Extension of Chronological Calculus for Dynamical Systems on Manifolds

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Abstract:

We present an extension of Chronological Calculus to the case of infinite-dimensional C^m-smooth manifolds. The original Chronological Calculus was developed by Agrachev and Gamkrelidze for the study of dynamical systems on C^∞-smooth finite-dimensional manifolds. The extension of this calculus allows for the study of control systems with merely measurable controls and may be applied to C^m-smooth manifolds modeled over Banach spaces. We apply our extension to establish a formula of Mauhart and Michor for the generation of Lie brackets of vector fields and we present a proof of the Chow-Rashevskii theorem on C^m-smooth manifolds modeled over Banach spaces.

Classical Chronological Calculus:

The $C^\infty(M)$ algebra: A central object of study in the Chronological Calculus of Agrachev and Gamkrelidze is the algebra $C^\infty(M)$ of C^∞-smooth functions $f{:}M \to \mathbf{R}$. An important observation is that inherently nonlinear objects such as diffeomorphisms of manifolds give rise to inherently linear objects such as automorphisms of this algebra. Indeed, given a diffeomorphism $A{:}M\to M$, one obtains an automorphism $\hat{A} : C^\infty(M)\to C^\infty(M)$ by $\hat{A}(f) = f \circ A$. There are similar correspondences for points in M, for tangent vectors, and for vector fields. These correspondences provide a means to study many of the nonlinear objects of control theory in a setting where they behave linearly.

The Whitney Topology: Agrachev and Gamkrelidze place a topology on $C^\infty(M)$ in which $f_n \to f$ if and only for any compact subset K of M, one has the uniform convergence over K of f_n and derivatives of all orders *to f*. The precise meaning of this statement can be formulated through the Whitney embedding theorem. Equipped with this topology, $C^\infty(M)$ has the structure of a Fréchet space and the correspondences described above lead to the study of nonlinear objects as linear operators on this space.

Challenges: The classic chronological calculus is unable to handle control problems in which the dynamics are merely C^m-smooth, or are merely measurable in time, or which take place on a manifold whose local structure is infinite dimensional. In addition, the use of Fréchet space structure seems to complicate proofs for a number of important results.

Main Results

Extension of Chronological Calculus: Given a C^m-smooth manifold M modeled over a Banach space E, let $C^r(M,E)$ denote the vector space of r-times differentiable functions $f : M\to E$. The principle setting for our extension is the study of families of operators on these vector spaces. In this way, we are able to develop results for C^m-smooth dynamics on manifolds modeled over Banach spaces. For example, the local flow $P_{s,t}$ of a nonautonomous vector field V_t gives rise under appropriate assumptions to a family of operators $C^r(M,E) \to C^r(M,E)$.

Calculus of Little o's: In order to facilitate use of the calculus, we have developed a calculus of remainder terms, so that one is able to refer to a family of operators Q_t as being *differentiable with derivative* V_t whenever one has $Q_{t+h} = Q_t+hV_t+o(h)$. This rule is satisfied, for example, when Q_t is the flow of an autonomous vector field V. We establish the following useful properties for these operators:

1. $o(t^n) + o(t^n) = o(t^n)$
2. $o(t^n)\circ o(t^m) = o(t^{n+m})$
3. For vector fields V_t and W_t with locally bounded derivatives, $V_t \circ o(t^n) \circ W_t = o(t^n)$
4. If P_t and Q_t are families of operators arising from flows of vector fields, $P_t \circ o(t^n) \circ Q_t = o(t^n)$

These properties lead to simplified proofs of important results such as the bracket formula of Mauhart and Michor.

Product Rule for Composition of Operators: We say that a family of operators is *differentiable at t with derivative* A_t if $P_{t+h} = P_t+h\,A_t+o(h)$. Using the above properties, one may check that if P_t and Q_t are differentiable at t with derivatives A_t and B_t, respectively, then the composition $P_t \circ Q_t$ is differentiable with derivative $A_t \circ Q_t + P_t \circ B_t$.

Flows of Perturbed Vector Fields: Given vector fields V_t and W_t we derive a formula for the flow of their sum as a correction to the flow of V_t. This is done in a general setting which allows the study of perturbations to nonautomomous C^m-smooth vector fields on Banach manifolds which are merely measurable in time.

Bracket Formula: Mauhart and Michor define a *bracket of flows* as $[P_t,Q_t]=P_t\circ Q_t\circ P_t^{-1}\circ Q_t^{-1}$. The calculus of remainder terms gives us an algebraic proof of the following formula of Mauhart and Michor:

$$B(P_t^1, P_t^2, \ldots, P_t^k) = Id + t^k B(X_1, X_2, \ldots, X_k) + o(t^k)$$

where B is a bracket expression.

Absolute Continuity: Given a family B_t of operators, we define its integral in a weak sense through its action on functions – a definition similar to the Dunford-Pettis integral of functional analysis. This in turn provides a definition of a *weak* or *distributional* derivative which is appropriate for an absolutely continuous family of operators. In particular, we say that A_t is absolutely continuous if

$$A_t = A_{t_0} + \int_{t_0}^{t} B_s\,ds$$

And we then say that A_t has derivative B_t in a *weak* or *distributional* sense. For example, if V_t is a nonautonomous vector field which is measurable in time then the flow P_t satisfies

$$P_t = P_{t_0} + \int_{t_0}^{t} P_s \circ V_s\,ds$$

We are able to prove that the composition of such operators is again absolutely continuous and that

$$\int_{t_0}^{t_1} \tfrac{d}{dt}(A_t \circ B_t)\,dt = \int_{t_0}^{t_1}\left(\frac{dA_t}{dt}\circ B_t + A_t \circ \frac{dB_t}{dt}\right)dt$$

Chow-Rashevskii Theorem: We apply the bracket formula of Mauhart and Michor, along with some nonsmooth analysis for manifolds, to prove a variant of the Chow-Rashevskii Theorem on Banach Manifolds. In particular, we prove that if M is modeled over a smooth Banach space, then a smooth affine control system is globally approximately controllable when the Lie algebra of the associated vector fields spans T_qM for any q.