Motion Planning Problem for some Control Systems Applied in Robotics

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Summary

The classic objects of study in robotics are mathematical models of wheeled mobile robots and robots-manipulators. In general, such systems are described by nonlinear nonholonomic control system linear with respect to control $\dot q = \sum_{i=1}^n u_i(t)X_i(q)$, where the state space $Q \ni q$ is a connected smooth manifold, the controls $(u_1,\dots,u_n) \in \mathbb{R}^n$ are measurable and locally bounded, and $X_1,\dots,X_n$ are smooth vector fields (see [1]). An interesting case occurs when the dimension of the state space exceeds the dimension of control $\dim Q > n > 1$. In generic case the minimal dimension of control $n = 2$ generates a completely controllable system which can reach any desired configuration from any initial configuration. A two-point boundary value problem for such systems is studied. The problem also known as the motion planning problem. The aim is to find controls $(u_1(t), u_2(t))$ which transfer the system from any given initial state $q^0 \in Q$ to any given terminal state $q^1 \in Q$: $q(0) = q^0, \quad q(T) = q^1$. A method of approximate solution based on the nilpotent approximation is used. The general method is concretized for solving the motion planning problem for five-dimensional systems with two-dimensional control: $\dot q = u_1(t)X_1(q) + u_2(t)X_2(q), \quad \dim(Q) = 5, \quad \rho(q(T), q^1) < \epsilon$, where ρ is a distance on manifold Q. Specific systems of the type under consideration is the kinematic model of mobile robot with two trailers and the ball rolling on a plane without slipping or twisting.

Statement of the problem

We consider the following motion planning problem

$$\dot q = u_1(t)X_1(q) + u_2(t)X_2(q), \quad (1)$$
$$q(0) = q^0, \; x(T) = q^1, \quad (2)$$

where the state space $Q \ni q$ is a connected five-dimensional smooth manifold, control takes values on a two-dimensional plane $(u_1, u_2) \in \mathbb{R}^2$, and the smooth vector fields X_1, X_2 satisfy Lie Algebra Rank condition (LARC) [2] on the manifold Q (i.e. system (1) is completely controllable). Nowadays there are no explicit methods to solve (1)-(2) in general case. Satisfactory solution exists only for certain special classes of systems. However, such problems arise in engineering, where approximate solution is enough, if the error does not exceed a prescribed value. We propose a method to construct the control $(u_1(t), u_2(t))$ that translates system (1) from any initial state q^0 to any terminal state q^1 with any desired precision $\varepsilon > 0$. That is, in such a state $\tilde q^1$, that $\rho(\tilde q^1, q^1) < \varepsilon$, where ρ is a distance on the manifold Q, for example, if $Q = \mathbb{R}^5$, then $\rho = \sqrt{\sum_{i=1}^5 (q^1 - \tilde q^1)^2}$.

Systems of the form (1) are characterized by the fact that the dimension of the control is less than the dimension of the state space $2 = \dim \mathbb{R}^2 < \dim Q = 5$ but any two points of the state space can be connected by trajectory of the system. In control theory such systems are called completely nonholonomic.

Nonlinear system (1), linear in controls, the number of which is less than dimension of the state space is characterized by different shifts in different directions. The value of displacement in the direction of the fields X_1 and X_2 in a small time t is $O(t)$, in the direction of a commutator $X_3 = [X_1, X_2]$ is $O(t^2)$ in the direction $X_4 = [X_1, X_3]$ and $X_5 = [X_2, X_3]$ is $O(t^3)$. Because of this anisotropy of the state space the control problem for such systems is highly nontrivial.

Controllability

Rashevsky-Chow theorem [2] claims that any two points q^0, $q^1 \in Q$ are reachable from each other if at any point $\tilde q \in Q$ linear span of elements of the Lie algebra Lie(X_1, X_2) coincides with the tangent space $T_{\tilde q}Q$ *(LARC)*: $\forall \tilde q \in Q$ span(Lie(X_1, X_2)) $= T_{\tilde q}Q$.

Let us fix $\tilde q \in Q$ and define by $\mathrm{L}^s(\tilde q)$ a vector space generated by the values of Lie brackets X_1, X_2 of length $\le s$, $s = 1, 2, \dots$ at $\tilde q$ (the fields X_i are brackets of length 1):

$$\mathrm{L}^1(\tilde q) = \mathrm{span}(X_1(\tilde q), X_2(\tilde q)),$$
$$\mathrm{L}^2(\tilde q) = \mathrm{span}(\mathrm{L}^1(\tilde q) + [X_1, X_2](\tilde q)),$$
$$\dots$$
$$\mathrm{L}^s(\tilde q) = \mathrm{span}(\mathrm{L}^{s-1}(\tilde q) + \{[X_{i_s}, [X_{i_{s-1}}, \dots [X_{i_2}, X_{i_1}] \dots]](\tilde q) | i_1, \dots, i_s \in \{1, 2\}\}).$$

LARC ensures that for every $\tilde q \in Q$ there exists a smallest integer $r = r(\tilde q)$ such that $\dim \mathrm{L}^r(\tilde q) = 5$. Define *Growth vector* as $(n_1(\tilde q), \dots, n_r(\tilde q))$, where $n_s(\tilde q) = \dim \mathrm{L}^s(\tilde q)$, $s = 1, \dots, r$. We consider system (1) in a neighborhood of regular points, where growth vector is equal to $(2, 3, 5)$.

Nilpotent approximation

We present a method for finding approximate solutions of the problem (1)–(2) based on nilpotent approximation. Local approximation of a control system by another (simpler) system is often used in control theory. Usually linearization of the control system is used as a local approximation. However, for control systems of the form (1) linearization gives too rough approximation. Since the number of controls less than the dimension of state space, the linearization can not be completely controlled. Natural replacement of the linear approximation in this case gives a nilpotent approximation — the most simple system that preserves the original structure of the control system and therefore controllability (in particular, it remains a growth vector).

We use algorithm of Bellaiche [3] to get nilpotent approximation of original system in a neighborhood of end point q^1 and then we make a change of variables in which the nilpotent approximation has the canonical form:

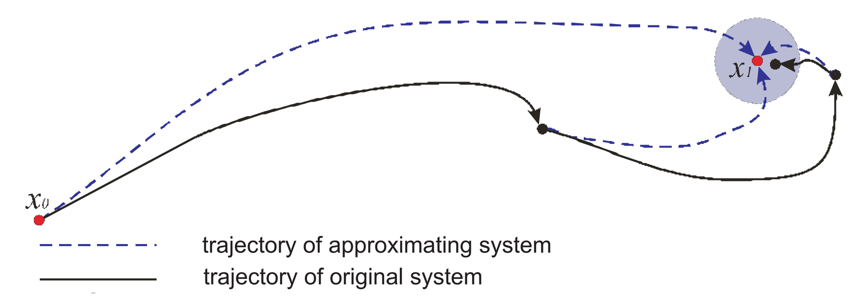
$$\begin{cases} \dot y_1 = u_1, \\ \dot y_2 = u_2, \\ \dot y_3 = \frac{1}{2}(y_1 u_2 - y_2 u_1), \\ \dot y_4 = \frac{1}{2}(y_1^2 + y_2^2)u_2, \\ \dot y_5 = -\frac{1}{2}(y_1^2 + y_2^2)u_1, \end{cases} \qquad y \in \mathbb{R}^5, \quad (3)$$

and boundary conditions are following:

$$Q(0) = Q_1, \qquad Q(T) = 0. \quad (4)$$

Motion Planning Iterative Algorithm

To solve problem (1), (2) we use Iterative algorithm based on the local approximation of the original system by nilpotent system (3), for which the control problem must be solved exactly in each iteration.



So, the problem is to find the control such that corresponding trajectory of system (1) starts from initial state q^0 and ends in a state $\tilde q^1$ that satisfied the inequality $\rho(q^1, \tilde q^1) < \varepsilon$ for any given $\varepsilon > 0$. To solve the problem we use the following iterative algorithm:

1. building nilpotent approximation at q^1 and computing the change of variables in which nilpotent approximation has form (3);
2. finding a control $(u_1(t), u_2(t))$ in given class of functions that solves problem (3)–(4) for nilpotent system exactly;
3. found control is applyed to the original system, and if the reached state misses the ε-neighborhood of the target state, then the required precision is not achieved, and the step 2 is repeated with the new boundary condition — the state reached by the previous iteration of the algorithm is chosen as new initial state, otherwise calculation stops.

We developed parallel software "MotionPlanning.m" that implements this algorithm as a package for Wolfram Mathematica. It solves the motion planning problem (1), (2) for sufficiently close q^0 and q^1 (it means $\rho(q^0, q^1) < \delta$, where $\delta > 0$ depends on the concrete form of the vector fields in right part of (1) and class of function in which control are to be found). δ must be estimated to establish convergence domain of the algorithm (work in progress). For the present we have a results of numerical experiments.

The software "MotionPlanning.m" solves the motion planning problem in the classes of piece-wise constant controls and optimal controls for nilpotent approximation.

Piece-wise constant control

For any $Q_1 \in \mathbb{R}^5$ exist $(\alpha_i, \beta_i, \gamma_i, \delta_i) \in \mathbb{R}^4$, $i \in \{1, 2\}$ and control

$$u_i = \begin{cases} \alpha_i, \text{ for } t \in [0, \frac{1}{4}], \\ \beta_i, \text{ for } t \in (\frac{1}{4}, \frac{1}{2}], \\ \gamma_i, \text{ for } t \in (\frac{1}{2}, \frac{3}{4}], \\ \delta_i, \text{ for } t \in (\frac{3}{4}, 1], \end{cases}$$

such that $Q(0) = Q_1$, $Q(1) = 0$

- algebraic equations for parameters $(\alpha_i, \beta_i, \gamma_i, \delta_i)$
- nonunique solution
- final fixing of parameters by criterion $\max |u_i| \to \min$

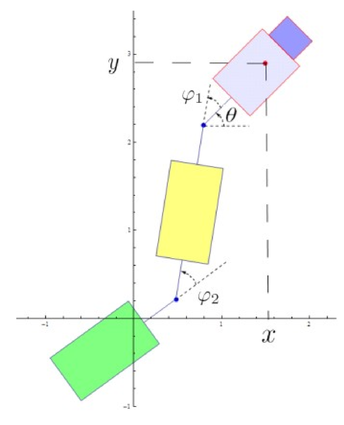
Optimal Control

- (3), (4), $\int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min$
- Nilpotent sub-Riemannian problem with growth vector (2,3,5) (Yu. Sachkov):
  - extremal trajectories, bounds on cut time, global structure of exponential mapping, symmetries, reduction to system of 3 algebraic equations in Jacobian functions of 3 variables
- Optimal synthesis algorithm
- Genetic algorithm for numerical solution of algebraic equations systems

Car with Two Trailers

- state variables
  $\xi = (x, y, \theta, \phi_1, \phi_2)$
  $\xi \in \mathbb{R}^2 \times S^1 \times (S^1 - \{\pi\})^2$
- control system

$$\begin{cases} \dot x &= \cos\theta\, u_1, \\ \dot y &= \sin\theta\, u_1, \\ \dot\theta &= u_2, \\ \dot\phi_1 &= -\sin\phi_1\, u_1 + \\ &\quad +(-1 - \cos\phi_1)\, u_2, \\ \dot\phi_2 &= (\sin(\phi_1 - \phi_2) + \sin\phi_1)\, u_1 + \\ &\quad +(\cos(\phi_1 - \phi_2) + \cos\phi_1) u_2. \end{cases}$$



Sphere rolling on a plane

Consider a sphere rolling on a plane whithout slipping or twisting (see [4]). State of the system is described by the contact point between the sphere and the plane and orientation of the sphere in three-dimensional space. One should roll the sphere from any initial contact configuration to any desired configuration. The problem has application in robotics: rotation of a solid body in robot's hand.

Let $(x, y) \in \mathbb{R}^2$ be the contact point of the sphere and the plane.

By $q = (q_0, q_1, q_2, q_3) \in S^3$ denote the unit quaternion (see [5]) representing the rotation of three-dimensional space, which translates the current orientation of the sphere to the initial orientation. The control system described rolling sphere has the following form:

$$\dot Q = u_1 X_1(Q) + u_2 X_2(Q),$$

where $X_1(Q) = (1, 0, q_2, q_3, -q_0, -q_1)^T$, $X_2(Q) = (0, 1, -q_1, q_0, q_3, -q_2)^T$ are smooth vector fields, state space is $Q = (x, y, q_0, q_1, q_2, q_3) \in M = \mathbb{R}^2 \times S^3$, and control $u = (u_1, u_2) \in \mathbb{R}^2$ is unbounded.

Since considered system is left-invariant problem on Lie Group $\mathbb{R}^2 \times S^3$ the motion planning problem for any boundary conditions is reduced to fixed initial position and arbitrary final one:

$$Q(0) = Q_0 = (0, 0, 1, 0, 0, 0), \qquad Q(t_1) = Q_1.$$

To apply the motion planning algorithm we choose local chart

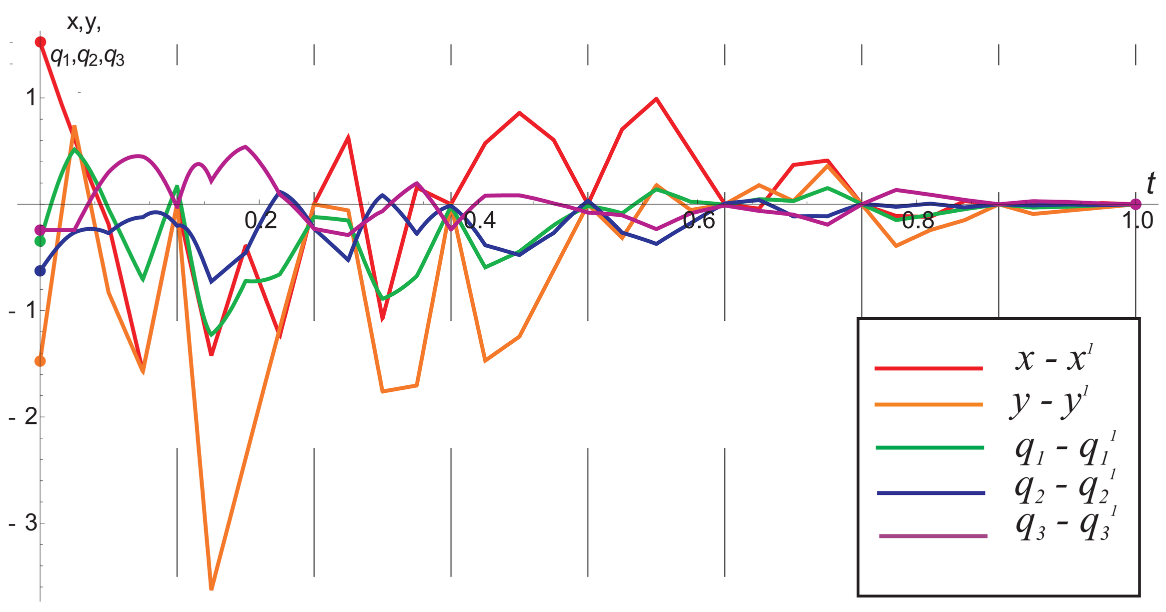
$$q_0 = \sqrt{1 - q_1^2 - q_2^2 - q_3^2} > 0.$$

MotionPlanning Package: Examples

Rolling the sphere using piecewise constant control from initial configuration $Q_0 = (0, 0, 0, 0, 0)$ to desired configuration

$$Q_1 = (-1.525, 1.475, 0.346, 0.626, 0.242)$$

with precision $\varepsilon = 10^{-5}$



Iterations: 8

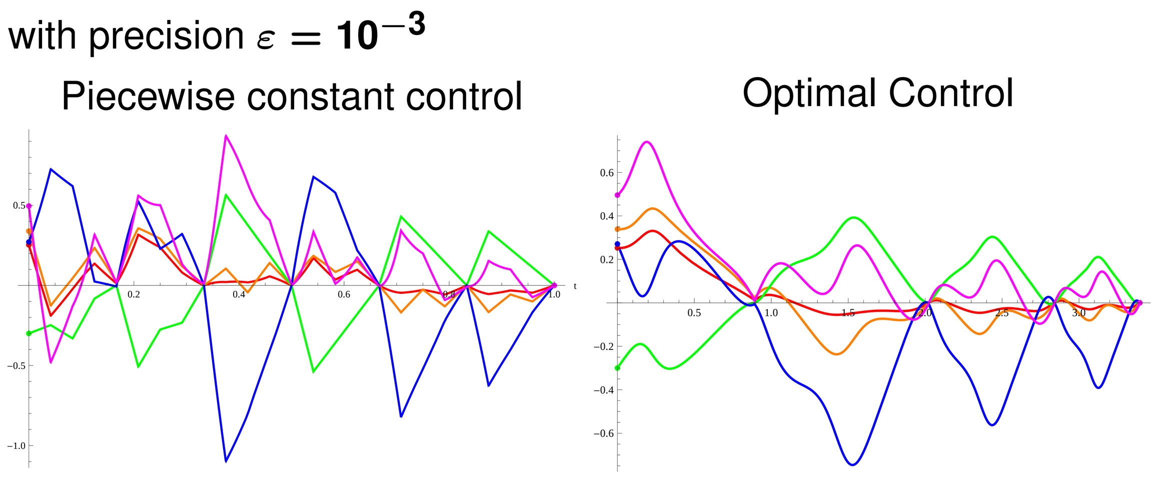
Transferring the car with two trailers from initial configuration

$$Q_0 = (0, 0, \frac{\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{4})$$

to desired configuration

$$Q_1 = (-0.252, -0.339, 1.085, 0.514, -1.281)$$

with precision $\varepsilon = 10^{-3}$



6 iterations   4 iterations

Bibliography

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Conclusion

We presented the iterative algorithm for solving motion planning problem (1), (2) whith any necessary precision. It has been implemented in a parallel software MotionPlaning.m. The software has been tested on two applications (problem of rolling of a sphere on a plane without slipping and twisting and the problem of steering the mobile robot with two trailers). In cases where the boundary conditions were not too far from each other, the software has been successfully solving the control problem. In cases of distant boundary conditions algorithm does not converge, which corresponds to the theoretical basis of the method (nilpotent approximation is the local approximation of the original system). In the future we plan to expand the functionality for solving the tasks with distant boundary conditions through its reduction to the sequence of local problems. Currently MotionPlaning.m is a convenient and reliable way to solve the local problem (1), (2).