Exponential mapping for the sub-Riemannian problem on the Engel group

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Summary

The left–invariant sub-Riemannian problem on the Engel group is considered. This problem is very important as nilpotent approximation of nonholonomic systems in four–dimensional space with two–dimensional control (see [1, 2]), for instance of a system which describes movement of mobile trailer robot.
Parameterization of extremal curves by elliptic Jacobi's functions was obtained. Discrete symmetries of Exponential mapping were considered and the corresponding Maxwell sets were constructed. Thus global bound of the cut time (i. e., the time of loss of *global* optimality) was found which gives necessary optimality conditions for extremal curves. The first conjugate time (i. e., the time of loss of *local* optimality) was investigated. It was shown that the function that gives the upper bound of the cut time provides the lower bound of the first conjugate time.

Optimal control problem statement

$$\dot q = \begin{pmatrix}\dot x\\ \dot y\\ \dot z\\ \dot v\end{pmatrix} = u_1\begin{pmatrix}1\\0\\-\frac{y}{2}\\0\end{pmatrix} + u_2\begin{pmatrix}0\\1\\\frac{x}{2}\\\frac{x^2+y^2}{2}\end{pmatrix},\quad q\in\mathbb{R}^4,\quad u\in\mathbb{R}^2, \tag{1}$$

$$q(0) = q_0 = (x_0, y_0, z_0, v_0),\quad q(t_1) = q_1 = (x_1, y_1, z_1, v_1), \tag{2}$$

$$l = \int_0^{t_1}\sqrt{u_1^2+u_2^2}\,dt \to \min. \tag{3}$$

Since the problem is invariant under left shifts on the Engel group, we can assume that the initial point is identity of the group $q_0 = (x_0, y_0, z_0, v_0) = (0, 0, 0, 0)$.

Hamiltonian system

Existence of optimal solutions of problem **(1)–(3)** is implied by Filippov's theorem [4]. By Cauchy–Schwarz inequality, it follows that sub-Riemannian length minimization problem **(3)** is equivalent to action minimization problem:

$$\int_0^{t_1}\frac{u_1^2+u_2^2}{2}\,dt \to \min. \tag{4}$$

Pontryagin's maximum principle [3, 4] was applied to the resulting optimal control problem **(1)**, **(2)**, **(4)**. Abnormal extremals were parameterized. Denote vector fields at the controls in the right-hand side of system **(1)**:

$$X_1 = (1, 0, -\frac{y}{2}, 0)^T,\qquad X_2 = (0, 1, \frac{x}{2}, \frac{x^2+y^2}{2})^T,$$

and the corresponding linear on fibers of the cotangent bundle T^*M Hamiltonians $h_i(\lambda) = \langle\lambda, X_i(q)\rangle$, $\lambda\in T^*M$, $i = 1, 2$.
Normal extremals satisfy the Hamiltonian system

$$\dot\lambda = \vec H(\lambda),\qquad \lambda\in T^*M, \tag{5}$$

where $H = \frac{1}{2}\left(h_1^2 + h_2^2\right)$.
The normal Hamiltonian system **(5)** is given, in certain natural coordinates, as follows on a level surface $\left\{\lambda\in T^*M \mid H = \frac{1}{2}\right\}$:

$$\begin{aligned}&\dot\theta = c,\qquad \dot c = -\alpha\sin\theta,\qquad \dot\alpha = 0,\\ &\dot q = \cos\theta\, X_1(q) + \sin\theta\, X_2(q),\qquad q(0) = q_0.\end{aligned} \tag{6}$$

Parameterization of normal extremal trajectories

The family of all normal extremals is parameterized by points of the phase cylinder of pendulum

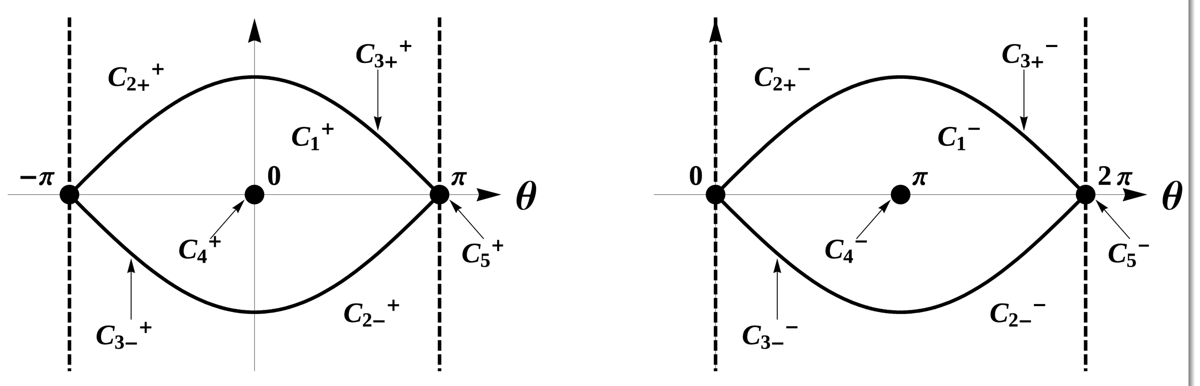
$$C = \left\{\lambda\in T^*_{q_0}M \mid H(\lambda) = \frac{1}{2}\right\} = \left\{(\theta, c, \alpha) \mid \theta\in S^1,\ c, \alpha\in\mathbb{R}\right\},$$

and is given by the exponential mapping

$$\begin{aligned}&\mathrm{Exp}\ :\ N = C\times\mathbb{R}_+\to M,\\ &\mathrm{Exp}(\lambda, t) = q_t = (x_t, y_t, z_t, v_t).\end{aligned}$$

Energy integral of pendulum **(6)** is expressed by $E = \frac{c^2}{2} - \alpha\cos\theta$. The cylinder C has the following stratification corresponding to the particular type of trajectories of the pendulum:

$$\begin{aligned}&C = \cup_{i=1}^7 C_i,\quad C_i\cap C_j = \emptyset,\ i\neq j,\quad \lambda = (\theta, c, \alpha),\\
&C_1 = \{\lambda\in C \mid \alpha\neq 0, E\in(-|\alpha|, |\alpha|)\},\\
&C_2 = \{\lambda\in C \mid \alpha\neq 0, E\in(|\alpha|, +\infty)\},\\
&C_3 = \{\lambda\in C \mid \alpha\neq 0, E = |\alpha|, c\neq 0\},\\
&C_4 = \{\lambda\in C \mid \alpha\neq 0, E = -|\alpha|\},\\
&C_5 = \{\lambda\in C \mid \alpha\neq 0, E = |\alpha|, c = 0\},\\
&C_6 = \{\lambda\in C \mid \alpha = 0,\ c\neq 0\},\\
&C_7 = \{\lambda\in C \mid \alpha = c = 0\}.\end{aligned}$$



Extremal trajectories were parameterized by elliptic Jacobi's functions for any $\lambda\in C$ in the paper [5]. This parameterization was obtained in natural coordinates (φ, k, α), which rectify the equations of pendulum: $\dot\varphi = 1$, $\dot k = 0$, $\dot\alpha = 0$.

Cut time

In order to investigate the optimality question for discovered extremal trajectories descrete group of symmetries of exponential mapping were considered:

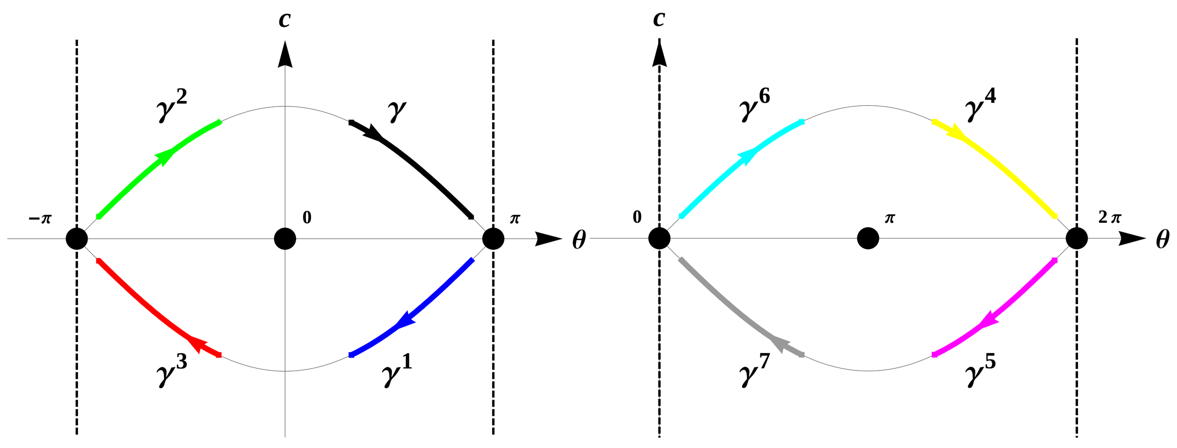


Figure: Action of the symmetries: $\varepsilon^i(\gamma) = \gamma^i$, $i = 1\dots7$

Thus the corresponding Maxwell sets were constructed. The point of sub-Riemannian geodesic is called Maxwell point if two different extremal trajectories come to this point at the same time called Maxwell time $t^1_{\mathrm{MAX}}\ :\ C\to(0, +\infty]$:

$$\begin{aligned}&\lambda\in C_1\quad\Rightarrow\quad t^1_{\mathrm{MAX}} = \min(2p^1_z, 4K)/\sigma,\\
&\lambda\in C_2\quad\Rightarrow\quad t^1_{\mathrm{MAX}} = 2Kk/\sigma,\\
&\lambda\in C_6\quad\Rightarrow\quad t^1_{\mathrm{MAX}} = \frac{2\pi}{|c|},\\
&\lambda\in C_3\cup C_4\cup C_5\cup C_7\quad\Rightarrow\quad t^1_{\mathrm{MAX}} = +\infty.\end{aligned}$$

where $\sigma = \sqrt{|\alpha|}$; $K(k) = \displaystyle\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-k^2\sin^2 t}}$;
$p^1_z(k)\in(K(k), 3K(k))$ is the first positive root of the function $f_z(p, k) = \mathrm{dn}\,p\ \mathrm{sn}\,p + (p - 2\,\mathrm{E}(p))\,\mathrm{cn}\,p$; $\mathrm{E}(p) = \int_0^p \mathrm{dn}^2\,t\,dt$; $\mathrm{sn}\,p$, $\mathrm{cn}\,p$ and $\mathrm{dn}\,p$ are Jacobi's functions [6].
It is well known that geodesic cannot be optimal after Maxwell point. Thus Maxwell time gives upper bound of the cut time:

$$t_{\mathrm{cut}}(\lambda) = \sup\{t > 0 \mid \mathrm{Exp}(\lambda, s)\text{ is global optimal for } s\in[0, t]\}.$$

Theorem (1)

For any $\lambda\in C$

$$t_{\mathrm{cut}}(\lambda)\leq t^1_{\mathrm{MAX}}(\lambda). \tag{7}$$

The bound of the cut time obtained in the Theorem (1) is sharp for the equilibrium of the pendulum, i.e. the corresponding trajectories are optimal to infinity. Analysis of the global structure of the exponential map shows that found estimate is not exact in general case.

Conjugate time

The local optimality of extremal trajectories was studied. A point $q_t = \mathrm{Exp}(\lambda, t)$ is called *a conjugate point* for q_0 if $\nu = (\lambda, t)$ is a critical point of the exponential mapping and that is why q_t is the corresponding critical value:

$$d_\nu\,\mathrm{Exp}\ :\ T_\nu N\to T_{q_t}M\text{ is degenerate},$$

i. e.,

$$\frac{\partial(x, y, z, v)}{\partial(\theta, c, \alpha, t)}(\nu) = 0.$$

Note that t in this case is called *a conjugate time* along extremal trajectory $q_s = \mathrm{Exp}(\lambda, s)$, $s\geq 0$.
Due to the strong Legendre condition, for any normal extremal there exists a countable family of conjugate points. Besides, conjugate times are separated from each other. The first conjugate time along the trajectory $\mathrm{Exp}(\lambda, s)$ is denoted by

$$t^1_{\mathrm{conj}} = \min\{t > 0 \mid t\text{ is a conjugate time along }\mathrm{Exp}(\lambda, s),\ s\geq 0\}.$$

The trajectory $\mathrm{Exp}(\lambda, s)$ loses local optimality at the moment $t = t^1_{\mathrm{conj}}(\lambda)$ (see [4]). The following lower bound of the first conjugate time was proved.

Theorem (2)

For any $\lambda\in C$

$$t^1_{\mathrm{conj}}(\lambda)\geq t^1_{\mathrm{MAX}}(\lambda). \tag{8}$$

Using the estimate of cut time, Theorem (1), and the estimate of conjugate time, Theorem (2), the global structure of the exponential map in sub-Riemannian problem on the Engel group was described. So this problem was reduced to solving the system of algebraic equations.

System of algebraic equations

In order to compute the optimal trajectory for a given terminal point (x_1, y_1, z_1, v_1), the following system of algebraic equations should be solved:

$$\begin{cases}x(u_1, u_2, k, \alpha) = x_1,\\ y(u_1, u_2, k, \alpha) = y_1,\\ z(u_1, u_2, k, \alpha) = z_1,\\ v(u_1, u_2, k, \alpha) = v_1.\end{cases} \tag{9}$$

Using one symmetry (dilations) the system (9) was reduced to the system with three algebraic equations in three unknowns variables:

$$Y(u_1, u_2, k) = Y_1,\quad Z(u_1, u_2, k) = Z_1,\quad V(u_1, u_2, k) = V_1, \tag{10}$$

where $Y = \frac{y}{x}$, $Z = \frac{z}{x^2}$, $V = \frac{v}{x^3}$.
Upper bound of cut time gives decomposition of the preimage $C = \cup_{i=1}^8 D_i$ of exponential map Exp into subdomains D_i.
The image of the exponential mapping was decomposed into subdomains respectively:

$$M = \cup_{i=1}^4 M_i, \tag{11}$$
$$M_1 = \{(x, y, z, v)\in\mathbb{R}^4 \mid x > 0, z > 0\}, \tag{12}$$
$$M_2 = \{(x, y, z, v)\in\mathbb{R}^4 \mid x < 0, z < 0\}, \tag{13}$$
$$M_3 = \{(x, y, z, v)\in\mathbb{R}^4 \mid x > 0, z < 0\}, \tag{14}$$
$$M_4 = \{(x, y, z, v)\in\mathbb{R}^4 \mid x < 0, z > 0\}. \tag{15}$$

There is a conjecture that restriction $\mathrm{Exp} : D_i\to M_i$, $\mathrm{Exp} : D_{i+4}\to M_i$ of the exponential map for these subdomains is a diffeomorphism, i. e. $\forall q_1\in M_i\ \ \exists!(\lambda, t)\in D_i, \mathrm{Exp}(\lambda, t) = q_1$ and $\forall q_1\in M_i\ \ \exists!(\lambda, t)\in D_{i+4}, \mathrm{Exp}(\lambda, t) = q_1, i\in\{1, \dots, 4\}$.

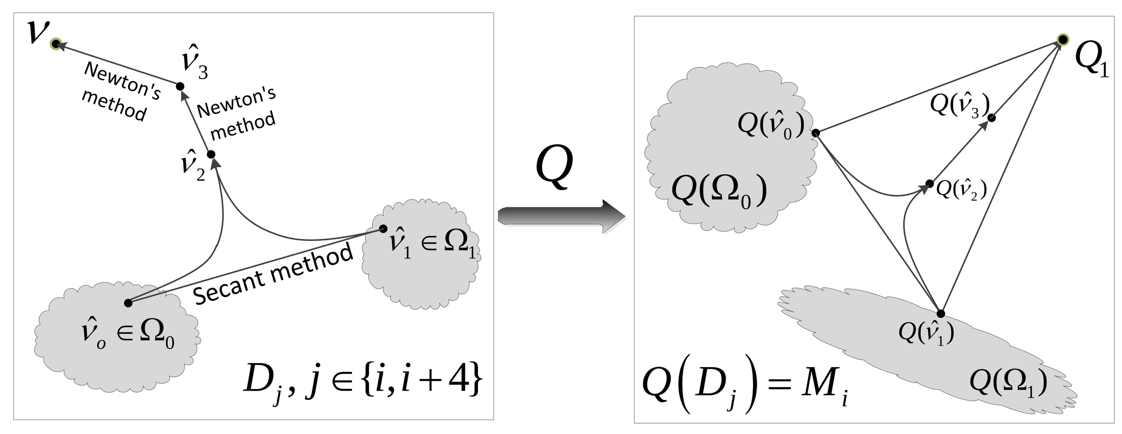


Figure: Hybrid method for solving system of algebraic equations (10)

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Conclusion

On the basis of these results, software for numerical computation of a global solution to the sub-Riemannian problem on a group of Engel was developed. So solution of the path-planning problem for mobile trailer robot via nilpotent approximation will be developed (this work is in progress).
The method for estimating a conjugate time used in this work was successfully applied earlier to Euler's elastic problem [7] and sub-Riemannian problem on the group of rototranslations [8]. There is no doubt that this method is also valid for nilpotent sub-Riemannian problem with the growth vector (2,3,5) [9, 10, 11, 12]. The method can be used for other invariant sub-Riemannian problems on Lie groups of low-dimensional integrable in non-elementary functions. The first natural step in this direction is investigation of invariant sub-Riemannian problem on 3D Lie groups which are classified by A.A. Agrachev and D.Barilari [13].