Quantum Feedback Control - Lecture 2

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QUANTUM COMPUTATION &
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Lecture 1 *Introduction and basic concepts*
Quantum technology, quantum control, postulates of quantum mechanics, quantum probability.

Lecture 2 *Measurement feedback quantum control*
Open quantum systems, quantum stochastic models, quantum filtering, optimal measurement feedback control, risk-sensitive quantum control.

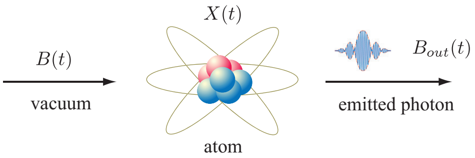
Lecture 3 *Coherent feedback quantum control*
Quantum feedback networks, quantum dissipative systems, control by interconnection, linear quantum systems.

Lecture 2 - Outline

1. Quantum Stochastic Models
2. Quantum Filtering
3. Optimal Measurement Feedback Quantum Control

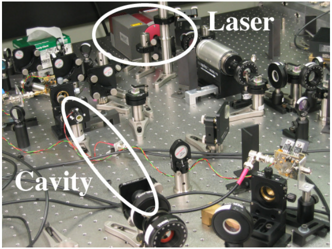
Quantum Stochastic Models

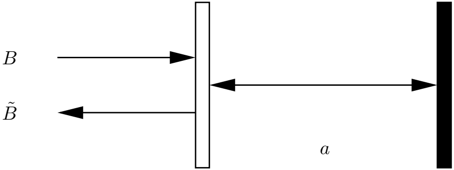
Recall that an *open quantum system* is a system interacting with an external environment. A basic example is an atom in an electromagnetic field.



We now describe dynamical models for open quantum systems in terms of quantum stochastic models in continuous time. Upon integration and expectation, these models yield quantum operation descriptions.

Quantum stochastic models describe open systems with inputs and outputs.





external free field with
input and output components

cavity mode

Recall that for a *closed* oscillator:

Unitary dynamics (Schrodinger equation)

$$\dot{U}(t) = -i\omega a^* a U(t), \quad U(0) = I$$

Heisenberg motion $a(t) = U^*(t) a U(t)$

$$\frac{d}{dt} a(t) = -i\omega a(t)$$

with solution

$$a(t) = e^{-i\omega t} a$$

The commutation relations are preserved:

$$[a(t), a^*(t)] = [a, a^*] = 1$$

Quantum fields (boson)

Infinitely many quantum oscillators $b(t)$ (or $b(x)$ or $b(\omega)$)

Singular commutation relations

$$[b(t), b^*(t')] = \delta(t - t')$$

Quantum stochastic representation

$$B(t) = \int_0^t b(s) ds$$

Ito product rule [more to come on this]

$$dB(t) dB^*(t) = dt$$

Open quantum harmonic oscillator

Single oscillator a interacting with field $b(t)$ - energy exchange:

$$H_{int} = i\sqrt{\gamma}(b^*(t)a - a^*b(t))$$

Dynamics (Ito form) [more to come on this]

$$dU(t) = \{\sqrt{\gamma} a dB^*(t) - \sqrt{\gamma} a^* dB(t) \\ -\frac{\gamma}{2} a^* a dt - i\omega a^* a dt\} U(t),$$

Motion of oscillator mode $a(t) = U^*(t) a U(t)$

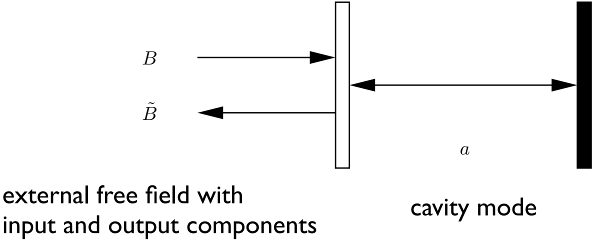
$$\dot{a}(t) = -(\frac{\gamma}{2} + i\omega) a(t) - \sqrt{\gamma}\, b(t)$$

Again, the commutation relations are preserved

$$[a(t), a^*(t)] = [a, a^*] = 1$$

The output field $B_{out}(t) = U^*(t)B(t)U(t)$ is given by

$$b_{out}(t) = \sqrt{\gamma}\, a(t) + b(t)$$



Quantum stochastic processes

The three fundamental (integrated) field operators are

$$B(t) = \int_0^t b(s)ds \quad \text{(annihilation)}$$

$$B^*(t) = \int_0^t b^*(s)ds \quad \text{(creation)}$$

$$\Lambda(t) = \int_0^t b^*(s)b(s)ds \quad \text{(counting)}$$

The amplitude quadrature

$$Q(t) = B(t) + B^*(t)$$

is self-adjoint, and commutes with itself at different times ($[Q(t), Q(s)] = 0$), and so by the spectral theorem it turns out that $Q(t)$ is equivalent to a classical Wiener process (with respect to the vacuum state).

The phase quadrature

$$P(t) = -i(B(t) - B^*(t))$$

which is also equivalent to a classical Wiener process, but note that $[Q(t), P(t)] \neq 0$.

Let $\alpha_1(t)$ and $\alpha_2(t)$ be operator-valued *adapted* processes, i.e. independent of future field operators.

Ito quantum stochastic integrals are defined in terms of forward increments:

$$\begin{aligned} I(t) &= \int_0^t \alpha_1(s) dB(s) + \int_0^t \alpha_2(s) dB^*(s) \\ &\approx \sum_j \alpha_1(s_j)(B(s_{j+1}) - B(s_j)) + \sum_j \alpha_2(s_j)(B^*(s_{j+1}) - B^*(s_j)) \end{aligned}$$

The Ito rule is expressed in terms of four products (ignoring $\Lambda(t)$)

$$
\begin{aligned}
dB(t)dB(t) = 0, \quad dB(t)dB^*(t) = dt, \\
dB^*(t)dB(t) = 0, \quad dB^*(t)dB^*(t) = 0.
\end{aligned}
$$

This Ito table is valid for vacuum and coherent field states. Ito tables for squeezed and thermal field states have more non-zero terms.
An important property is that for an adapted process $\alpha(t)$, we have

$$
[\alpha(t), dB(t)] = 0 = [\alpha(t), dB^*(t)].
$$

The expected value of the above stochastic integral is zero in the vacuum state $|\phi\rangle$:

$$
\mathbb{E}_\phi[I(t)] = 0.
$$

Suppose

$$J(t) = \int_0^t \beta_1(s) dB(s) + \int_0^t \beta_2(s) dB^*(s)$$

The product rule is

$$\begin{aligned} d(IJ) &= (dI)J + I(dJ) + (dI)(dJ) \\ &= \alpha_1 J dB + \alpha_2 J dB^* + I\beta_1 dB + I\beta_2 dB^* + \alpha_1\beta_2 dt \\ &= (\alpha_1 J + I\beta_1) dB + (\alpha_2 J + I\beta_2) dB^* + \alpha_1\beta_2 dt; \end{aligned}$$

that is,

$$\begin{aligned} I(t)J(t) &= \int_0^t (\alpha_1(s)J(s) + I(s)\beta_1(s)) dB(s) \\ &\quad + \int_0^t (\alpha_2(s)J(s) + I(s)\beta_2(s)) dB^*(s) \\ &\quad + \int_0^t \alpha_1(s)\beta_2(s) ds. \end{aligned}$$

Note that the order is important in these expressions.

Open quantum systems

An open system G coupled to n field channels is specified by three physical parameters:

$$G = (S, L, H)$$

Here:

- H is the intrinsic Hamiltonian
- L is a vector of coupling operators
- S is a unitary matrix of operators.

Two level system:

$$G_{qbit} = (I, \sqrt{\kappa}\, \sigma_-, \frac{\omega}{2}\sigma_z)$$

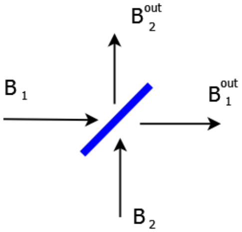
Harmonic oscillator:

$$G_{osc} = (I, \sqrt{\gamma}\, a, \frac{\omega}{2} a^* a)$$

Beamsplitter:

$$G_{bs} = (S, 0, 0)$$

$$\begin{bmatrix} B_1^{out}(t) \\ B_2^{out}(t) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}$$



Quantum Stochastic Differential Equation

$$dU(t) = \{L dB^*(t) - L^* dB(t) - (\frac{1}{2}L^*L + iH)dt\}U(t)$$

$$dX = (-i[X,H] + \mathscr{L}_L(X))dt + dB^*[X,L] + [L^*,X]dB$$

$$dY = (L + L^*)dt + d(B + B^*)$$

quantum expectation

quantum conditional expectation

Master Equation

$$\frac{d\rho}{dt} = -i[H,\rho] + \mathscr{L}_L^*(\rho)$$

classical expectation

Quantum Filter (Quantum Trajectories)

$$\begin{aligned} d\hat{\rho} \quad &= \quad (-i[H,\hat{\rho}] + \mathscr{L}_L^*(\hat{\rho}))dt \\ &\quad + (L\hat{\rho} + \hat{\rho}L^* - \mathrm{tr}[(L+L^*)\hat{\rho}]\hat{\rho})(dY - \mathrm{tr}[(L+L^*)\hat{\rho}]dt) \end{aligned}$$

Schrodinger equation: (take $n = 1$ for simplicity)

$$dU(t) = \{(S - I)d\Lambda(t) + dB^*(t)L - L^*SdB(t) - (\frac{1}{2}L^*L + iH)dt\}U(t)$$

[Hudson-Parthasarathy (1984), Gardiner-Collett (1985)]

Heisenberg dynamics of system operators:

$$X(t) = U^*(t)(X \otimes I)U(t).$$

Using the quantum Ito rule, we have

$$\begin{aligned} dX(t) &= (-i[X(t), H(t)] + \mathcal{L}_L(X(t)))dt \\ &\quad + dB^*(t)S^\dagger(t)[X(t), L(t)] + [L^*(t), X(t)]S(t)dB(t) \\ &\quad + (S^\dagger(t)X(t)S(t) - X(t))d\Lambda(t) \end{aligned}$$

Here, $H(t) = U^*(t)(H \otimes I)U(t)$ and $L(t) = U^*(t)(L \otimes I)U(t)$.

The master equation for a system density operator ρ is obtained by averaging out the quantum noise:

$$\mathrm{tr}[\rho(t)X] = \mathbb{E}[X(t)]$$

Master equation: (vacuum state)

$$\dot{\rho}(t) = \mathcal{L}^*(\rho(t))$$

where

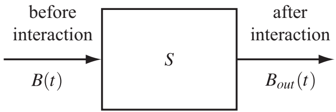
$$\mathcal{L}^*(\rho) = \frac{1}{2}[L, \rho L^*] + \frac{1}{2}[L\rho, L^*] + i[\rho, H].$$

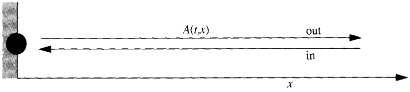
The output field is defined by

$$B_{out}(t) = U^*(t)(I \otimes B(t))U(t).$$

By the quantum Ito rule,

$$dB_{out}(t) = L(t)dt + dB(t).$$





Quantum probability model for open quantum systems:

- A quantum probability space $(\mathscr{S} \otimes \mathscr{F}, \rho_0 \otimes \Phi)$, where
  - $\mathscr{S}$ are system operators (acting on $\mathfrak{H}$),
  - $\mathscr{F}$ are field (environment) operators (acting on $\mathfrak{F}$),
  - ρ_0 is the initial system state, and
  - Φ is the vacuum state for the field.
- A Schrodinger equation with unitary solution $U(t)$.

Two-level system

$$G_{qbit} = (I, \sqrt{\kappa}\,\sigma_-, \frac{\omega}{2}\sigma_z)$$

Equations of motion for Pauli matrices:

$$\begin{aligned}
d\sigma_x(t) &= (-\omega\sigma_y(t) - \frac{\kappa}{2}\sigma_x(t))dt \\
&\quad +\sqrt{\kappa}(dB^*(t)\sigma_z(t) + \sigma_z(t)dB(t)) \\
d\sigma_y(t) &= (\omega\sigma_x(t) - \frac{\kappa}{2}\sigma_y(t))dt \\
&\quad -i\sqrt{\kappa}(\sigma_z(t)dB^*(t) - \sigma_z(t)dB(t)) \\
d\sigma_z(t) &= (-\kappa\sigma_z(t) - \kappa)dt \\
&\quad -2\sqrt{\kappa}(dB^*(t)\sigma_-(t) + \sigma_+(t)dB(t))
\end{aligned}$$

Output field:

$$dB_{out}(t) = \sqrt{\kappa}\,\sigma_-(t)dt + dB(t).$$

The above equations are quantum stochastic differential equations.

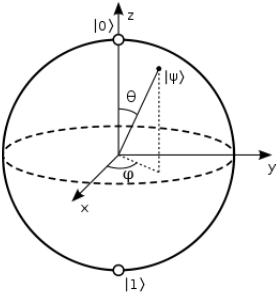
Bloch vector

$$r(t) = (x(t), y(t), z(t))$$

representation for density operator

$$\rho(t) = \frac{1}{2}(I + x(t)\sigma_x + y(t)\sigma_y + z(t)\sigma_z),$$

where $x(t) = \mathbb{E}[\sigma_x(t)] = \mathrm{tr}[\rho(t)\sigma_x]$, $y(t) = \mathbb{E}[\sigma_y(t)] = \mathrm{tr}[\rho(t)\sigma_y]$, and $z(t) = \mathbb{E}[\sigma_z(t)] = \mathrm{tr}[\rho(t)\sigma_z]$.

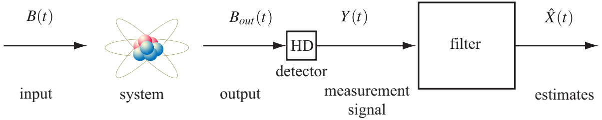


Master equation:

$$
\begin{aligned}
\dot{x}(t) &= -\frac{\kappa}{2}x(t) - \omega y(t), \\
\dot{y}(t) &= -\frac{\kappa}{2}y(t) + \omega x(t), \\
\dot{z}(t) &= -\kappa z(t) - \kappa.
\end{aligned}
$$

The master equation is an ordinary differential equation.

Quantum Filtering



Continuous monitoring of the output field (e.g. homodyne detection) approximates measurement of the observable

$$Y(t) = B_{out}(t) + B^*_{out}(t)$$

In differential form,

$$dY(t) = (L(t) + L^*(t))dt + dZ(t),$$

where $Z(t) = B(t) + B^*(t)$.

The quantum conditional expectation

$$\hat{X}(t) = \pi_t(X) = \mathbb{E}[X(t) | Y(s), 0 \leq s \leq t]$$

is well-defined, since $X(t)$ commutes with $Y(s)$, $0 \leq s \leq t$.

Using the quantum stochastic calculus, the conditional expectation is given by the quantum filter:

$$\begin{aligned} d\pi_t(X) &= \pi_t(\mathcal{L}(X))dt \\ &\quad +(\pi_t(XL + L^*X) - \pi_t(X)\pi_t(L + L^*))(dY(t) - \pi_t(L + L^*)dt) \end{aligned}$$

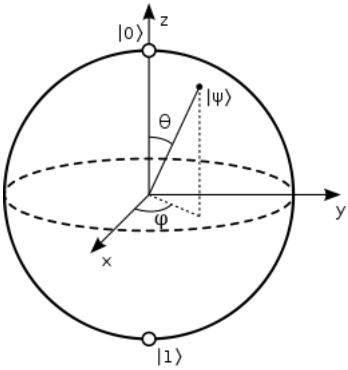
[Belavkin (1993), Carmichael (1993)]

Conditional density operator $\hat{\rho}(t)$ is defined by

$$\pi_t(X) = \mathrm{tr}[\hat{\rho}(t)X]$$

For a two-level system, we use Bloch sphere coordinates:

$$\hat{\rho}(t) = \frac{1}{2}(I + \hat{x}(t)\sigma_x + \hat{y}(t)\sigma_y + \hat{z}(t)\sigma_z),$$



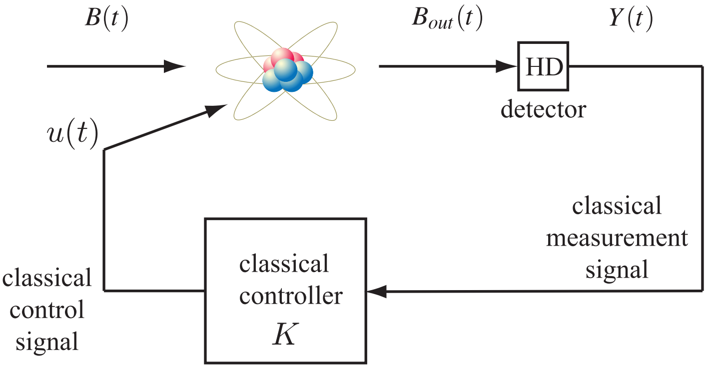
The quantum filter is then given by

$$
\begin{aligned}
d\hat{x}(t) &= (-\omega\hat{y}(t) - \frac{\kappa}{2}\hat{x}(t))dt \\
&\qquad + \sqrt{\kappa}\,(1 + \hat{z}(t) - \hat{x}^2(t))dW(t) \\
d\hat{y}(t) &= (\omega\hat{x}(t) - \frac{\kappa}{2}\hat{y}(t))dt \\
&\qquad + \sqrt{\kappa}\,\hat{x}(t)\hat{y}(t)dW(t), \\
d\hat{z}(t) &= (-\kappa\hat{z}(t) - \kappa)dt \\
&\qquad - \sqrt{\kappa}\;\hat{x}(t)(1 + \hat{x}(t))dW(t).
\end{aligned}
$$

The innovations process is given by $dW(t) = dY(t) - \hat{x}(t)dt$.

The quantum filter is driven by the measurement signal $Y(t)$.

Optimal Measurement Feedback Quantum Control



Quantum optimal control (measurement feedback)

For a measurement feedback controller K define

$$J(K) = \mathbb{E}[\int_0^T C_1(u(s))ds + C_2(T)]$$

where [two-level system]

$$C_1(u) = \begin{pmatrix} \frac{c_1}{2}|u|^2 & 0 \\ 0 & 1 + \frac{c_1}{2}|u|^2 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix},$$

and $\mathbb{E}$ denotes quantum expectation with respect to the underlying states for the system and field (vacuum).

The *measurement feedback quantum optimal control problem* is to minimize $J(K)$ over all measurement feedback controllers K.

Using properties of conditional expectation, the cost function can be expressed in terms of the quantum conditional expectation

$$\begin{aligned} J(K) &= \mathbb{E}[\int_0^T \pi_s(C_1(u(s)))ds + \pi_T(C_2)] \\ &= \mathbf{E}[\frac{1}{2}\int_0^T (1 - \hat{z}(t) + c_1|u(t)|^2)dt + \frac{c_2}{2}(1 - \hat{z}(T))]. \end{aligned}$$

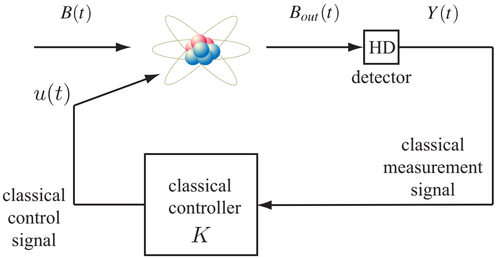
This converts a quantum measurement feedback problem to a classical full information control problem that can be solved using standard classical optimal control methods.

Optimal measurement feedback controller:

$$
\begin{aligned}
d\pi_t(X) &= \pi_t(\mathcal{L}^{u(t)}(X))dt \\
&\quad +(\pi_t(XL+L^*X) - \pi_t(X)\pi_t(L+L^*))(dY(t) - \pi_t(L+L^*)dt) \\
u(t) &= \mathbf{u}^\star(\pi_t, t)
\end{aligned}
$$

Note the *separation structure*:

- estimation part (filter, the equation for π_t)
- control part ($\mathbf{u}^\star$)



[Belavkin (1983), Doherty-Jacobs (1999), James (2005)]

Risk-sensitive quantum optimal control (measurement feedback)

For a measurement feedback controller K define the risk-sensitive performance index

$$J(K) = \mathbb{E}[R^*(T)e^{\mu C_2(T)}R(T)]$$

where

$$\dot{R}(t) = \frac{\mu}{2}C_1(u(t))R(t), \quad R(0) = I.$$

The solution to the problem involves a *quantum information state* given by a modified Schrodinger equation that includes the cost term. This appears to be new to physics. This state depends on:

- information gained as the system evolves (knowledge), and
- the objective of the closed loop feedback system (purpose).