Optimal Control and Mean Field Games (Part 1)

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Mean field game theory analyzes

- optimal control problems
- with (infinitely) many identical controlers

In other words it is a mathematical modeling approach to continuous-time systems which involve a great number of “agents".

Ideas introduced by

- Lasry-Lions ’06,
- Huang-Caines-Malhame, ’06

Motivations :

- Problems arising in economy
  - financial markets (Price formation and dynamic equilibria, Formation of volatility) (Lasry, Lions, 2006)
  - general economic equilibrium with rational expectations (Guéant, Lasry, and Lions, 2007)
- Dynamics of population models
  - crowd motion : mexican wave "la ola", ... (Guéant, Lasry, Lions - Lachapelle, ...)
  - academic behavior (Besancenot, Courtault, El Dika...)
- Engineereing literature : Large Population Stochastic Wireless Power Control Problem
  (Huang, Caines, Malhamé, 2003, Mériaux, Lasaulce...)

Different approaches

- limit of $N-$player (stochastic) differential games as $N \to +\infty$,
  $\longrightarrow$ analogy with the Mean Field theories in statistical physics (kinetic theory of gases, Boltzmann and Vlasov equations) and quantum mechanics and quantum Chemistry (Hartree-Fock models...)
- direct definition of (stochastic) differential games with infinitely many identical players (applications to $N-$player games),
  $\longrightarrow$ approach from game theory
- potential games : games arising as necessary conditions for optimal control problems of PDE equations.

  $\longrightarrow$ related to optimal transportation problems

The Mean Field Game system

The 3 approaches yield the same MFG system with unknown $(u,m):[0,T]\times\mathbb{R}^d\to\mathbb{R}^2$:

$$(MFG)\quad\left\{\begin{array}{ll} (i) & -\partial_t u-\sigma^2\Delta u+H(x,Du,m)=0 \\ & \qquad\qquad\qquad \text{in } [0,T]\times\mathbb{R}^d \\ (ii) & \partial_t m-\sigma^2\Delta m-\text{div}(m\, D_pH(x,Du,m))=0 \\ & \qquad\qquad\qquad \text{in } [0,T]\times\mathbb{R}^d \\ (iii) & m(0)=m_0,\ u(x,T)=G(x,m(T)) \qquad \text{in } \mathbb{R}^d \end{array}\right.$$

where

- $\sigma\in\mathbb{R}$,
- $H=H(x,p,m)$ is a convex Hamiltonian (in p) depending on the density m,
- $G=G(x,m(T))$ is a function depending on the position x and the density $m(T)$ at time T.
- m_0 is a probability density on $\mathbb{R}^d$.

System introduced in

- Lasry-Lions ’06,
- Huang-Caines-Malhame, ’06

to model large population differential games.

Aim of the lectures

- Describe the MFG model and its interpretations
  (Part 1)
- Existence of the MFG system by fixed point arguments
  (First order, non local MFG - Part 2)
- The MFG system as optimality condition for optimal control problems of PDEs
  (First order, local MFG - Part 3)

Part 1

Interpretations of the MFG system

Outline

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Example : Several swimmers on a beach.

They want to be

- close from the sea,
- not too far from their car
- far from each other

What is the optimal repartition of the swimmers ?

Key assumption : swimmers are identical (same tastes).

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Formalization

Let N be a (large) number of players.

We consider a one-shot game with N symmetric players :

- the players have the same set of actions Q,
- the cost of player i (where $i \in \{1, \dots, N\}$) is given by $F_i^N = F^N(x_i, (x_j)_{j \neq i})$ where $F^N : Q^N \to \mathbb{R}$ is symmetric in the last variables.

Back to the “beach" example :

- A strategy for player i is a position x_i on the beach. The A strategy set $Q = \mathrm{Beach}$.

- The cost of player i, F_i^N, can be given, for instance, by

$$F_i^N = F^N(x_i, (x_j)_{j \neq i}) = \alpha\, \mathrm{dist}(x_i, \mathrm{Sea}) + \beta\, \mathrm{dist}(x_i, \mathrm{Parking}) - \gamma \frac{1}{N-1} \sum_{j \neq i} |x_j - x_i|$$

  where $\alpha, \beta, \gamma > 0$ are the same for all swimmers.

A solution of the game is a Nash equilibrium :

Definition

A Nash equilibrium for the game $(F_1^N, \dots, F_N^N)$ is an element $(\bar{x}_1^N, \dots, \bar{x}_N^N) \in Q^N$ such that

$$F_i^N(y_i, (\bar{x}_j^N)_{j\neq i}) \geq F_i^N(\bar{x}_i^N, (\bar{x}_j^N)_{j\neq i}) \qquad \forall y_i \in Q\ .$$

In other worlds, $\bar{x}_i^N$ minimizes the map $y_i \to F_i^N(y_i, (\bar{x}_j^N)_{j\neq i})$.

Problem

Understand the behavior of Nash equilibria $(\bar{x}_1^N, \ldots, \bar{x}_N^N) \in Q^N$ as $N \to +\infty$.

Symmetric functions of many variables

Let Q be a compact metric space. We denote by $\mathcal{P}(Q)$ be the set of Borel probability measures on Q, endowed with the Kantorowich-Rubinstein distance

$$\mathbf{d}_1(\mu,\nu) = \sup\left\{ \int_Q f\, d(\mu-\nu) \text{ where } f : Q \to \mathbb{R} \text{ is } 1-\text{Lipschitz continuous} \right\} .$$

Facts :

- $\mathbf{d}_1$ metricizes the weak-* convergence on $\mathcal{P}(Q)$.
- $\mathcal{P}(Q)$ is compact for $\mathbf{d}_1$.

Let, for any $N \in \mathbb{N}$, $w_N : Q^N \to \mathbb{R}$ be a symmetric function such that

1. (Uniform bound) $\exists C_0 > 0$ with

$$\|w_N\|_{L^\infty(Q)} \leq C_0 \qquad \forall N \in \mathbb{N}\,,$$

2. (Uniform Lipschitz continuity) $\exists C_1 > 0$ such that

$$|w_N(X) - w_N(Y)| \leq C_1 \mathbf{d}_1(m_X^N, m_Y^N) \qquad \forall X, Y \in Q^N,\ \forall N \in \mathbb{N},$$

where $m_X^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and $m_Y^N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$ if $X = (x_1, \dots, x_N)$ and $Y = (y_1, \dots, y_N)$.

Lemma

There is a subsequence (w_{N_k}) of (w_N) and a Lipschitz continuous map $W : \mathcal{P}(Q) \to \mathbb{R}$ such that

$$\lim_{k \to +\infty} \sup_{X \in Q^{N_k}} |w_{N_k}(X) - W(m_X^{N_k})| = 0\,.$$

Remarks :

- The result holds if one replaces the “uniform Lipschitz continuity" assumption by a uniform modulus.

- This condition still holds if

$$\|\partial_{x_j} w_N\|_\infty \leq \frac{C_1}{N} \ .$$

Indeed, fix $X, Y \in Q^N$ and let σ permutation s.t.

$$\mathbf{d}_1(m_X^N, m_Y^N) = \frac{1}{N}\sum_{i=1}^N d(x_i, y_{\sigma(i)}) \ .$$

Then

$$\begin{array}{rcl} |w_N(X) - w_N(Y)| & = & |w_N(X) - w_N((y_{\sigma(i)}))| \\ & \leq & \displaystyle\sum_{i=1}^N \frac{C_1}{N} d(x_i, y_{\sigma(i)}) = C_1 \mathbf{d}_1(m_X^N, m_Y^N) \end{array}$$

Proof of the Lemma : Let $W^N : \mathcal{P}(Q) \to \mathbb{R}$ be defined by

$$W^N(m) = \inf_{X \in Q^N} \left\{ w_N(X) + C_1 \mathbf{d}_1(m_X^N, m) \right\} \qquad \forall m \in \mathcal{P}(Q) \ .$$

Then

- $|W^N(m)| \leq C_0 + C_1 \mathrm{diam}(Q) \qquad \forall m \in \mathcal{P}(Q)$,
- $W^N(m_X^N) = w_N(X)$ for any $X \in Q^N$,
- the W^N are C_1-Lipschitz continuous on $\mathcal{P}(Q)$.

By Ascoli, $\exists (W_{N_k})$ which converges uniformly to a limit W. Then

$$\limsup_{k \to +\infty} \sup_{X \in Q^{N_k}} |w_{N_k}(X) - W(m_X^{N_k})| \leq \lim_{k \to +\infty} \sup_{m \in \mathcal{P}(Q)} |W_{N_k}(m) - W(m)| = 0 \ .$$

Back to the game with many players

We consider a symmetric $N-$player one-shot game :

- the set of actions Q is the same for each players and Q is compact,
- the cost of player i (where $i \in \{1, \dots, N\}$) is $F_i^N = F^N(x_i, (x_j)_{j\neq i})$ where $F^N : Q^N \to \mathbb{R}$ is symmetric in the last variables.

In view of the previous discussion, we can assume that there is $F : Q \times \mathcal{P}(Q) \to \mathbb{R}$, continuous, such that, for any $i \in \{1, \dots, N\}$

$$F^N(x_i, (x_j)_{j\neq i}) = F\left(x_i, \frac{1}{N-1}\sum_{j\neq i} \delta_{x_j}\right) \qquad \forall (x_1, \dots, x_N) \in Q^N .$$

For instance, in the “beach" example : The map F is given by

$$F(x,m) = \alpha \,\mathrm{dist}(x,\mathrm{Sea}) + \beta \,\mathrm{dist}(x,\mathrm{Parking}) - \gamma \int_{\mathrm{Beach}} |y-x| dm(y)$$

Then

$$\begin{aligned} F^N(x_i,(x_j)_{j\neq i}) &:= F\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right) \\ &= \alpha \,\mathrm{dist}(x_i,\mathrm{Sea}) + \beta \,\mathrm{dist}(x_i,\mathrm{Parking}) - \gamma \frac{1}{N-1}\sum_{j\neq i}|x_j - x_i| \end{aligned}$$

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Recall that a Nash equilibrium for the game $(F_1^N, \dots, F_N^N)$ is an element $(\bar{x}_1^N, \dots, \bar{x}_N^N) \in Q^N$ such that

$$F_i^N(y_i, (\bar{x}_j^N)_{j \neq i}) \geq F_i^N(\bar{x}_i^N, (\bar{x}_j^N)_{j \neq i}) \qquad \forall y_i \in Q \,.$$

For $(\bar{x}_1^N, \dots, \bar{x}_N^N) \in Q^N$, we set

$$X^N = (\bar{x}_1^N, \dots, \bar{x}_N^N) \qquad \text{and} \qquad \bar{m}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\bar{x}_i^N} \,.$$

Proposition

Assume that, for any N, $X^N = (\bar{x}_1^N, \dots, \bar{x}_N^N)$ is a Nash equilibrium for the game $F_1^N, \dots, F_N^N$.

Then up to a subsequence, the sequence of measures $(\bar{m}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\bar{x}_i^N})$ converges to a measure $\bar{m} \in \mathcal{P}(Q)$ such that

$$(*) \qquad \int_Q F(y, \bar{m}) d\bar{m}(y) = \inf_{m \in \mathcal{P}(Q)} \int_Q F(y, \bar{m}) dm(y) \ .$$

Remark : $(*)$ is equivalent to saying that the support of $\bar{m}$ is contained in the set of minima of $F(y, \bar{m})$:

$$\bar{m}\left(\{x \ ; \ F(x, \bar{m}) \leq F(x', \bar{m}) \ \forall x' \in Q\}\right) = 1 \ .$$

Definition

Given a continuous map $F : Q \times \mathcal{P}(Q) \to \mathbb{R}$, we say that $\bar{m} \in \mathcal{P}(Q) \to \mathbb{R}$ is a Nash equilibrium of the continuous game if $\bar{m}$ satisfies

$$(*) \qquad \int_Q F(y, \bar{m}) d\bar{m}(y) = \inf_{m \in \mathcal{P}(Q)} \int_Q F(y, \bar{m}) dm(y) \ ,$$

or, equivalently,

$$\bar{m}\left(\{x \ ; \ F(x, \bar{m}) \leq F(y, \bar{m}) \qquad \forall y \in Q\}\right) = 1 \ .$$

Proof of the Proposition : We can assume that the sequence $(\bar{m}^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\bar{x}_i^N})$ converges to some $\bar{m}$. Let us check that $\bar{m}$ satisfies $(*)$.

- By definition, the measure $\delta_{\bar{x}_i^N}$ is a minimum of the problem

$$\inf_{m\in\mathcal{P}(Q)} \int_Q F(y, \frac{1}{N-1}\sum_{j\neq i} \delta_{\bar{x}_j^N}) dm(y) .$$

- Since $d\left(\frac{1}{N-1}\sum_{j\neq i} \delta_{\bar{x}_j^N}, \bar{m}^N\right) \leq \frac{2}{N}$ and since F is continuous, the measure $\delta_{\bar{x}_i^N}$ is also $\varepsilon-$optimal for the problem

$$\inf_{m\in\mathcal{P}(Q)} \int_Q F(y, \bar{m}^N) dm(y)$$

  for N is large enough.

- By linearity, $\bar{m}^N$ is also $\varepsilon-$optimal for the problem

$$\int_Q F(y,\bar{m}^N)d\bar{m}^N(y) \leq \inf_{m\in\mathcal{P}(Q)} \int_Q F(y,\bar{m}^N)dm(y) + \varepsilon \ .$$

for N is large enough.

- Letting $N \to +\infty$ gives the result.

Existence of a solution to $(*)$

Assume $F : Q \times \mathcal{P}(Q) \to \mathbb{R}$ is continuous.

Proposition

There is at least one Nash equilibrium of the continuous game, i.e., a measure $\bar{m}$ such that

$$(*) \qquad \int_Q F(y, \bar{m}) d\bar{m}(y) = \inf_{m \in \mathcal{P}(Q)} \int_Q F(y, \bar{m}) dm(y)$$

holds.

Remark : In general, no Nash equilibria for the $N-$person game. So the above Proposition is not a consequence of the passage to the limit.

Proof : Recall Ky Fan fixed point Theorem :
Let X be a convex compact set of a locally convex Hausdorff space and $G : X \to 2^X$ be a multiapplication with convex compact values and closed graph. Then G has a fixed point :

$$\exists \bar{x} \in X \text{ such that } \bar{x} \in G(\bar{x}) \ .$$

Let $X = \mathcal{P}(Q)$ and $G : X \to 2^X$ defined by

$$G(m) = \text{argmin}_{m' \in X} \int_Q F(x, m) dm'(x)$$

Then G is upper-semicontinuous multi-application with convex compact values.
So G has a fixed point :

$$\exists \bar{m} \in X \text{ such that } \int_Q F(y, \bar{m}) d\bar{m}(y) = \inf_{m \in \mathcal{P}(Q)} \int_Q F(y, \bar{m}) dm(y) \ .$$

From MFG to Nash equilibria in the $N-$player game

Problem : Given a Nash equilibrium of the continuous game, is it possible to derive a Nash equilibrium for the $N-$player game (for large N) ?

Yes, but in mixed strategies.

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Existence of $\varepsilon-$Nash equilibria

Mixed extension : Recall that $F_i^N(x_i,(x_j)_{j\neq 0}) = F\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right)$. The mixed extension of F_i^N is the map $\tilde F_i^N : (\mathcal{P}(Q))^N \to \mathbb{R}$ defined by

$$\tilde F_i^N(m_i,(m_j)_{j\neq i}) = \int \dots \int F_i^N(y_i,(y_j)_{j\neq i})dm_1(y_1)\dots dm_N(y_N)$$

for all $(m_1,\dots,m_N)\in(\mathcal{P}(Q))^N$.

Proposition

Let $\bar m$ be a Nash equilibrium of the continuous game associated with the continuous cost $F : Q\times\mathcal{P}(Q)\to\mathbb{R}$.

Then, $\forall \varepsilon>0$, $\exists \bar N$ such that, for $N\geq \bar N$, the random strategy $(\bar m,\dots,\bar m)\in(\mathcal{P}(Q))^N$ is an $\varepsilon-$Nash equilibrium of the $N-$player game :

$$\tilde F(m,(\bar m,\dots,\bar m)) \geq \tilde F(\bar m,(\bar m,\dots,\bar m)) - \varepsilon \qquad \forall m\in\mathcal{P}(Q) .$$

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$$\tilde{F}(m,(\bar{m},\dots,\bar{m})) \geq \tilde{F}(\bar{m},(\bar{m},\dots,\bar{m})) - \varepsilon \qquad \forall m \in \mathcal{P}(Q) .$$

Potential games

Case of local interactions : We now assume that $F : Q \times [0, +\infty) \to \mathbb{R}$.

In this context, a Nash equilibrium of the continuous game is an a.c. probability measure $\bar{m}$ satisfying

$$(*) \qquad \int_Q F(y, \bar{m}(y))\bar{m}(y)dy = \inf_{m \in \mathcal{P}(Q)} \int_Q F(y, \bar{m}(y))dm(y) \ ,$$

or, equivalently,

$$F(x, \bar{m}(x)) = \inf_{y \in Q} F(y, \bar{m}(y)) \qquad \text{for a.e. } x \ .$$

Proposition

Let $\Phi(x,m) = \int_0^m F(x,r)dr$. Assume that $\bar{m}$ is an a.c. probability measure on Q minimizing

$$m \to \int_Q \Phi(x, m(x))dx$$

Then $\bar{m}$ is a Nash equilibrium of the continuous game.

Indeed, the necessary conditions read

$$\int_Q \frac{\partial \Phi}{\partial m}(x,\bar{m})(m-\bar{m}) \geq 0 \qquad \forall m \in \mathcal{P}(Q) .$$

So

$$\int_Q F(x,\bar{m})dm \geq \int_Q F(x,\bar{m})d\bar{m} \qquad \forall m \in \mathcal{P}(Q) ,$$

which shows that $\bar{m}$ is an equilibrium.

An example : People in a concert want to be

- as close as possible from the stage ($= 0 \in \mathbb{R}^2$)
- not too packed

Modelized by the map $F(x,m) = \frac{|x|^2}{2} + \log(m(x))$.

The optimality condition $(*)$ reads :

$$F(\cdot, \bar{m}) = \min_{y \in \mathbb{R}^2} F(y, \bar{m}) \qquad \text{in } \{\bar{m} > 0\}\ .$$

Let $\bar{\lambda} = \min_{y \in \mathbb{R}^2} F(y, \bar{m})$. Then $\frac{|x|^2}{2} + \log(\bar{m}(x)) = \bar{\lambda} \text{ if } \bar{m}(x) > 0$, i.e.,

$$\bar{m}(x) = e^{\bar{\lambda}} e^{-\frac{|x|^2}{2}}\ .$$

So $\bar{m}$ is a Gaussian.

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Generalization (Mas-Colell, 1984)

Goal : deal with several populations problems.

Back to the swimmers' example : Several swimmers on a beach.

They want to be

- close from the sea,
- not too far from their car,
- far from each other,
- they can have different priorities.

What is the optimal repartition of the swimmers according to their priorities ?

Description of the game :

- The set of strategies is a compact metric space Q. Let $\mathcal{P}(Q)$ be the set of Borel probability measures on Q.
- A cost function is a continuous map $u : Q \times \mathcal{P}(Q) \to \mathbb{R}$. We set $C_Q := \mathcal{C}^0(Q \times \mathcal{P}(Q))$ the set of cost functions.
- A game is a Borel probability measure μ over the set C_Q of cost functions.

Remarks :

- Each cost function describes a type of population.
- A game represents the distribution of each type in the overall population.
- The previous model consisted in a single type : $\mu = \delta_F$ where $F \in C_Q$.

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Definition

Let μ be a game. A Nash equilibrium of the game is a measure σ over $C_Q \times Q$ such that, if σ_C and σ_Q are the marginals of σ, then

1. $\sigma_C = \mu$ and
2. $\sigma\left(\{(u,x)\ ;\ u(x;\sigma_Q) \geq u(x';\sigma_Q)\ \forall x' \in Q\}\right) = 1.$

Remarks :

- In principle, we would like a map $\Phi : C_Q \to Q$ saying what strategy $\phi(u)$ a player of type u should play. This map seldom exists.
- In the single type model (i.e., $\mu = \delta_F$ where $F \in C_Q$), a Nash equilibrium σ must satisfy :
  1. $\sigma = \delta_F \otimes \tau$ where $\tau \in \mathcal{P}(Q)$,
  2. $\tau\left(\{x\ ;\ F(x;\tau) \geq F(x';\tau)\ \forall x' \in Q\}\right) = 1.$

Theorem (Mas-Colell, 1984)

Given a game μ there exists a Nash equilibrium.

Proof : Again by the Ky Fan fixed point Theorem.

Conclusion

Nash equilibrium of the continuous games

- arise as limit of $N-$player games as $N \to +\infty$,
- can be intrinsically defined by an equilibrium condition
- can be derived as equilibrium position for potential games
- also allows to formalize the behavior of several populations

Mean field games

- generalize the above approaches **when the optimization problem is replace by an optimal control problem.**
- (however seldom deal with several population games)

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Some references on games with many players (nonatomic games)

- Cooperative games and Shapley value
  - Shapley (1961), Shapley and Shubik (1963), Aumann-Shapley (1974), Aumann (1975),
  - Hildenbrand and Mertens (1972), Hildenbrand (1974), Dubey (1975), Hart (1977), Neyman (1977), Mertens (1980), Dubey-Neyman (1984), Monderer (1986), Haimanko (2000), ...
- Noncooperative games
  - Schmeidler (1973), Novshek and Sonnenschein (1983) Mas-Colell (1983, 1984), Green (1984), Fudenberg-Levine (1986), Sandholm (2001), Kalai (2004) ...

Outline

The Mean Field Game system

We study solutions $(u, m) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^2$ to

$$(MFG) \quad \begin{cases} (i) & -\partial_t u - \sigma^2 \Delta u + H(x, Du, m) = 0 \\ & \qquad \text{in } [0, T] \times \mathbb{R}^d \\ (ii) & \partial_t m - \sigma^2 \Delta m - \mathrm{div}(m\, D_p H(x, Du, m)) = 0 \\ & \qquad \text{in } [0, T] \times \mathbb{R}^d \\ (iii) & m(0, x) = m_0(x),\ u(x, T) = G(x, m(T)) \qquad \text{in } \mathbb{R}^d \end{cases}$$

where

- $\sigma \in \mathbb{R}$,
- $H = H(x, p, m)$ is a convex Hamiltonian (in p) depending on the density m,
- $G = G(x, m(T))$ is a function depending on the position x and the density $m(T)$ at time T.
- m_0 is a probability density on $\mathbb{R}^d$.

We want to understand this system as a Nash equilibrium of a continuous game where the payoff is of optimal control type.

Heuristic interpretation of (i)

Given a family $(m_t)_{t\in[0,T]}$ of probability densities, an average agent controls the stochastic differential equation

$$dX_s = \alpha_s ds + \sqrt{2}\sigma^2 dB_s, \qquad X_t = x$$

where (α_s) is the control and (B_s) is a standard B.M. He aims at minimizing the cost

$$J(x,(\alpha_s),(m_s)) := \mathbf{E}\left[\int_t^T L(X_s,\alpha_s,m(s))\, ds + G(X_T,m(T))\right] .$$

where $L(x,q,m) = \sup_{p\in\mathbb{R}^d} \{-\langle p,q\rangle - H(x,p,m)\}$.

His value function u is given by

$$u(t,x) = \inf_{(\alpha_s)} J(x,(\alpha_s),(m_s)) .$$

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- The value function u then satisfies

$$\begin{cases} (i) & -\partial_t u - \sigma^2 \Delta u + H(x, Du, m) = 0 \qquad \text{in } \mathbb{R}^d \times (0, T) \\ (iii) & u(x, T) = G(x, m(T)) \qquad \text{in } \mathbb{R}^d \end{cases}$$

- The optimal control is given by

$$\alpha^*(t, x) = -D_p H(x, Du(t, x), m(t, x)) \ .$$

Proof by verification : *If u solves (i) and (iii), we have by Itô's formula,*

$$\begin{aligned} \frac{d}{ds}\mathbf{E}\left[u(s, X_s) - \int_s^T L(X_\tau, \alpha_\tau, m(\tau))d\tau\right] \\ = \mathbf{E}\left[\partial_s u(s, X_s) + \langle Du, \alpha_s\rangle + \sigma^2 \Delta u + L(X_s, \alpha_s, m(s))\right] \\ \geq \mathbf{E}\left[\partial_s u(s, X_s) + \sigma^2 \Delta u - H(X_s, Du, m(s))\right] = 0 \end{aligned}$$

with equality only for $\alpha = \alpha^*$.

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with equality only for $\alpha = \alpha^*$*.*

Integrating between 0 *and* T *:*

$$\mathbf{E}\left[u(T, X_T) - u(t,x) + \int_t^T L(X_\tau, \alpha_\tau, m(\tau))d\tau\right] \geq 0$$

with equality for $\alpha = \alpha^*$.

By (iii), $u(T, X_T) = G(X_T, m(T))$, *so that*

$$u(t,x) \leq \mathbf{E}\left[\int_t^T L(X_\tau, \alpha_\tau, m(\tau))d\tau + G(X_T, m(T))\right]$$

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Therefore u *is the value function.*

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Therefore u is the value function.

To summarize : Given a family $(m_t)_{t\in[0,T]}$ of probability densities,

- the value function u of an average agent is the solution to the HJ eq

$$\begin{cases} (i) & -\partial_t u - \sigma^2 \Delta u + H(x, Du, m) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T) \\ (iii) & u(x,T) = G(x, m(T)) \qquad \text{in } \mathbb{R}^d \end{cases}$$

- The optimal control is given by

$$\alpha^*(t,x) = -D_p H(x, Du(t,x), m(t)) \ .$$

- Therefore its optimal dynamics solves the SDE

$$dX_s = -D_p H(X_s, Du(t, X_s), m(s))ds + \sqrt{2}\sigma^2 dB_s, \qquad X_t = x$$

Heuristic interpretation of (*ii*)

Assume that the initial distribution of the players is the probability m_0.

Then the distribution $\tilde{m}(s)$ of the players at time s is the law of $\tilde{X}_s$, where $(\tilde{X}_s)$ solves the SDE

$$d\tilde{X}_s = -D_pH(\tilde{X}_s, Du(t,\tilde{X}_s), m(s))ds + \sqrt{2}dB_s, \qquad \tilde{X}_0 \sim m_0$$

Equation satisfied by $(\tilde{m}(s))$ **:** Let $\phi = \phi(s,x) \in \mathcal{C}_c^\infty$. Then, by Itô's formula,

$$\begin{aligned} \frac{d}{ds}\int_{\mathbb{R}^d} \phi \tilde{m}(s) &= \frac{d}{ds}\mathbf{E}\left[\phi(s,\tilde{X}_s)\right] \\ &= \mathbf{E}\left[\partial_s\phi - \langle D_pH(\tilde{X}_s, Du, m(s)), D\phi\rangle + \sigma^2\Delta\phi\right] \\ &= \int_{\mathbb{R}^d} \left(\partial_s\phi - \langle D_pH(x, Du, m(s)), D\phi\rangle + \sigma^2\Delta\phi\right)\tilde{m}(s) \end{aligned}$$

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Integrate in time to get :

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^d} \left(\partial_s \phi - \langle D_p H(x, Du, m(s)), D\phi\rangle + \sigma^2 \Delta \phi\right) \tilde m(s) \\ &= \int_0^T \int_{\mathbb{R}^d} \left(-\partial_s \tilde m(s) + \mathrm{div}(\tilde m(s) D_p H(x, Du, m(s))) + \sigma^2 \Delta \tilde m(s)\right) \phi \end{aligned}$$

where $\mathrm{div} = \sum_{i=1}^d \frac{\partial}{\partial x_i}$.

So $\tilde m$ solves the Kolmogorov equation

$$\left\{ \begin{array}{ll} (ii) & \partial_t \tilde m - \sigma^2 \Delta \tilde m - \mathrm{div}(\tilde m \, D_p H(x, Du, m)) = 0 \qquad \text{in } \mathbb{R}^d \times (0, T) \\ (iii) & \tilde m(0) = m_0 \qquad \text{in } \mathbb{R}^d \end{array} \right.$$

To summarize : Given a family $(m_t)_{t\in[0,T]}$ of probability densities,

- the value function u of an average agent is the solution to the HJ eq

$$\left\{\begin{array}{ll} (i) & -\partial_t u - \sigma^2\Delta u + H(x,Du,m) = 0 \qquad \text{in } \mathbb{R}^d\times(0,T) \\ (iii) & u(x,T) = G(x,m(T)) \qquad \text{in } \mathbb{R}^d \end{array}\right.$$

- The distribution $\tilde m(s)$ of the players solves the Kolmogorov equation

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A solution (u,m) of the mean filed game is a fixed point of the map $m \to \tilde m$.

Structure of MFG system

Namely, the pair $(u, m) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^2$ solves

$$(MFG) \quad \begin{cases} (i) & -\partial_t u - \sigma^2 \Delta u + H(x, Du, m) = 0 \\ (ii) & \partial_t m - \sigma^2 \Delta m - \mathrm{div}(m\, D_p H(x, Du, m)) = 0 \\ (iii) & m(0) = m_0, \; u(x, T) = G(x, m(T)) \end{cases}$$

where

- $H(x, p, m) = \sup_{q \in \mathbb{R}^d} \{-\langle p, q\rangle - L(x, q, m)\}$ is convex in p.
- $m \to H(x, p, m)$ can be
  - a local map : e.g.,

$$H(x, p, m) = \frac{1}{2}|p|^2 - F(x, m(x)) \qquad \text{or} \qquad H(x, p, m) = \frac{|p|^2}{(m(x))^\alpha}$$

  - or a nonlocal map : e.g., $H(x, p, m) = \frac{1}{2}|p|^2 - (\rho \star m(t, \cdot)) \star \rho$.
- $m(t, \cdot)$ is a probability density on $\mathbb{R}^d$ for all $t \in [0, T]$.

An example : the mexican wave (Guéant-Lasry-Lions)

The stadium is formalized as a $1-D$ torus $T^1 = \mathbb{R}\backslash\mathbb{Z}$.

Each individual is characterized by

- its geographic position in the stadium $x \in T^1$
- a position $z \in [0,1]$ describing if he is seated ($z = 0$), standing ($z = 1$), or in an intermediate position ($z \in (0,1)$).

Individuals

- cannot change their geographic position
- prefer either being seated or standing
- avoid to change too often their position
- wants to look like their neighbors

If the position of individual seated at place $y \in T^1$ and at time t is $\tilde{z}(t,y)$, the optimal control problem for the individual at place x is :

$$\inf_{(z(s))} \int_0^T \left\{ \ell(z(s)) + \frac{1}{2}(\dot{z}(s))^2 + F(x, z(s), \tilde{z}(s,\cdot)) \right\} ds$$

where $z(0) = z_0(x)$,

$$\ell(z) = K z^\alpha (1-z)^\beta \qquad (\alpha, \beta > 0)$$

$$F(x, z(s), \tilde{z}(s,\cdot)) = F(z(s), \tilde{z}) = \int_{\mathbb{R}} ((z(s) - \tilde{z}(s, x-y))^2 G(y) dy$$

(where G is, e.g., a Gaussian).

This yields to the (first order) MFG system for (u, m) where $m = \delta_{\tilde{z}(t,x)}(z)$:

$$\left\{ \begin{array}{ll} (i) & -\partial_t u + \frac{1}{2}(\partial_z u)^2 - \ell(z) = F(x, z, \tilde{z}(s, \cdot)) \\ & \qquad\qquad\qquad\qquad\qquad \text{in } [0, T] \times \mathbb{R}^2 \\ (ii) & \partial_t m - \mathrm{div}(m \partial_z u) = 0 \\ & \qquad\qquad\qquad\qquad\qquad \text{in } [0, T] \times \mathbb{R}^2 \\ (iii) & \tilde{z}(0) = \tilde{z}_0, \;\; u(x, T) = 0 \qquad \text{in } \mathbb{R}^2 \end{array} \right.$$

$\longrightarrow$ Guéant, Lasry, Lions prove the existence of an explicit periodic solution in an asymptotic regime.

Summary

Heuristic argument show that the (*MFG*) system represents a Nash equilibrium for a continuous game.

This raises several questions :

- Existence, uniqueness for the MFG system,
- Link with games with a large number of players,
- Asymptotic behavior of the system
- MFG as optimality conditions for optimal control problems of EDPs.
- ...

Outline

We discuss here

- Existence and uniqueness results for second order (*MFG*) systems,
- Link with Nash equilibria for differential games with a large number of players
- The asymptotic limit as $T \to +\infty$ of the (*MFG*) system.

Existence of solutions (nonlocal, second order MFG)

We consider the system

$$(MFG)\quad \left\{\begin{array}{cl} (i) & -\partial_t u - \Delta u + H(x, Du) = F(x, m(t,\cdot)) \\ & \qquad\qquad\qquad\qquad \text{in } \mathbb{R}^d \times (0,T) \\ (ii) & \partial_t m - \Delta m - \operatorname{div}(D_pH(x,Du)m) = 0 \\ & \qquad\qquad\qquad\qquad \text{in } \mathbb{R}^d \times (0,T) \\ (iii) & m(0) = m_0\ ,\ u(x,T) = u_f(x) \end{array}\right.$$

where

- data are periodic in space,
- $F : \mathbb{R}^d \times M \to \mathbb{R}$ (where M is the set of probability measures on $\mathbb{T}^d$).

Assumptions on the data

1. F maps $\mathbb{R}^d \times L^1_\sharp(\mathbb{R}^d)$ into a bounded subset of $W^{1,\infty}_\sharp(\mathbb{R}^d)$,
2. F is continuous from $\mathbb{R}^d \times L^1_\sharp(\mathbb{R}^d)$ to $\mathcal{C}^0_\sharp(\mathbb{R}^d)$,
3. F is bounded from $\mathcal{C}^{k,\alpha}_\sharp(\mathbb{R}^d)$ into $\mathcal{C}^{k+1,\alpha}_\sharp(\mathbb{R}^d)$,
4. $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth, periodic in x, convex in p, with
$$\left|\frac{H(x,p)}{\partial p}\right| \leq C(1+|p|)$$
5. m_0 and u_f are smooth.

Theorem (Lasry-Lions, 06)

Under the above conditions, there exists a classical solution (u, m) of (MFG).

Proof : By fixed point. Fix $m \in L^\infty([0,T], L^1_\sharp(\mathbb{R}^d))$.

- Solve
$$-\partial_t u - \Delta u + H(x, Du) = F(x, m(t,\cdot)), \qquad u(x,T) = u_f(x)$$
Then u is bounded Lipschitz in x, Hölder in t and Du is Hölder (unif. w.r.t. m)
- Let $\tilde{m}$ solve
$$\partial_t \tilde{m} - \Delta \tilde{m} - \mathrm{div}\,(D_p H(x, Du)\tilde{m}) = 0, \qquad \tilde{m}(0) = m_0 \ .$$
Then $\tilde{m}$ is Hölder continuous (unif. w.r.t. m).
- The map $m \to \tilde{m}$ is continuous on $L^\infty([0,T], L^1_\sharp(\mathbb{R}^d))$ with pre-compact range : conclusion by Schauder fixed point theorem.

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Then $\tilde{m}$ is Hölder continuous (unif. w.r.t. m).
- The map $m \to \tilde{m}$ is continuous on $L^\infty([0, T], L^1_\sharp(\mathbb{R}^d))$ with pre-compact range : conclusion by Schauder fixed point theorem.

Assume further that

$$\int_{Q_1} (F(m) - F(m'))(m - m') \geq 0 \qquad \forall m, m' \in L^1_\sharp(\mathbb{R}^d) ,$$

and, either H uniformly convex in p or

$$\int_{Q_1} (F(m) - F(m'))(m - m') = 0 \qquad \Rightarrow \qquad F(m) = F(m') .$$

Proposition

The solution to (MFG) is unique.

Typical example : $F(m) = (\rho \star m) \star \rho$, where ρ is smooth and symmetric. Then

$$\begin{aligned} \int_{Q_1} (F(m) - F(m'))(m - m') &= \int_{Q_1} (\rho \star (m - m'))^2 \\ &\geq \left(\int_{Q_1} \rho \star (m - m') \right)^2 = (F(m) - F(m'))^2 . \end{aligned}$$

From MFG to Nash equilibria with many players

Let (u, m) be a solution to

$$(MFG) \quad \left\{ \begin{array}{ll} (i) & -\partial_t u - \Delta u + H(x, Du) = F(x, m) \\ (ii) & \partial_t m - \Delta m - \mathrm{div}(m\, D_p H(x, Du)) = 0 \\ (iii) & m(0) = m_0, \ u(x, T) = G(x) \end{array} \right.$$

Aim

Build (almost-)Nash equilibria from (u, m).

References :

- Huang-Caines-Malhame, 2006.
- Carmona-De la Rue-Lachapelle, preprint 2012

The game
We consider a $N-$Player differential games, where each player i controls his dynamics

$$dX_s^i = \alpha_s^i ds + \sqrt{2} dW_s^i, \qquad X_0^i \sim m_0$$

with $\alpha^i : [0, T] \to \mathbb{R}^d$ control of Player i, W^i are independent $d-$dimensional BM and X_0^i are independent.

Players aim at minimizing the cost function, given by

$$J^i(\alpha^1, \dots, \alpha^N) = \mathbf{E}\left[\int_t^T L(X_s^i, \alpha_s^i) ds + F(X_s^i, \frac{1}{N-1}\sum_{j \neq i} \delta_{X_s^j})\ ds + G(X_T^i)\right]$$

where $X = (X^1, \dots, X^N)$ and $L(x, \alpha) = \sup_p \{\langle p, \alpha\rangle - H(x, p)\}$.

Assumptions :

- Data are periodic in space
- $G(x, m) = G(x)$ (for simplicity)
- $F = F(x, m)$ is smoothing
- $H = H(x, p)$ is convex in p and satisfies standard regularity and growth conditions
  (e.g., $H(x, p) = \frac{1}{2}|p|^2$)
- (u, m) is a smooth solution to (MFG)

Controls associated with the MFG

For any $i = 1, \ldots, N$, let $\tilde{X}^i$ be the solution to

$$\left\{ \begin{array}{l} d\tilde{X}^i_s = -\partial_p H(\tilde{X}^i_s, Du(s, \tilde{X}^i_s))ds + \sqrt{2}dW^i_s \\ \tilde{X}^i_0 \sim m_0 \end{array} \right.$$

We set $\tilde{\alpha}^i_s = -\partial_p H(\tilde{X}^i_s, Du(s, \tilde{X}^i_s))$.

Lemma

For any $i = 1, \ldots, N$, the law of $\tilde{X}^i_s$ is $m(s)$.

Approximate Nash equilibrium

Theorem

For $\epsilon > 0$, there is N_ϵ such that : $\forall N \geq N_\epsilon$, $(\tilde{\alpha}^1, \dots, \tilde{\alpha}^N)$ is an $\epsilon-$Nash equilibrium :

$$J^i(\tilde{\alpha}^1, \dots, \tilde{\alpha}^N) \leq J^i(\alpha^i, (\tilde{\alpha}^j)_{j\neq i}) + \epsilon$$

for any control α^i.

Remark : $(\tilde{\alpha}^1, \dots, \tilde{\alpha}^N)$ is an open-loop Nash equilibrium : no need to observe the other players.

Idea of proof : the $(\tilde{X}^j_s)_{j\neq i}$ are iid with law $m(s)$, so that, by the law of large numbers,

$$\frac{1}{N-1}\sum_{j\neq i} \delta_{\tilde{X}^j_s} \sim m(s) \ .$$

Estimate for N_ϵ

Theorem (Louzada)

One can take

$$N_\epsilon \sim \log\left(\frac{1}{\epsilon}\right)\frac{1}{\epsilon^{d+2}}$$

Idea of proof : Quantitative concentration inequalities (Bolley-Guillin-Villani, 2007) allow to estimate the difference

$$\left| F(x, \frac{1}{N-1}\sum_{j\neq i}\delta_{\tilde{X}^j_s}) - F(x, m(s)) \right| .$$

Existence of solutions (local, second order MFG)

We study the model equation

$$(MFG)\quad \begin{cases} (i) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x,m) \\ & \qquad \text{in } \mathbb{R}^d \times (0,T) \\ (ii) & \partial_t m - \Delta m - \mathrm{div}\,(mDu) = 0 \\ & \qquad \text{in } \mathbb{R}^d \times (0,T) \\ (iii) & m(0) = m_0\ ,\ u(x,T) = u_f(x) \end{cases}$$

where

- data are periodic in space,
- $F : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$ is smooth.

Theorem (C.-Lasry-Lions-Porretta, 2012)

Assume that $F : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}$ *is* $\mathcal{C}^1$, $\mathbb{Z}^d-$*periodic in* x *and bounded below.*

Then there exists a classical solution (u, m) *of (MFG).*

It is unique if F *is increasing.*

Remarks :

- No growth condition on F...
- ... but a strong structure condition on the Hamiltonian.
- Existence of classical solutions for more general equations but bounded F : Lasry-Lions, 06.
- Existence of weak solutions for more general equations : Lasry-Lions, 06.

Theorem (C.-Lasry-Lions-Porretta, 2012)

Assume that $F : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}$ is $\mathcal{C}^1$, $\mathbb{Z}^d-$periodic in x and bounded below.

Then there exists a classical solution (u, m) of (MFG).

It is unique if F is increasing.

Remarks :

- No growth condition on F...
- ... but a strong structure condition on the Hamiltonian.
- Existence of classical solutions for more general equations but bounded F : Lasry-Lions, 06.
- Existence of weak solutions for more general equations : Lasry-Lions, 06.

Asymptotic behavior

Goal : Investigate the long-time behavior of the solution (u^T, m^T) to the Mean Field Game system

$$(MFG) \quad \left\{ \begin{array}{ll} (i) & -\partial_t u^T - \Delta u^T + \frac{1}{2}|Du^T|^2 = F(x, m^T) \\ (ii) & \partial_t m^T - \Delta m^T - \mathrm{div}(m^T Du^T) = 0 \\ (iii) & m^T(0) = m_0, \; u^T(x, T) = u_f(x) \end{array} \right.$$

Motivation : Hope to reduce the system to a stationary equation.

The expected limit is the ergodic system :

$$(MFG-ergo)\quad \left\{\begin{array}{ll} (i) & \bar\lambda - \Delta \bar u + \frac{1}{2}|D\bar u|^2 = F(x,\bar m) \qquad \text{in } \mathbb{R}^d \\ (ii) & -\Delta \bar m - \mathrm{div}(\bar m D\bar u) = 0 \qquad \text{in } \mathbb{R}^d \end{array}\right.$$

Note that

- $\bar m = e^{-\bar u} / \left(\int_{Q_1} e^{-\bar u}\right)$ solves (MFG-ergo)(ii)
- the map
$$(x,t) \to (\bar u(x) + \bar\lambda t\ ,\ \bar m(x))$$
satisfies (MFG)(i-ii).

Assumptions on the data

- $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, $\mathbb{Z}^d-$periodic in x, and increasing with respect to m.
- $m_0 : \mathbb{R}^d \to \mathbb{R}$ is smooth, $\mathbb{Z}^d-$periodic, $m_0 > 0$ and $\int_{Q_1} m_0 = 1$.
- $G : \mathbb{R}^d \to \mathbb{R}$ is smooth, $\mathbb{Z}^d-$periodic.

Convergence under mild monotony condition

Recall the definition of the scaled functions on $\mathbb{R}^d \times [0,1]$:

$$\upsilon^T(t,x) := u^T(tT,x) \qquad ; \qquad \mu^T(t,x) := m^T(tT,x)$$

Theorem (C.-Lasry-Lions-Porretta, '12)

As $T \to +\infty$,

1. $\upsilon^T(t,\cdot)/T$ *converges to* $t \to (1-t)\bar\lambda$ *in* $L^2(Q_1)$ *for any* $t \in [0,1]$,
2. $\upsilon^T - \int_{Q_1} \upsilon^T(t)$ *converges to* $\bar u$ *in* $L^2(Q_1 \times (0,1))$,
3. μ^T *converges to* $\bar m$ *in* $L^p(Q_1 \times (0,1))$, *for any* $p < \frac{N+2}{N}$.

Convergence rate (strong monotony)

Assume, furthermore, there is $\gamma > 0$ with

$$F(x,s) - F(x,t) \geq \gamma(s-t) \qquad \forall s \geq t,\ \forall x \in \mathbb{R}^d .$$

Set $\tilde{u}^T(t,x) = u^T(t,x) - \int_{Q_1} u^T(t,y)dy$.

Theorem (C.-Lasry-Lions-Porretta, '12)

There is $\kappa > 0$ such that

1. $\|\tilde{u}^T(t) - \bar{u}\|_{L^1(Q_1)} \leq \frac{C}{T-t}\left(e^{-\kappa(T-t)} + e^{-\kappa t}\right)$
2. $\|m^T(t) - \bar{m}\|_{L^1(Q_1)} \leq \frac{C}{t}\left(e^{-\kappa(T-t)} + e^{-\kappa t}\right)$
3. $\left\|\frac{u^T(t)}{T} - \bar{\lambda}\left(1 - \frac{t}{T}\right)\right\|_{L^1(Q_1)} \leq \frac{C}{T}$

The proofs relies on two ingredients :

- The Hamiltonian structure of the (MFG) equation
- A main energy equality

The Hamiltonian structure :
Set $\Phi(x,m) = \int_0^m F(x,\rho)\,d\rho$ and

$$\mathcal{E}(u,m) = \int_{Q_1} m\,\frac{1}{2}|Du|^2 + \langle Du, Dm\rangle - \Phi(x,m)\,dx$$

Lemma

(u^T, m^T) solution of (MFG) $\Leftrightarrow$ (u^T, m^T) satisfies

$$\left\{\begin{array}{ll} (i) & -\partial_t u^T = -\dfrac{\partial \mathcal{E}}{\partial m}(u^T, m^T) \\[2ex] (ii) & \partial_t m^T = -\dfrac{\partial \mathcal{E}}{\partial u}(u^T, m^T) \\[2ex] (iii) & m^T(0) = m_0,\ u^T(x,T) = G(x, m^T(T)) \end{array}\right.$$

In particular the energy $\mathcal{E}(u^T(t), m^T(t))$ is constant along the flow.

Main energy equality :

Lemma (Lasry-Lions, 06)

For any $t \in [0, T]$

$$-\frac{d}{dt}\int_{Q_1}(u^T(t)-\bar{u})(m^T(t)-\bar{m})dx =$$

$$\int_{Q_1}\frac{(m^T(t)+\bar{m})}{2}|Du^T(t)-D\bar{u}|^2+(F(x,m^T(t))-F(x,\bar{m}))(m^T(t)-\bar{m})$$

Proof : Multiply (MFG)(i)-(MFG-ergo)-(i) by $(m^T - \bar{m})$ and substract to (MFG)(ii)-(MFG-ergo)(ii) multiplied by $(u^T - \bar{u})$.

Why the convergence ?

We define the scaled functions on $\mathbb{R}^d \times [0,1]$:

$$\upsilon^T(x,t) := u^T(x,tT) \qquad ; \qquad \mu^T(x,t) := m^T(x,tT)$$

Integrate in time the main energy equality :

$$\int_0^1 \int_{Q_1} \frac{(\mu^T+\bar m)}{2}|D\upsilon^T - D\bar u|^2 + (F(x,\mu^T)-F(x,\bar m))(\mu^T-\bar m)\ dxdt$$
$$= -\frac{1}{T}\left[\int_{Q_1} (\upsilon^T-\bar u)(\mu^T-\bar m)dx\right]_0^1$$

Then

- The Hamiltonian structure implies that the RHS $\to 0$ as $T \to +\infty$,
- $D\upsilon^T \to D\bar u$,
- which implies that $D\upsilon^T \to D\bar u$

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$$= -\frac{1}{T} \left[\int_{Q_1} (v^T - \bar{u})(\mu^T - \bar{m}) dx \right]_0^1$$

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- which implies that $Dv^T \to D\bar{u}$

Conclusion and comments for second order problems

- Existence and uniqueness results for second order (*MFG*) systems :
  - Well understood for nonlocal equations, work to be done for local ones (unbounded RHS),
  - Little is known for systems of the form

$$\left\{ \begin{array}{ll} (i) & -\partial_t u - \Delta u + \frac{|Du|^2}{2\ m^{\alpha}} = 0 \\ (ii) & \partial_t m - \Delta m - \mathrm{div}\left(m^{1-\alpha} Du\right) = 0 \\ (iii) & m(0) = m_0\ ,\ u(x,T) = u_f(x) \end{array} \right.$$

- Link with Nash equilibria for differential games with a large number of players
  - OK nonlocal setting,
  - Nothing written in the local setting
- The asymptotic limit as $T \to +\infty$ of the (*MFG*) system : known only for quadratic Hamiltonians.

Some references

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  - Lasry-Lions : CRAS 06, Jpn. J. Math. 2 (2007), Lions' lecture at Collège de France
  - Huang-Caines-Malhamé : Com. Information Systems '06, ...
  - Related works : Guéant, Gomes-Pires-Sanchez Morgado.
- Discrete model
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  - Gomes-Mohr-Souza J. Math. Pures Appl. (9) 93 (2010)
  - Guéant (preprint)
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  - Achdou-Capuzzo Dolcetta : SIAM J. Numer. Anal. 48 (2010).
  - Lachapelle-Salomon-Turinici : Math. Models Methods Appl. Sci. (2010).
  - Achdou-Camilli-Capuzzo Dolcetta : SIAM J. Control Opt. (2012).
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Some references (continued)

- Long-time behavior
  - Gomes-Mohr-Souza (discrete setting)
  - C.-Lasry-Lions-Porretta : NHM 2012.
- Linear-quadratic MFG
  - Bardi, NHM 2012
  - Bensoussan-Sung-Yam-Yung, pre-print.
  - Carmona, Delarue, Lachapelle
- Related works :
  - Price formation : Lasry-Lions, Chayes-González-Gualdani, Markowich-Matevosyan-Pietschmann-Wolfram, Caffarelli-Markowich-Pietschmann
  - Formalization of human crowds : Lachapelle, Santambrogio
- Lecture notes on MFG : Guéant-Lasry-Lions, Achdou, C., Tao.

Outline

Warning : this part is mostly heuristic.

Differential games with many players (1/3)

We consider a $N-$Player differential games, where each player i controls his velocity

$$\frac{d}{ds}X_s^i = \alpha_s^i, \qquad X_t^i = x^i$$

with $\alpha^i : [0,T] \to \mathbb{R}^d$ control of Player i.

Players aim at minimizing the cost function, given by

$$J_i^N(t,x,\alpha) = \int_t^T L_i^N(X_s, \alpha_s^i)ds + G_i^N(X_T)$$

where $X = (X^1, \ldots, X^N)$, $L_i^N : \mathbb{R}^{Nd} \times \mathbb{R}^d \to \mathbb{R}$ and $G_i^N : \mathbb{R}^{Nd} \to \mathbb{R}$.

Nash equilibrium

Fix an initial condition $(t,x) \in [0,T] \times \mathbb{R}^{Nd}$. We say that the controls $(\alpha_1^*, \dots, \alpha_N^*)$ is a Nash equilibrium at (t,x) if

$$J_i^N(t,x,\alpha_1^*, \dots, \alpha_N^*) \leq J_i^N(t,x,\alpha_i,(\alpha_j^*)_{j \neq i})$$

for any $i = 1, \dots, N$ and any control α_i.

The “controls" are

- either “open loop" = depend only on time : $\alpha_i = \alpha_i(t)$
  $\longrightarrow$ Nash equilibria seldom exist in this framework
- or “closed loop" = depend on time and on the position of the other players : $\alpha_i = \alpha_i(t, x_1, \dots, x_N)$
  $\longrightarrow$ Existence of Nash equilibria in this framework is more likely, but difficult to implement when N is large.

Key assumption : Players are identical and, for a player i, the other players are undistinguishable :

$$L_i^N(x,\alpha) = L(x_i, \alpha^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j})$$

and

$$G_i^N(x) = G(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j})$$

where $L : \mathbb{R}^d \times \mathbb{R}^d \times M \to \mathbb{R}$ and $G : \mathbb{R}^d \times M \to \mathbb{R}$, where M is the set of Borel probability measures on $\mathbb{R}^d$. Let

$$H(x,p,m) = \sup_{\alpha\in\mathbb{R}^d}\{-\langle \alpha, p\rangle - L(x,\alpha,m)\}$$

The PDE system associated with the differential game

Finding a “good" *Nash equilibrium payoff* boils down to solve the following system of Hamilton-Jacobi equations :

$$
(NE)\quad \left\{ \begin{array}{l}
-\dfrac{\partial u_i^N}{\partial t} + H(x_i, D_{x_i} u_i^N, \dfrac{1}{N-1}\sum_{j\neq i}\delta_{x_j}) \\
\qquad + \sum_{j\neq i} \langle \dfrac{\partial H}{\partial p}(x_j, D_{x_j} u_j^N, \dfrac{1}{N-1}\sum_{k\neq j}\delta_{x_k}), D_{x_j} u_i^N\rangle = 0 \\
\qquad\qquad i = 1, \dots, N,\ (t,x) \in (0,T)\times \mathbb{R}^{Nd} \\
u_i^N(T,x) = G(x_i, \dfrac{1}{N-1}\sum_{j\neq i}\delta_{x_j}) \\
\qquad\qquad i = 1, \dots, N,\ x \in \mathbb{R}^{Nd}
\end{array} \right.
$$

Interpretation of the PDE system

Lemma

If (u_i^N) is a smooth solution to (NE), then the feedback strategies

$$\alpha_i^*(t,x) = -\frac{\partial H}{\partial p}(x_i, D_{x_i}u_i^N(t,x), \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j})$$

provide a feedback Nash equilibrium for the game. Namely

$$u_i^N(t,x) = J_i^N(t,x,(\alpha_j^*)_{j=1,\dots,N}) \leq J_i^N(t,x,\alpha_i,(\hat{\alpha}_j^*)_{j\neq i})$$

for any i and any control α_i.

Remark : Payers need to observe all the other players to play in optimal way.

Existence of solutions for the PDE system

- System (NE) is ill-posed in general, even for small N (Bressan-Shen, 2004)
- In the second order setting, system (NE) has at least one symmetric solution.
  (Bensoussan-Frehse, Lasry-Lions)
- No uniqueness in general
- Solution impossible to compute in practice when N is large.

The MFGf

We consider a symmetric solution u_i^N to the PDE system (NE) :
$u_i^N(t,x) = u_i^N(t, x_i, (x_j)_{j\neq i})$ where $u_i^N(t, x_i, \cdot)$ is a symmetric functions of many variables.

In view of the previous discussion, we expect that

$$u_i^N(t,x) = u_i^N\left(t, x_i, (x_j)_{j\neq i}\right) \sim U\left(t, x_i, \frac{1}{N-1}\sum_{j\neq i} \delta_{x_j}\right)$$

where $U : [0, T] \times \mathbb{R}^d \times M \to \mathbb{R}$.

This requires estimates of the form

$$\sup_{j\neq i} \|\partial_{x_j} u_i^N\|_\infty \leq \frac{C_1}{N} \ ,$$

which seems to be known only for T small and in the second order case (Lasry-Lions).

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which seems to be known only for T small and in the second order case (Lasry-Lions).

Since the $u_i^N \sim U\left(t, x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right)$ solve

$$(NE)\quad \begin{cases} -\dfrac{\partial u_i^N}{\partial t} + H(x_i, D_{x_i}u_i^N, \dfrac{1}{N-1}\displaystyle\sum_{j\neq i}\delta_{x_j}) \\ \qquad + \displaystyle\sum_{j\neq i}\langle \dfrac{\partial H}{\partial p}(x_j, D_{x_j}u_j^N, \dfrac{1}{N-1}\displaystyle\sum_{k\neq j}\delta_{x_k}), D_{x_j}u_i^N\rangle = 0 \\ u_i^N(T,x) = G(x_i, \dfrac{1}{N-1}\displaystyle\sum_{j\neq i}\delta_{x_j}) \end{cases}$$

one expects that $U = U(t,x,m)$ satisfies

$$(MFGf)\quad \begin{cases} -\dfrac{\partial U}{\partial t} + H(x,m,D_xU) + \langle \dfrac{\partial U}{\partial m}, \dfrac{\partial H}{\partial p}(x,m,D_xU)\nabla\cdot\rangle = 0 \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times M \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times M \end{cases}$$

Notation : If $B : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector field, we have set

$$\langle \frac{\partial U}{\partial m}, B\,\nabla \cdot \rangle := \frac{d}{ds} U(t,x,m(s))_{|s=0}$$

where $m(s)$ solves

$$\partial_s m(s) - \mathrm{div}\,(B\,m(s)) = 0, \qquad m(0) = m$$

(MFG) as characteristics of (MFGf)

Fix the initial repartition m_0 and let $m(t)$ solve

$$\left\{ \begin{array}{l} \frac{\partial m}{\partial t} - \mathrm{div}(m \frac{\partial H}{\partial p}(x, D_x U, m)) = 0 \\ m(0) = m_0 \end{array} \right.$$

Set $u(x,t) = U(x, m(t), t)$. We “claim" that u solves

$$-\frac{\partial u}{\partial t} + H(x, Du, m) = 0$$

"Indeed",

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} - \langle \frac{\partial U}{\partial m}, \frac{\partial H}{\partial p}(x, m, D_x U)) \nabla \cdot \rangle$$

where

$$\frac{\partial U}{\partial t} - \langle \frac{\partial U}{\partial m}, \frac{\partial H}{\partial p}(x, m, D_x U) \nabla \cdot \rangle = H(x, m, D_x U)$$

Therefore the pair (u, m) is a solution of the MFG system

$$\begin{cases} \frac{\partial u}{\partial t} - H(x, Du, m) = 0 \\ \frac{\partial m}{\partial t} - \mathrm{div}(m \frac{\partial H}{\partial p}(x, Du, m)) = 0 \\ u(x, T) = G(x, m(T)), \; m(x, 0) = m_0(x) \end{cases}$$

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Conclusion and comments

1. Very little is known on (MFGf) : recent analysis by Lasry-Lions for a discretized system (hyperbolic equation in non-divergence form)

2. The above arguments are heuristic : the link between the system of PDEs related to Nash (NE) and (MFGf) is not clear yet.

3. However, the limit of (NE) is known in particular cases (second order and stationary or short time).
(Lasry-Lions)