Optimal Control and Mean Field Games (Part 3)

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Joint work with P.-M. Cannarsa (in progress)

Aim of the lectures

- $\surd$ Describe the model and its interpretations
  (Part 1)
- $\surd$ Existence of the MFG system by fixed point arguments
  (Non local MFG - Part 2)
- $\rightarrow$ The MFG system as optimality condition for an optimal control problem of HJ equations
  (Local MFG - Part 3)

Part 3

The MFG system and optimal control problems for PDE equations

The Mean Field Game system

We are interested in the MFG system

$$(MFG) \qquad \begin{cases} (i) & -\partial_t u + H(x, Du) = f(x, m(x,t)) \\ (ii) & \partial_t m - \mathrm{div}(m D_p H(x, Du)) = 0 \\ (iii) & m(0) = m_0, \; u(x,T) = u_T(x) \end{cases}$$

where $f : \mathbb{R}^d \times [0, +\infty) \to [0, +\infty)$ is a local coupling term.

Problem : Existence/uniqueness of a solutions - smoothness properties.

Similar systems

- The Monge-Kantorovich optimal transport problems : minimize the cost to transport a probability density $\bar{m}_0$ onto a probability density $\bar{m}_1$. The dual problem reads

$$\max\left\{\int_{\mathbb{R}^d} u(x)d(\bar{m}_1-\bar{m}_0)(x),\ u:\mathbb{R}^d\to\mathbb{R}\ 1-\text{Lipschitz continuous}\right\}.$$

If u is optimal, the system of necessary conditions reads

$$\left\{\begin{array}{l} |Du|\leq 1 \text{ in } \mathbb{R}^d, \qquad |Du|=1 \text{ in } \{m>0\}\\ -\text{div}(mDu)=\bar{m}_1-\bar{m}_0 \text{ in } \mathbb{R}^d \end{array}\right.$$

where m is the transport density.
$\longrightarrow$ Existence of solutions, uniqueness of m and uniqueness (up to additive constants) of u on each connected component of $\{m>0\}$.
(Evans-Gangbo 1999, Feldman-Mc Cann 2002, Ambrosio 2003)

- Sandpile model : Introduced by Hadeler, Kuttler (1999). $\Omega \subset \mathbb{R}^2$ is a bounded table, on which one pours sand with a rate $f = f(x)$. In the stationary regime, the heap of sand consists in a standing layer u and a rolling layer m.

$$\left\{ \begin{array}{l} |Du| \leq 1 \text{ in } \Omega, \qquad |Du| = 1 \text{ in } \{m > 0\} \\ -\mathrm{div}(mDu) = f \text{ in } \Omega \\ u = 0 \text{ on } \ \partial\Omega \end{array} \right.$$

$\longrightarrow$ existence and representation of solutions, uniqueness of m and uniqueness of u in $\{m > 0\}$.
(Feldman (1999), Carnarsa-C. (2004), Crasta-Finzi Vita (2008), Cannarsa-C.-Sinestrari (2009)).

- Adjoint methods for Hamilton-Jacobi PDE : analysis of the vanishing viscosity limit :

$$\left\{\begin{array}{l}\partial_t u^\varepsilon + H(Du^\varepsilon) = \epsilon\Delta u^\varepsilon,\\ -\partial_t m^\varepsilon - \mathrm{div}\Big(DH(Du^\varepsilon)\Big) = \epsilon\Delta m^\varepsilon,\\ u^\varepsilon(0,x) = u_0(x),\ m^\varepsilon(1,.) = m_1.\end{array}\right.$$

$\longrightarrow$ better understand the convergence of the vanishing viscosity method when H is nonconvex.
(Evans (2010))

- A congestion model : An optimal transport problem related to congestion yields to the following system of PDEs : for $\alpha \in (0,1)$,

$$\left\{\begin{array}{l} \partial_t u + \frac{\alpha}{4} m^{\alpha-1}|Du|^2 = 0, \\ \partial_t m + \operatorname{div}\left(\frac{1}{2} m^{\alpha} Du\right) = 0, \\ m(0,.) = m_0,\ m(1,.) = m_1. \end{array}\right.$$

(Benamou-Brenier formulation of Wasserstein distance : $\alpha = 1$ - Dolbeault-Nazaret-Savaré, 2009)
$\longrightarrow$ analysis of the system : Existence and uniqueness of a weak solution
(C.-Carlier-Nazaret 2012.)

Back to the MFG system

$$(MFG) \qquad \left\{ \begin{array}{rl} (i) & -\partial_t u + H(x, Du) = f(x, m(x,t)) \\ (ii) & \partial_t m - \mathrm{div}(m D_p H(x, Du)) = 0 \\ (iii) & m(0) = m_0, \; u(x,T) = u_T(x) \end{array} \right.$$

under the assumptions that

- $f : \mathbb{R}^d \times [0, +\infty) \to [0, +\infty)$ is a local coupling term.
- all the data are $\mathbb{Z}^d-$periodic in space : denoted by $\sharp$.

Difficulties for the MFG with local coupling

- **For nonlocal coupling,** existence/regularity of the MFG system come from
  - semi-concavity estimates for HJ equations with $\mathcal{C}^2$ RHS,
  - preservation of the absolute continuity of m_0 by Kolmogorov equation for gradient flows of semi-concave functions.
- **For local equations,**
  - the RHS is a priori only measurable (or integrable),
  - the HJ with discontinuous RHS is poorly understood,
  - the Kolmogorov equation with $Du \in L^\infty$ is ill-posed as well.

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Idea for the MFG system

Inspired with similar problems in optimal transport, we look for optimization problems with the MFG system as necessary condition.

This is the case

- For an optimal control problem for an Hamilton-Jacobi equation
- For an optimal control problem for a Kolmogorov equation
- The second one is the dual of the first one.

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Outline

1. Preliminary estimates on HJ equations
2. Optimal control of the HJ equation
3. Optimal control of the Kolmogorov equation
4. Existence/uniqueness of solutions for the (*MFG*) system
5. Conclusion

Outline

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Aim

We study the optimal control of Hamilton-Jacobi equations :

$$(HJ) \qquad \left\{ \begin{array}{ll} -\partial_t u + H(x, Du) = \alpha & \text{in } (0,T)\times\mathbb{R}^d \\ u(T,x) = u_T(x) & \text{in } \mathbb{R}^d \end{array}\right.$$

by the control $\alpha = \alpha(t,x)$.

$\longrightarrow$ Find estimates for the solution u in terms of integral norms of α.

- L^∞ estimates
- Hölder estimates

Discussion

The viscosity solution of

$$(HJ) \qquad \left\{ \begin{array}{ll} -\partial_t u + H(x, Du) = \alpha & \text{in } (0,T)\times\mathbb{R}^d \\ u(T,x) = u_T(x) & \text{in } \mathbb{R}^d \end{array} \right.$$

is represented by the optimal control problem

$$u(t,x) = \inf_{\xi(\cdot),\ \xi(t)=x} \left\{ \int_t^T \left(L(\xi(s),\xi'(s)) + \alpha(s,\xi(s))\right) ds + u_T(\xi(T)) \right\}$$

where $L(x,y) := \sup_z \langle z, y\rangle - H(x,-z)$ is as smooth as H.

- The optimal control avoids places where α is large...
- but if $\alpha << -1$ even at some small places, then $u << -1$. So smallness of α leads to instabilities.

Assumptions

- The Hamiltonian $H:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is of class $\mathcal{C}^1$ and has a superlinear growth in the gradient variable : there is $\mathbf{r}>\mathbf{1}$ such that

$$\frac{1}{\bar C}|\xi|^{\mathbf{r}}-\bar C\leq H(x,\xi)\leq \bar C(|\xi|^{\mathbf{r}}+1)\qquad \forall (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d\ .$$

- $x\to H(x,z)$ is $\mathbb{Z}^d$ periodic for any z.
- $\alpha\in L^p_\sharp([0,T]\times\mathbb{R}^d)$ where $p>1+d/r$.

Upper bounds

Let $\alpha \in \mathcal{C}^1_\sharp([0,T]\times\mathbb{R}^d)$ and u be a viscosity subsolution to

$$-\partial_t u + H(x, Du) \leq \alpha \qquad \text{in } (0,T)\times\mathbb{R}^d$$

Lemma (Cannarsa, C.)

There is a universal constant C such that

$$u(t_1, x) \leq u(t_2, x) + C(t_2 - t_1)^{\frac{p-1-d/r}{p+d/r}} \|(\alpha)_+\|_p$$

for any $0 \leq t_1 < t_2 \leq T$ (where $p - 1 - d/r > 0$ by assumption).

Remark : the result actually holds if $u \in BV_\sharp$, $Du \in L^r$ and $\alpha \in L^p_\sharp$, where the inequality holds in the sense of distribution.

Proof : We assume that u and α are of class $\mathcal{C}^1$.
Fix $\beta \in (1/r, \frac{1}{d(q-1)})$. For $\sigma \in \mathbb{R}^d$ with $|\sigma| \leq 1$, we define the arc

$$x_\sigma(s) = \begin{cases} x + \sigma(s - t_1)^\beta & \text{if } s \in [t_1, \frac{t_1+t_2}{2}] \\ x + \sigma(t_2 - s)^\beta & \text{if } s \in [\frac{t_1+t_2}{2}, t_2] \end{cases}$$

Let L be the convex conjugate of $p \to H(x, -p)$, i.e., $L(x, \xi) = H^*(x, -\xi)$.
Then

$$\begin{array}{l} \frac{d}{dt}\left[u(s, x_\sigma(s)) - \int_s^{t_2} L(x_\sigma(\tau), x'_\sigma(\tau))d\tau \right] \\ \qquad = \partial_t u(s, x_\sigma(s)) + \langle Du(s, x_\sigma(s)), x'_\sigma(s)\rangle + L(x_\sigma(s), x'_\sigma(s)) \\ \qquad \geq \partial_t u(s, x_\sigma(s)) - H(x_\sigma(s), Du(s, x_\sigma(s))) \geq -\alpha(s, x_\sigma(s)) \end{array}$$

So

$$\frac{d}{dt}\left[u(s,x_\sigma(s)) - \int_s^{t_2} L(x_\sigma(\tau),x'_\sigma(\tau))d\tau\right] \geq \ -\alpha(s,x_\sigma(s))$$

We integrate in time on $[t_1, t_2]$:

$$u(t_2,x)-u(t_1,x)+\int_s^{t_2} L(x_\sigma(\tau),x'_\sigma(\tau))d\tau \geq -\int_s^{t_2}\alpha(\tau,x_\sigma(\tau))d\tau$$

Then we integrate on $\sigma \in B_1$:

$$u(t_1,x) \leq u(t_2,x)+\frac{1}{|B_1|}\int_{B_1}\int_{t_1}^{t_2}\left[L(x_\sigma(s),x'_\sigma(s))+\alpha(s,x_\sigma(s))\right] dsd\sigma \ .$$

By the growth assumption on L (=coercivity assumption on H), we have

$$\begin{array}{rcl}\frac{1}{|B_1|}\int_{B_1}\int_{t_1}^{t_2} L(x_\sigma(s),x'_\sigma(s))\ dsd\sigma & \leq & \bar{C}\left[\int_{B_1}\int_{t_1}^{t_2}|x'_\sigma(s))|^{r'}\ dsd\sigma+(t_2-t_1)\right] \\ & \leq & C(t_2-t_1)^{1-r'(1-\beta)}\end{array}$$

where $1-r'(1-\beta)>0$ since $\beta>1/r$.

Using Hölder's inequality, we get, on another hand,

$$\int_{B_1}\int_{t_1}^{t_2} \alpha(s, x_\sigma(s))dsd\sigma \;\leq\; C(t_2-t_1)^{(1-d\beta(q-1))/q}\|\alpha\|_p$$

where $1-d\beta(q-1)>0$ since $\beta<\frac{1}{d(q-1)}$.

For a suitable choice of $\beta\in(\frac{1}{r},\frac{1}{d(q-1)})$, we obtain finally that

$$u(t_1,x)\leq u(t_2,x)+C(t_2-t_1)^{\frac{p-1-d/r}{d+d/r}}\|\alpha\|_p\ .$$

□

Regularity

Theorem (C.-Silvestre, 2012)

Let u be a bounded viscosity solution of

$$\begin{cases} -\partial_t u + H(x, Du) = \alpha & \text{in } (0,T) \times \mathbb{R}^d \\ u(T,x) = u_T(x) & \text{in } \mathbb{R}^d \end{cases}$$

where $\alpha \geq 0$, $\alpha \in L^p$ with $p > 1 + d/r$.

Then, for any $\delta > 0$, u is Hölder continuous in $[0, T-\delta] \times \mathbb{R}^d$:

$$|u(t,x) - u(s,y)| \leq C|(t,x) - (s,y)|^\gamma$$

where

$$\gamma = \gamma(\|u\|_\infty, \|\alpha\|_p, d, r), \ \ C = C(\|u\|_\infty, \|\alpha\|_m, d, r, \delta).$$

Related results

- Capuzzo Dolcetta-Leoni-Porretta (2010), Barles (2010) : stationary equations, bounded RHS,
- C. (2009), Cannarsa-C. (2010), C. Rainer (2011) : evolution equations, bounded RHS,
- Dall'Aglio-Porretta (preprint) : stationary setting, unbounded RHS,
- C.-Silvestre (2012) : evolution equations, unbounded RHS.

Idea of proofs :

- For the stationary setting, by comparison with suitable test functions. (Capuzzo Dolcetta-Leoni-Porretta, Barles, Dall'Aglio-Porretta)
- For evolution equations, two techniques
  - By representation formula and reverse Hölder inequalities, (C., Cannarsa-C., C. Rainer)
  - Comparison + improvement of oscillations techniques (C.-Silvestre)

Summary

Let (u,α) solve

$$\left\{\begin{array}{ll} -\partial_t u + H(x,Du) = \alpha & \text{in } (0,T)\times\mathbb{R}^d \\ u(T,\cdot) = u_T & \text{in } \mathbb{R}^d \end{array}\right.$$

with $\alpha = \alpha(t,x) \geq 0$. Then

- (Upper bound)

$$u(t_1,\cdot) \leq u(t_2,\cdot) + C(t_2-t_1)^{\frac{p-1-d/r}{d+d/r}} \|\alpha\|_p \qquad \text{a.e.}$$

  for any $0 \leq t_1 < t_2 \leq T$ (where $r - d(q-1) > 0$ by assumption (H3)).
- (Lower bound) $u(t,x) \geq u_T(x) - C(T-t)$.
- (Regularity) u is locally Hölder continuous in $[0,T)\times\mathbb{R}^d$ with a modulus depending only on $\|\alpha\|_p$ and $\|u_T\|_\infty$.

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Back to the MFG system

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under the assumptions that

- $f : \mathbb{R}^d \times [0, +\infty) \to [0, +\infty)$ is a local coupling term.
- all the data are $\mathbb{Z}^d-$periodic in space : denoted by $\sharp$.

We study the optimal control of the HJ equation :

$$(\mathbf{HJ}-\mathbf{Pb}) \qquad \inf\left\{\int_0^T\int_Q F^*(x,\alpha(t,x))\,dxdt - \int_Q u(0,x)dm_0(x)\right\}$$

where u is the solution to the HJ equation

$$\begin{cases} -\partial_t u + H(x,Du) = \alpha \qquad \text{in } (0,T)\times\mathbb{R}^d \\ u(T,\cdot) = u_T \qquad \text{in } \mathbb{R}^d \end{cases}$$

and $F^*(x,a) = \sup_{m\in\mathbb{R}}(am - F(x,m))$ where

$$F(x,m) = \begin{cases} \int_0^m f(x,m')dm' & \text{if } m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Note that $F^*(x,a) = 0$ for $a \leq 0$.

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Heuristics :

- α must be small in order to minimize $\displaystyle\int_0^T \int_Q F^*(x, \alpha(t,x))\, dxdt$.
- However, if α is small, then u is also small (by comparison).
- This contradicts the smallness of the term $-\int_Q u(0,x)dm_0(x)$ (because $m_0 \geq 0$).

In order to prove existence of optimal solutions

- We look for estimates on minimizing sequences,
- and derive from them a relaxed problem.

Assumptions

(H1) $f : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}$ is smooth and increasing with respect to the second variable m with $f(x, 0) = 0$,

(H2) There exists $q > 1$ and a constant $\bar{C}$ such that

$$-\bar{C} + \frac{1}{\bar{C}}|m|^{q-1} \leq f(x, m) \leq \bar{C}(1 + |m|^{q-1}) \qquad \forall m \, .$$

(H3) The Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is of class $\mathcal{C}^1$ and has a superlinear growth in the gradient variable : there is $r > d(q - 1)$ such that

$$\frac{1}{\bar{C}}|\xi|^r - \bar{C} \leq H(x, \xi) \leq \bar{C}(|\xi|^r + 1) \qquad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \, .$$

(H4) We also assume that there is $\theta \in [0, \frac{r}{d+1})$ such that

$$|H(x,\xi) - H(y,\xi)| \leq \bar{C}|x-y|\,(|\xi| \vee 1)^{\theta} \qquad \forall x, y, \xi \in \mathbb{R}^d$$

(H5) $u_T : \mathbb{R}^d \to \mathbb{R}$ is a smooth, periodic map, while $m_0 : \mathbb{R}^d \to \mathbb{R}$ is a smooth, periodic map, with $m_0 \geq 0$ and $\int_Q m_0 dx = 1$.

Remark : Recall that $F^*(x,a) = \sup_{m \in \mathbb{R}} F(x,m)$ where

$$F(x,m) = \begin{cases} \displaystyle\int_0^m f(x,m')dm' & \text{if } m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

With our assumptions, F^* is $\mathcal{C}^1$, nondecreasing, with $F^*(x,a) = 0$ for $a \leq 0$.

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With our assumptions, F^* is $\mathcal{C}^1$, nondecreasing, with $F^*(x,a) = 0$ for $a \leq 0$.

Estimates on the minimizing sequence

Let (u_n, α_n) be a minimizing sequence for the optimal control of HJ equation :

$$\left\{ \int_0^T \int_Q F^*(x, \alpha_n)\, dxdt - \int_Q u_n(0, x) dm_0(x) \right\} \to \inf$$

where (u_n, α_n) is the solution to the HJ equation

$$\begin{cases} -\partial_t u_n + H(x, Du_n) = \alpha_n & \text{in } (0, T) \times \mathbb{R}^d \\ u_n(T, \cdot) = u_T & \text{in } \mathbb{R}^d \end{cases}$$

Let $\tilde{\alpha}_n = \alpha_n \vee 0$ and $\tilde{u}_n$ solve

$$\left\{ \begin{array}{l} -\partial_t \tilde{u}_n + H(x, D\tilde{u}_n) = \tilde{\alpha}_n \qquad \text{in } (0,T)\times\mathbb{R}^d \\ \tilde{u}_n(T,\cdot) = u_T \qquad \text{in } \mathbb{R}^d \end{array} \right.$$

Then :

- by comparison, $u_n \leq \tilde{u}_n$,
- $F^*(x, \alpha_n(t,x)) \geq F^*(x, \tilde{\alpha}_n(t,x))$ because $F^*(x,0) \leq F^*(x,a)$ for all a.

Therefore

$$\int_0^T \int_Q F^*(x, \tilde{\alpha}_n) - \int_Q \tilde{u}_n(0) m_0 \leq \int_0^T \int_Q F^*(x, \alpha_n) - \int_Q u_n(0) m_0$$

and $(\tilde{u}_n, \tilde{\alpha}_n)$ is still a minimizing sequence.

So we now assume that (u_n, α_n) be a minimizing sequence for the optimal control of HJ equation with $\alpha_n \geq 0$.

We use the results of the first part :

- (Upper bound)

$$u_n(t_1,\cdot) \leq u_n(t_2,\cdot) + C(t_2-t_1)^{\frac{p-1-d/r}{d+d/r}} \|\alpha_n\|_p$$

  for any $0 \leq t_1 < t_2 \leq T$ (where $r - d(q-1) > 0$ by assumption (H3)).

- (Lower bound) $u_n(t,x) \geq u_T(x) - C(T-t)$.

- (Regularity) u_n is locally Hölder continuous in $[0,T) \times \mathbb{R}^d$ with a modulus depending only on $\|\alpha_n\|_p$ and $\|u_T\|_\infty$.

In particular

$$u_n(0,\cdot) \le u_T + C\|\alpha_n\|_p$$

By assumption on F, we have

$$\frac{1}{\bar{C}}|a|^p - \bar{C} \le F^*(x,a) \le \bar{C}(1+|a|^p) \qquad \forall a \ge 0 \; ,$$

(where $1/p + 1/q = 1$). Therefore

$$C \ge \int_0^T \int_Q F^*(x,\alpha_n) - \int_Q u_n(0)m_0 \ge \int_0^T \int_Q |\alpha_n|^p - C\|\alpha_n\|_p - C$$

This shows that (α_n) is bounded in L^p.

Estimates on the minimizing sequence

Proposition

If (u_n, α_n) is a minimizing sequence, then

- (α_n) is bounded in L^p.
- (uniform bounds on the u_n)

$$u_T(x) - C(T-t) \leq u_n(t,x) \leq u_T(x) + C(T-t)^{\frac{p-1-d/r}{d+d/r}}$$

- (Regularity) the u_n are uniformly locally Hölder continuous in $[0, T) \times \mathbb{R}^d$.
- (Integral bounds) Du_n is bounded in $L^r_\sharp$ and $(\partial_t u_n)$ is bounded in $L^1_\sharp$.

Remark : In particular, (Du_n) is bounded in $BV_\sharp$.

Explanation of the integral bound
From the equation satisfied by the (u_n) we have

$$\begin{aligned} \int_0^T \int_Q H(x, Du_n) &= \int_0^T \int_Q (\alpha_n + \partial_t u_n) \\ &\leq \|\alpha_n\|_p + \int_Q (u_n(T) - u_n(0)) \leq C \end{aligned}$$

As H is coercive, (Du_n) is bounded in $L^r_\sharp$.

Now

$$\partial_t u_n = H(x, Du_n) - \alpha_n$$

where $(H(x, Du_n))$ is bounded in $L^1_\sharp$ while (α_n) is bounded in $L^p_\sharp$. So $(\partial_t u_n)$ is bounded in $L^1_\sharp$. □

The relaxed problem

Let $\mathcal{K}$ be the set $(u, \alpha) \in BV_\sharp((0,T) \times \mathbb{R}^d) \times L^p_\sharp((0,T) \times \mathbb{R}^d)$ such that

- $Du \in L^r_\sharp((0,T) \times \mathbb{R}^d)$
- $\alpha \geq 0$ a.e.
- $u(T, x) = u_T(x)$ and

$$-\partial_t u + H(x, Du) \leq \alpha \qquad \text{in } (0,T) \times \mathbb{R}^d \ .$$

  holds in the sense of distribution.

Note that $\mathcal{K}$ is a convex set.

The relaxed problem is

$$(\mathbf{HJ} - \mathbf{Rpb}) \qquad \inf_{(u,\alpha) \in \mathcal{K}} \left\{ \int_0^T \int_Q F^*(x, \alpha(t,x))\, dxdt - \int_Q u(0,x) dm_0(x) \right\}$$

The relaxed problem

Let $\mathcal{K}$ be the set $(u, \alpha) \in BV_\sharp((0,T)\times\mathbb{R}^d) \times L^p_\sharp((0,T)\times\mathbb{R}^d)$ such that

- $Du \in L^r_\sharp((0,T)\times\mathbb{R}^d)$
- $\alpha \geq 0$ a.e.
- $u(T,x) = u_T(x)$ and

$$-\partial_t u + H(x, Du) \leq \alpha \qquad \text{in } (0,T)\times\mathbb{R}^d\ .$$

  holds in the sense of distribution.

Note that $\mathcal{K}$ is a convex set.

The relaxed problem is

$$(\mathbf{HJ}-\mathbf{Rpb}) \qquad \inf_{(u,\alpha)\in\mathcal{K}} \left\{ \int_0^T \int_Q F^*(x, \alpha(t,x))\, dxdt - \int_Q u(0,x) dm_0(x) \right\}$$

Theorem

- *The relaxed problem has (**HJ-Rpb**) at least one minimum* (u, α), *where u is continuous and satisfies in the viscosity sense*

$$-\partial_t u + H(x, Du) \geq 0 \qquad \text{in } (0, T) \times \mathbb{R}^d$$

- *The value of the relaxed problem (**HJ-Rpb**) is equal to the value of the optimal control of the HJ equation (**HJ-Pb**) .*

Outline

The optimal control problem of HJ equation (**HJ-Pb**) can be rewritten as

$$\inf\left\{\int_0^T\int_Q F^*(x,-\partial_t u+H(x,Du))\;dxdt-\int_Q u(0,x)dm_0(x)\right\}$$

with constraint $u(T,\cdot)=u_T$.

This is a convex problem.

$\longrightarrow$ Suggests a duality approach.

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This is a convex problem.

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The Fenchel-Rockafellar duality Theorem

Let

- E and F be two normed spaces,
- $\Lambda \in \mathcal{L}_c(E, F)$ and
- $\mathcal{F} : E \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{G} : F \to \mathbb{R} \cup \{+\infty\}$ be two lsc proper convex maps.

Theorem

Under the qualification condition : there is x_0 such that $\mathcal{F}(x_0) < +\infty$ and $\mathcal{G}$ continuous at $\Lambda(x_0)$, we have

$$\inf_{x \in E} \{\mathcal{F}(x) + \mathcal{G}(\Lambda(x))\} = \max_{y^* \in F'} \{-\mathcal{F}^*(\Lambda^*(y^*)) - \mathcal{G}^*(-y^*)\}$$

as soon as the LHS is finite.

Proof of one inequality : for any $x \in E$ and $y^* \in F'$ we have

$$\mathcal{F}(x) + \mathcal{F}^*(\Lambda^*(y^*)) \geq \langle \Lambda^*(y^*), x \rangle$$

and

$$\mathcal{G}(\Lambda(x)) + \mathcal{G}^*(-y^*) \geq \langle -y^*, \Lambda(x) \rangle = -\langle \Lambda^*(y^*), x \rangle$$

Adding both inequalities gives

$$\mathcal{F}(x) + \mathcal{F}^*(\Lambda^*(y^*)) + \mathcal{G}(\Lambda(x)) + \mathcal{G}^*(-y^*) \geq 0 \ .$$

Rearranging

$$\mathcal{F}(x) + \mathcal{G}(\Lambda(x)) \geq -\{\mathcal{F}^*(\Lambda^*(y^*)) + \mathcal{G}^*(-y^*)\}$$

Therefore

$$\inf_{x \in E} \{\mathcal{F}(x) + \mathcal{G}(\Lambda(x))\} \geq \sup_{y^* \in F'} \{-\mathcal{F}^*(\Lambda^*(y^*)) - \mathcal{G}^*(-y^*)\}$$

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Remark : If $\bar{x}$ and $\bar{y}^*$ are respectively optimal in

$$\inf_{x\in E}\{\mathcal{F}(x)+\mathcal{G}(\Lambda(x))\} \text{ and } \sup_{y^*\in F'}\{-\mathcal{F}^*(\Lambda^*(y^*))-\mathcal{G}^*(-y^*)\}$$

then

$$\mathcal{F}(\bar{x})+\mathcal{F}^*(\Lambda^*(\bar{y}^*))=\langle\Lambda^*(\bar{y}^*),\bar{x}\rangle$$

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i.e.,

$$\Lambda^*(\bar{y}^*)\in\partial\mathcal{F}(\bar{x})$$

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The optimal control problem of HJ equation (**HJ-Pb**)

$$\inf\left\{\int_0^T\int_Q F^*(x,-\partial_t u+H(x,Du))\;dxdt-\int_Q u(0,x)dm_0(x)\right\}$$

(with constraint $u(T,\cdot)=u_T$) can be rewritten as

$$\inf_{u\in E}\{\mathcal{F}(u)+\mathcal{G}(\Lambda(u))\}$$

where

- $E=\mathcal{C}^1_\sharp([0,T]\times\mathbb{R}^d)$,
- $F=\mathcal{C}^0_\sharp([0,T]\times\mathbb{R}^d,\mathbb{R})\times\mathcal{C}^0_\sharp([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$,
- $\mathcal{F}(u)=-\int_Q m_0(x)u(0,x)dx \qquad \text{if } u(T,\cdot)=u_T \qquad (+\infty \text{ otherwise}).$
- $\mathcal{G}(a,b)=\int_0^T\int_Q F^*(x,-a(t,x)+H(x,b(t,x)))\;dxdt$.
- $\Lambda(u)=(\partial_t u,Du)$.

Then $\mathcal{F}$ is convex and lower semi-continuous on E while $\mathcal{G}$ is convex and continuous on F. Moreover Λ is bounded and linear.

The qualification condition is satisfied by $u(t,x) = u_T(x)$.

By Fenchel-Rockafellar duality theorem we have

$$\inf_{u \in E} \{\mathcal{F}(u) + \mathcal{G}(\Lambda(u))\} = \max_{(m,w) \in F'} \{-\mathcal{F}^*(\Lambda^*(m,w)) - \mathcal{G}^*(-(m,w))\}$$

where F' is the set of Radon measures $(m,w) \in M_\sharp([0,T] \times \mathbb{R}^d, \mathbb{R} \times \mathbb{R}^d)$ and $\mathcal{F}^*$ and $\mathcal{G}^*$ are the convex conjugates of $\mathcal{F}$ and $\mathcal{G}$.

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Recall that

$$\mathcal{F}(u) = -\int_Q m_0(x)u(0,x)dx \qquad \text{if } u(T,\cdot) = u_T \qquad (+\infty \text{ otherwise})$$

and

$$\mathcal{G}(a,b) = \int_0^T \int_Q F^*(x, -a(t,x) + H(x, b(t,x)))\, dxdt\ .$$

Lemma

$$\mathcal{F}^*(\Lambda^*(m,w)) = \begin{cases} \int_Q u_T(x)dm(T,x) & \text{if } \partial_t m + \text{div}(w) = 0,\ m(0) = m_0 \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\mathcal{G}^*(m,w) = \begin{cases} \int_0^T \int_Q -F(x,m) - mH^*(x, -\frac{w}{m})\, dtdx & \text{if } (m,w) \in L^1_\sharp \\ +\infty & \text{otherwise} \end{cases}$$

Consequence of the Lemma :

$$\max_{(m,w)\in F'} \{-\mathcal{F}^*(\Lambda^*(m,w)) - \mathcal{G}(m,w)\}$$
$$= \max \left\{ \int_0^T \int_Q -F(x,m) - mH^*(x, -\frac{w}{m})\, dtdx - \int_Q u_T(x) m(T,x)\, dx \right\}$$

where the maximum is taken over the $L^1_\sharp$ maps (m,w) such that $m \geq 0$ a.e. and

$$\partial_t m + \mathrm{div}(w) = 0, \ m(0) = m_0$$

Idea of proof :

$$\begin{aligned}
\mathcal{F}^*(\Lambda^*(m,w)) &= \sup_{u(T)=u_T} \langle \Lambda^*(m,w), u\rangle - \mathcal{F}(u) \\
&= \sup_{u(T)=u_T} \langle (m,w), \Lambda(u)\rangle + \int_Q m_0(x)u(0,x)dx \\
&= \sup_{u(T)=u_T} \int_0^T\int_Q (m\partial_t u + \langle w, Du\rangle) + \int_Q m_0(x)u(0,x)dx \\
\text{``}&= \sup_{u(T)=u_T} \int_0^T\int_Q -u(-\partial_t m + \mathrm{div}(w)) \\
&\qquad + \int_Q m(T)u_T + \int_Q (m_0 - m(0))u(0)dx\text{''} \\
&= \begin{cases} \displaystyle\int_Q u_T(x)dm(T,x) & \text{if } \partial_t m + \mathrm{div}(w) = 0,\ m(0) = m_0 \\ +\infty & \text{otherwise} \end{cases}
\end{aligned}$$

□

The dual of the optimal control of HJ eqs

Theorem

*The dual of the optimal control of HJ (**HJ-Pb**) equation is given by*

$$(\mathbf{K}-\mathbf{Pb}) \qquad \inf\left\{\int_0^T\int_Q mH^*(x,-\frac{w}{m})+F(x,m)\;dxdt+\int_Q u_T(x)m(T,x)dx\right\}$$

where the infimum is taken over the pairs
$(m,w)\in L^1_\sharp((0,T)\times\mathbb{R}^d)\times L^1_\sharp((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ *such that*

$$\partial_t m+\mathrm{div}(w)=0,\;\; m(0)=m_0$$

in the sense of distributions. Moreover this dual problem has a unique minimum.

(**K-Pb**) as an optimal control problem for Kolmogorov equation :

Set $v = w/m$. Then (**K-Pb**) becomes

$$\inf \left\{ \int_0^T \int_Q mH^*(x, -v) + F(x, m)\, dxdt + \int_Q u_T(x) m(T, x) dx \right\}$$

where the infimum is taken over the pairs (m, v) such that

$$\partial_t m + \mathrm{div}(mv) = 0, \;\; m(0) = m_0$$

in the sense of distributions.

(**K-Pb**) as an optimal control problem for Kolmogorov equation :

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$$\partial_t m+\mathrm{div}(mv)=0,\ \ m(0)=m_0$$

in the sense of distributions.

Outline

Back to the (*MFG*) system

We now study the weak solutions of the (*MFG*) system

$$(MFG) \qquad \left\{ \begin{array}{ll} (i) & -\partial_t u + H(x, Du) = f(x, m(x,t)) \\ (ii) & \partial_t m - \mathrm{div}(m D_p H(x, Du)) = 0 \\ (iii) & m(0) = m_0, \; u(x,T) = u_T(x) \end{array} \right.$$

and explain the relation with the two optimal control problems

- for the HJ equation (problem (**HJ-Pb**))
- for the Kolmogorov equation (problem (**K-Pb**))

Definition

A pair $(m, u) \in L^1((0,T)\times\mathbb{R}^d) \times BV((0,T)\times\mathbb{R}^d)$ is a weak solution of (MFG) if

(i) u is continuous in $[0,T]\times\mathbb{R}^d$ with $Du \in L^r((0,T)\times\mathbb{R}^d, m)$ and $mD_pH(x,Du) \in L^1_\sharp$

(ii) Inequality $\quad -\partial_t u + H(x,Du) \leq f(x,m) \quad$ holds in the sense of distribution, with $u(T,x) = u_T(x)$ in the sense of trace,

(iii) Equality $\quad \partial_t m - \mathrm{div}(mD_pH(x,Du)) = 0 \quad$ holds in the sense of distribution in $(0,T)\times\mathbb{R}^d$ and $m(0) = m_0$,

(iv) Equality $\displaystyle\int_0^T\int_Q m\left(\partial_t u^{ac} - \langle Du, D_pH(x,Du)\rangle\right) = \int_Q m(T)u_T - m_0 u(0)$ holds.

(where $\partial_t u^{ac}$ is the a.c. part of the measure $\partial_t u$).

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A pair $(m, u) \in L^1((0,T) \times \mathbb{R}^d) \times BV((0,T) \times \mathbb{R}^d)$ is a weak solution of (*MFG*) if

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Remarks

- If (ii) holds with an equality and if u is in $W^{1,1}$, then (iii) implies (iv).
- Conditions (ii) and (iv) imply that
  $-\partial_t u^{ac}(t,x) + H(x, Du(t,x)) = f(x, m(t,x)) \quad$ holds a.e. in $\{m > 0\}$.

Existence for the (*MFG*) system

Theorem

There is a weak solution (m, u) of (MFG) such that u is locally Hölder continuous in $[0, T) \times \mathbb{R}^d$ and which satisfies in the viscosity sense

$$-\partial_t u + H(x, Du) \geq 0 \qquad \text{in } (0, T) \times \mathbb{R}^d \ .$$

Idea of proof :

- Let (m, w) is a minimizer of (**K-Pb**) and (u, α) is a minimizer of (**HJ-Rpb**) such that u is continuous. Then one can show that (m, u) is a solution of mean field game system (*MFG*) and $w = -mD_pH(\cdot, Du)$ while $\alpha = f(\cdot, m)$ a.e..
- Conversely, any solution of (*MFG*) such that u is continuous is such that the pair $(m, -mD_pH(\cdot, Du))$ is the minimizer of (**K-Pb**) while $(u, f(\cdot, m))$ is a minimizer of (**HJ-Rpb**).

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Uniqueness for the (*MFG*) system

Theorem

Let (m, u) and (m', u') be two weak solutions of (MFG). Then $m = m'$ and $u = u'$ in $\{m > 0\}$.

Moreover, if u satisfies the additional condition

$$(*) \qquad -\partial_t u + H(x, Du) \geq 0 \qquad \text{in } (0, T) \times \mathbb{R}^d ,$$

in the viscosity sense, then $u \geq u'$.

Remark : In particular, if we add condition (*) to the definition of weak solution of (*MFG*), then the weak solution exists and is unique.

Uniqueness for the (*MFG*) system

Theorem

Let (m, u) and (m', u') be two weak solutions of (MFG). Then $m = m'$ and $u = u'$ in $\{m > 0\}$.

Moreover, if u satisfies the additional condition

$$(*) \qquad -\partial_t u + H(x, Du) \geq 0 \qquad \text{in } (0, T) \times \mathbb{R}^d ,$$

in the viscosity sense, then $u \geq u'$.

Remark : In particular, if we add condition (*) to the definition of weak solution of (*MFG*), then the weak solution exists and is unique.

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Idea of proof :

- In a suitable sense,

$$u(t,x) = \inf_{\xi(\cdot),\ \xi(t)=x} \left\{ \int_t^T \left(L(\xi(s),\xi'(s)) + \alpha(s,\xi(s))\right) ds + u_T(\xi(T)) \right\}$$

  where $L(x,y) := \sup_z \langle z, y\rangle - H(x,-z)$ is as smooth as H.

- the optimal solutions of the above problems can be built by the pair (m,w) where $w = -mD_pH(\cdot, Du)$.

Outline

Summary : Throughout these lectures, we have seen

- how the MFG system is a natural formulation for Nash equilibria for differential games with infinitely many players,
- that existence/uniqueness of MFG systems are well understood for second order problems,
- the relation between the 1rst order, nonlocal MFG systems and semiconcavity,
- the relation between the 1rst order, local MFG systems and optimal control of PDEs

Open problems

- Regularity of solutions for 1rst order, local MFG systems.
- Existence/uniqueness for the MFG system of congestion type ($\alpha \in (0,2)$)

$$\left\{\begin{array}{ll} (i) & -\partial_t u + \dfrac{|Du|^2}{2m^\alpha} = f(x, m(x,t)) \\ (ii) & \partial_t m - \mathrm{div}(m^{1-\alpha} Du)) = 0 \\ (iii) & m(0) = m_0,\ u(x,T) = u_T(x) \end{array}\right.$$

- Application to $N-$player games,
- Long-time behavior as $T \to +\infty$,
- Periodic solutions...

Thank you !