(u)$. Fix $h > 0$ small and let $A = B_h(\gamma(s))$. Then $\gamma(t) \in A$ for $t \in [s-h, s+h]$. In addition, $d_{\partial\Omega}(x) \leq \delta(s) + h$ for $x \in A$.

Therefore $\phi(d_{\partial\Omega}(\cdot))$ is semiconcave in A with constant given by

$$C = \phi''(\delta(s) + h)$$

The monotonicity of the superdifferential of $\phi(d_{\partial\Omega}(\cdot))$ implies:

$$\begin{aligned} & \langle\ \phi'(\delta(s+h))p(s+h) - \phi'(\delta(s))\,p(s)\ ,\ \gamma(s+h)-\gamma(s)\ \rangle \\ \le\ & \phi''(\delta(s)+h)\ \mathrm{dist}(\gamma(s),\gamma(s+h))^2. \end{aligned}$$

(with the abuse of notation as before). We rewrite it as

$$\begin{aligned} & \phi'(\delta(s+h))\ \langle\, p(s+h)-p(s)\,,\,\gamma(s+h)-\gamma(s)\,\rangle \\ \le\ & \phi''(\delta(s)+h)\ \langle\,\gamma(s+h)-\gamma(s)\,,\,\gamma(s+h)-\gamma(s)\,\rangle \\ & -\left[\phi'(\delta(s+h))-\phi'(\delta(s))\right]\langle p(s),\gamma(s+h)-\gamma(s)\rangle. \end{aligned}$$

Now divide by h^2 and let $h\to 0$.

We have

$$\frac{\gamma(s+h)-\gamma(s)}{h} \longrightarrow \dot\gamma(s) = p(s),$$

$$\frac{\phi'(\delta(s+h))-\phi'(\delta(s))}{h} \longrightarrow \phi''(\delta(s))\frac{d}{ds}\delta(s) = \phi''(\delta(s))|p(s)|^2.$$

Therefore we find at the RHS

$$\begin{aligned}\lim_{h\to 0} \quad & \phi''(\delta(s)+h)\left\langle \frac{\gamma(s+h)-\gamma(s)}{h}, \frac{\gamma(s+h)-\gamma(s)}{h}\right\rangle \\ & - \left[\frac{\phi'(\delta(s+h))-\phi'(\delta(s))}{h}\right]\left\langle p(s), \frac{\gamma(s+h)-\gamma(s)}{h}\right\rangle \\ = \quad & \phi''(\delta(s))\left(|p(s)|^2 - |p(s)|^4\right).\end{aligned}$$

We obtain the *same factor* $\phi''(\delta(s))$.

The left hand side gives

$$\begin{aligned}\lim_{h\to 0} \quad & \phi'(\delta(s+h)) \left\langle \frac{p(s+h)-p(s)}{h}, \frac{\gamma(s+h)-\gamma(s)}{h} \right\rangle \\ = \quad & \phi'(\delta(s)) \langle \dot p(s), p(s)\rangle = \frac{\phi'(\delta(s))}{2} \frac{d}{ds}|p(s)|^2.\end{aligned}$$

In conclusion, we obtain

$$\begin{aligned}\frac{d}{ds}|p(s)|^2 \quad & \leq \quad 2\frac{\phi''(\delta(s))}{\phi'(\delta(s))} \left(|p(s)|^2 - |p(s)|^4\right) \\ & = \quad \frac{2\alpha}{\tanh(\alpha\,\delta(s))}|p(s)|^2 \left(1-|p(s)|^2\right).\end{aligned}$$

Thus $|p(s)|<1 \implies |p(t)|<1$, for all $t \geq s$. $\Box$

An application

Consider the control system in $\mathbb{R}^n$ governed by the equation

$$y'(t) = F(y(t))\,\alpha(t)$$

where $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a smooth matrix-valued function such that $\det F(x) \neq 0$ for all x, and the control $\alpha(t) \in B_1 \subset \mathbb{R}^n$.

For any bounded open set $\Omega \subset \mathbb{R}^n$, let $T : \Omega \to \mathbb{R}$ be the *minimum time function* associated with the system, with $\mathbb{R}^n \setminus \Omega$ as a target.

Then $T(x)$ can be regarded as the distance function from $\partial\Omega$ with respect to the riemannian metric on $\mathbb{R}^n$ induced by $F(x)$.

$\Longrightarrow$ the singular set $\Sigma(T)$ is invariant under the generalized characteristic flow

$\Longrightarrow$ the singular set $\Sigma(T)$ is homotopically equivalent to Ω.

Thank you for your attention!