NON-LIPSCHITZ POINTS AND THE SBV REGULARITY OF THE MINIMUM TIME FUNCTION

ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

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INTRODUCTION

Consider the control dynamics:

$$\begin{cases} \dot{y} &= F(y) + G(y)u \\ u &\in \mathcal{U} \\ y(0) &= x \in \mathbb{R}^N, \end{cases} \quad , \qquad (1)$$

where F, G are smooth enough and the control set $\mathcal{U} \subset \mathbb{R}^M$ is compact.

The **minimum time** $T(x)$ to reach the origin from x:

$$T(x) = \inf\{t : y(t) = 0, y \text{ is a solution of } (1)\}$$

In general, T is **nonsmooth** and even **non-Lipschitz**. Under a controllability assumption, T is **semiconcave/convex** and thus satisfies several regularity properties. In particular,

- T is a.e. twice differentiable.
- The singular set of T has a structure.
- T is locally BV (Bounded Variation).

By weakening the controllability assumptions (e.g., assuming T merely continuous), one can prove that T, although not Lipschitz, satisfies essentially the same properties of a semiconcave/convex function, including **a.e. twice differentiable** and **locally BV**. Under such assumption, we show that non-Lipschitz points of T lie exactly where the Hamiltonian vanishes. Our main result is the $\mathcal{H}^{N-1}$-rectifiability of the set $\mathcal{S}$ of non-Lipschitz points of T for the linear single input case and the $\mathcal{H}^1$ -rectifiability for the nonlinear two dimensional case. As a consequence we obtain that **T is locally SBV** .

CONCEPTS AND NOTIONS

- $\mathcal{R}_\tau = \{x : T(x) \le \tau\}$.
- A closed set $K \subset \mathbb{R}^N$ is said to have **positive reach** iff there exists a continuous function $f : K \to [0, +\infty)$ such that for all $x, y \in K$ and $v \in N_K(x)$
$$\langle v, y - x\rangle \le f(x)\|v\|\,\|y - x\|^2.$$
- Let $0 \le k < \infty$ and let $\mathcal{H}^k$ denote the Hausdorff k-dimensional measure. Let E be $\mathcal{H}^k$-measurable. We say that E is **countably $\mathcal{H}^k$-rectifiable** if there exist countably many sets $A_i \subseteq \mathbb{R}^k$ and countably many Lipschitz functions $f_i : A_i \to \mathbb{R}^N$ be such that
$$\mathcal{H}^k\left(E \setminus \bigcup_{i=1}^{\infty} f_i(A_i)\right) = 0.$$
- A BV function φ is **SBV (Special Bounded Variation)** if its distributional derivative $D\varphi$ has no Cantor part.

MINIMIZED HAMILTONIAN AND NON-LIPSCHITZ POINTS

The Minimized Hamiltonian: $h(x,\zeta) = \langle F(x), \zeta\rangle + \min_{u\in\mathcal{U}}\langle G(x)u, \zeta\rangle$.
We prove: Under assumptions which imply epi(T) has positive reach, T is non-Lipschitz at x iff there exists $0 \ne \zeta \in \mathbb{R}^N$ such that $h(x,\zeta) = 0$ and $\zeta \in N_{\mathcal{R}_{T(x)}}(x)$.

RESULTS FOR LINEAR SYSTEMS IN $\mathbb{R}^N$

Consider the linear control dynamics: $\dot{x} = Ax + bu, \quad |u| \le 1,$ where $A \in \mathbb{M}_{N\times N}$, $b \in \mathbb{R}^N$, satisfies the Kalman rank condition $\text{rank}[b, Ab, \ldots, A^{N-1}b] = N$.
Then epi(T) has positive reach and the set of all non-Lipschitz points of T is

$$\mathcal{S} = \left\{x : \exists r > 0, \zeta \in \mathbb{S}^{N-1} \text{ such that } x = \int_0^r e^{A(t-r)} b \,\text{sign}\left(\langle \zeta, e^{At}b\rangle\right) dt \text{ and } \langle \zeta, b\rangle = 0\right\}$$

We prove also:

- $\mathcal{S}$ is closed, countably $\mathcal{H}^{N-1}$-rectifiable.
- $T \in SBV_{\text{loc}}(\mathbb{R}^N)$.
- (Propagation result) For $\mathcal{H}^{N-1}-$ a.e $x \in \mathcal{S}$ there exists a neighborhood V of x such that
$$\mathcal{H}^{N-1}(V \cap \mathcal{S}) > 0.$$

RESULTS FOR NONLINEAR SYSTEMS IN $\mathbb{R}^2$

Consider the nonlinear control dynamics: $\dot{x} = F(x) + G(x)u, \quad |u| \le 1,$ where $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ are of class $\mathcal{C}^{1,1}$ satisfying $F(0) = 0$, $G(0) = 0$ and $\text{rank}[G(0), DF(0)G(0)] = 0$.

There exists $\mathcal{T} > 0$ depending only on the data of the dynamics such that for all $0 < \tau < \mathcal{T}$, epi(T) has positive reach in $\mathcal{R}_\tau$ and the set of all non-Lipschitz points of T within $\mathcal{R}_\tau$ is

$$\mathcal{S} = \{x \in \mathcal{R}_\tau : \exists \zeta \in \mathbb{S}^1 \cap N_{\mathcal{R}_{T(x)}}(x) \text{ such that } h(x,\zeta) = 0\}$$

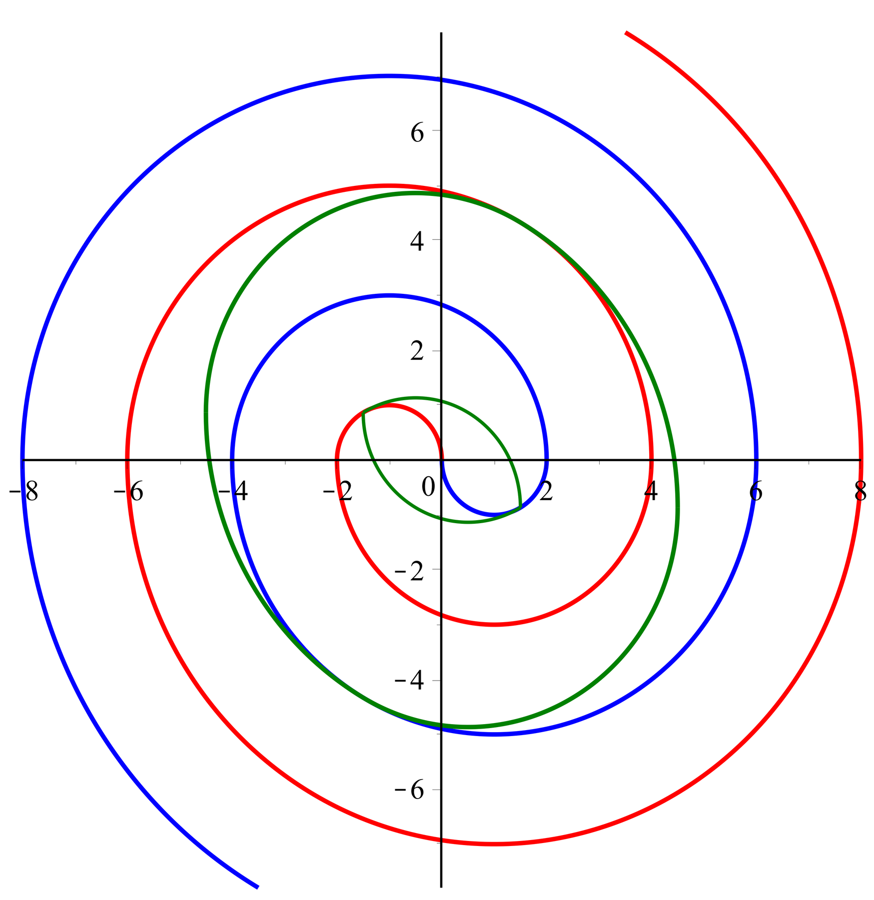
We prove also:

- $\mathcal{S}$ is closed, countably $\mathcal{H}^1$ - rectifiable.
- $T \in SBV_{\text{loc}}(\text{int}(\mathcal{R}_\mathcal{T}))$.
- (Propagation result) For all $\bar{x} \in S$ there exists $\delta > 0$ such that $\mathcal{H}^1\left(S \cap B(\bar{x}, \delta)\right) > 0$.

EXAMPLES

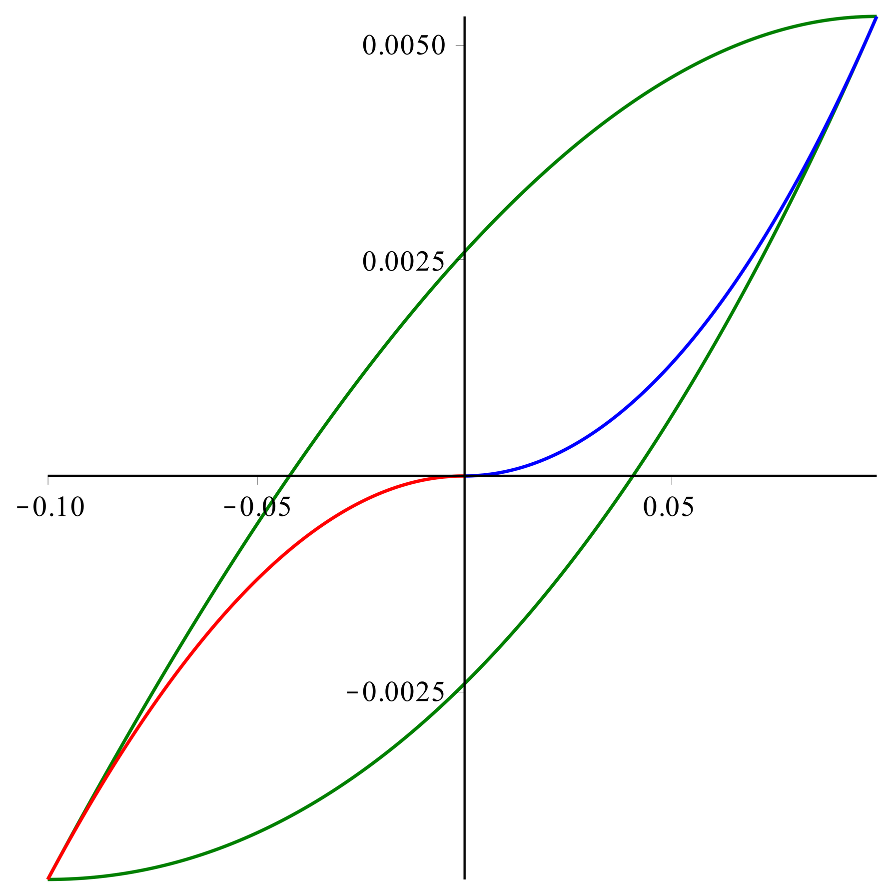
Consider the linear control system:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -x + u \end{cases}, u \in [-1, 1]$$



Consider the nonlinear control system:

$$\begin{cases} \dot{x} &= x^2 - u \\ \dot{y} &= -x^2 - x \end{cases}, u \in [-1, 1]$$



Remark:

- The set $\mathcal{S}$ of non-Lipschitz points of T consists of two optimal trajectories (the **red** and **blue** curves) starting from the origin.
- The non-Lipschitz trajectories are tangent to sublevels of T (the **green** curves).

FUTURE WORK

Extending the above results to higher dimensional nonlinear control systems.