A semi-Lagrangian scheme for a first order Mean Field Game problem

F. Camilli, E. Carlini and F. J. Silva
Dipartimento di Matematica Guido Catelnuovo, Universita La Sapienza.

1 Introduction

We consider the following first order Mean Field Game problem

$$\begin{aligned} -\partial_t v(x,t) + \tfrac{1}{2}|Dv(x,t)|^2 &= F(x, m(t)), \text{ in } \mathbb{R}^d \times (0,T), \\ \partial_t m(x,t) - \operatorname{div}\big(Dv(x,t)m(x,t)\big) &= 0, \quad \text{in } \mathbb{R}^d \times (0,T), \\ v(x,T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d &, \quad m(0) = m_0 \in L^\infty(\mathbb{R}^d). \end{aligned} \tag{1}$$

The above equations have been introduced by J.M. Lasry and P. L. Lions in [4, 3] in order to model a deterministic differential game with an infinite number of players. The main assumption is that the players are indistinguishable and each one of them has a small influence on the overall system. Existence of a solution, where the first equation is satisfied in the viscosity sense and the second one in the distributional sense, can be proved under rather general assumptions. The uniqueness is also satisfied if the following assumption holds true

$$\left.\begin{aligned} \int_{\mathbb{R}^d} [F(x,m_1) - F(x,m_2)]\,\mathrm{d}[m_1 - m_2](x) > 0 \quad \text{for all } m_1, m_2 \in \mathcal{P}_1, \ m_1 \neq m_2, \\ \int_{\mathbb{R}^d} [G(x,m_1) - G(x,m_2)]\,\mathrm{d}[m_1 - m_2](x) > 0 \quad \text{for all } m_1, m_2 \in \mathcal{P}_1, \ m_1 \neq m_2. \end{aligned}\right\} \tag{M}$$

In this poster we present

- The semi-discrete in time scheme introduced in [1] (joint work with F. Camilli).
- A fully-discrete semi-Lagrangian scheme introduced in [2] (joint work with E. Carlini) which depends on the discretization parameters $\rho > 0$ and $h > 0$ for the state and time, respectively, and a regularization parameter $\varepsilon > 0$.
- An existence result for the fully discrete scheme.
- In the case $d = 1$ a convergence result.
- A numerical simulation.

For precise assumptions over the data, see [1, 2].

2 The semi-discrete scheme [1] (with F. Camilli)

For $h > 0$ and $N \in \mathbb{N}$, with $Nh = T$, and $t_k := kh$ for $k = 0, \ldots, N$, we set

$$\mathcal{K}_h := \left\{ \mu = (\mu(t_k))_{k=0}^N : \ \text{ such that } \mu(t_k) \in \mathcal{P}_1 \quad \text{for all } k = 0, \ldots, N \right\}.$$

For $\mu \in \mathcal{K}_h$ and $n = [t/h]$, we define recursively the sequence

$$\begin{aligned} v_h[\mu](x, t_k) &= \inf_{\alpha \in \mathbb{R}^d} \left\{ v_h[\mu](x - h\alpha, t_{k+1}) + \tfrac{1}{2}h|\alpha|^2 \right\} + hF(x, \mu(t_k)), \\ v_h[\mu](x, T) &= G(x, \mu(T)). \end{aligned} \tag{2}$$

Given $x \in \mathbb{R}^d$ and $t_{n_1} \leq t_{n_2}$, the *discrete flow* $\Phi_h[\mu](\cdot, t_{n_1}, \cdot)$ is defined recursively as

$$\begin{aligned} \Phi_h[\mu](x, t_{n_1}, t_{n_2+1}) &:= \Phi_h[\mu](x, t_{n_1}, t_{n_2}) - h\alpha_h[\mu](x, t_{n_2}), \\ \Phi_h[\mu](x, t_{n_1}, t_{n_1}) &:= x, \end{aligned}$$

where for every (x, t_k), the discrete control $\alpha_h[\mu](x, t_k)$ solves (2). It can be proved that $\Phi_h[\mu](x, t_{n_1}, \cdot)$ is uniquely defined a.e. in $\mathbb{R}^d$. Now, we define

$$m_h[\mu](t_n) := \Phi_h[\mu](\cdot, 0, t_n)\sharp m_0.$$

The semi-discrete approximation of (1) is defined as

$$\text{Find } m_h \in \mathcal{K}_h \text{ such that } \ m_h(t_n) = \Phi_h[m_h](\cdot, 0, t)\sharp m_0 \ \text{ for all } n = 0, \ldots, N. \tag{3}$$

Theorem 1 *Problem* (3) *admits at least one solution* m_h. *Moreover, if* **(M)** *holds then the solution is unique.*

We also have the following convergence result, which, in particular, provides another proof for the existence of a solution of problem (1).

Theorem 2 *Every limit point of* m_h *(there exists at least one) solves* (1). *In particular, if* **(M)** *holds we have that* $m_h \to m$ *(the unique solution of* (1)*) in* $C([0,T]; \mathcal{P}_1)$ *and in* $L^\infty\left(\mathbb{R}^d \times [0,T]\right)$*-weak-*$*$.

The key elements in the proof are optimal control techniques and the fact that m_0 is absolutely continuous w.r.t. the Lebesgue measure.

3 The fully-discrete scheme [2] (with E. Carlini)

For $h, \rho > 0$, let $\mathcal{G}_\rho := \{x_i = i\rho, \ i \in \mathbb{Z}^d\}$ and $\mathcal{G}^{\rho,h} := \{t_n\}_{n=0}^N \times \mathcal{G}_\rho$ be the *time-space grid*. Given the hypercube $Q(x_i) := [x_i \pm \rho e_1] \times \ldots \times [x_i \pm \rho e_d]$, set $\beta_i(x) = 1 - \frac{\|x - x_i\|_1}{\rho}$ if $x \in Q(x_i)$ and 0 if not. Given $\mu \in C([0,T], \mathcal{P}_1)$, define

$$v_i^n = S^{\rho,h}[\mu](v^{n+1}, i, n) \quad \text{and} \quad v_i^N = G(x_i, \mu(T)),$$

where $S^{\rho,h}[\mu]$ is defined as

$$S^{\rho,h}[\mu](w, i, n) := \inf_{\alpha \in \mathbb{R}^d} \left[\sum_{j \in \mathbb{Z}^d} \beta_j(x_i - h\alpha) w_j + \tfrac{1}{2}h|\alpha|^2 \right] + hF(x_i, \mu(t_n)).$$

We set

$$v^{\rho,h}[\mu](x,t) := \sum_{i \in \mathbb{Z}^d} \beta_i(x) v_i^{\left[\frac{t}{h}\right]} \quad \text{for all } (x,t) \in \mathbb{R}^d \times [0,T].$$

Let $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\rho \geq 0$ and $\int_{\mathbb{R}^d} \rho(x)\mathrm{d}x = 1$. For $\varepsilon > 0$, we consider the *mollifier* $\rho_\varepsilon(x) := \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)$ and define

$$v_\varepsilon^{\rho,h}[\mu](\cdot, t) := \rho_\varepsilon * v^{\rho,h}[\mu](\cdot, t) \quad \text{for all } t \in [0,T].$$

Consider the set

$$\mathcal{S} := \left\{ (z_i)_{i \in \mathbb{Z}^d} \, ; \ z_i \in \mathbb{R}_+ \text{ and } \sum_{i \in \mathbb{Z}^d} z_i = 1 \right\}.$$

The coordinates of $\mu \in \mathcal{S}^N$ are denoted as μ_i^k, with $i \in \mathbb{Z}^d$ and $k = 0, \ldots, N$. Each $\mu \in \mathcal{S}^N$ is identified with $\mu \in C([0,T]; \mathcal{P}_1)$ defined as

$$\mu(x,t) := \frac{1}{\rho^d} \left[\frac{t_{k+1} - t}{h} \sum_{i \in \mathbb{Z}^d} \mu_i^k \mathbb{I}_{E_i}(x) + \frac{t - t_k}{h} \sum_{i \in \mathbb{Z}^d} \mu_i^{k+1} \mathbb{I}_{E_i}(x) \right] \quad \text{if } t \in [t_k, t_{k+1}],$$

where $E_i := [x_i \pm \frac{1}{2}\rho e_1] \times \ldots \times [x_i \pm \frac{1}{2}\rho e_d]$. Let us define

$$\Phi_\varepsilon^{\rho,h}[\mu](x_i, t_k, t_{k+1}) := x_i - hDv_\varepsilon^{\rho,h}[\mu](x_i, t_k).$$

We define $m_\varepsilon^{\rho,h}[\mu] \in \mathcal{S}^{N+1}$ recursively as

$$\begin{aligned} (m_\varepsilon^{\rho,h}[\mu])_i^{k+1} &:= \textstyle\sum_j \beta_i\left(\Phi_\varepsilon^{\rho,h}[\mu](x_j, t_k, t_{k+1})\right)(m_\varepsilon^{\rho,h}[\mu])_j^k, \quad \text{for } i \in \mathbb{Z}^d, \\ (m_\varepsilon^{\rho,h}[\mu])_i^0 &:= \textstyle\int_{E_i} m_0(x)\mathrm{d}x, \quad \text{for } i \in \mathbb{Z}^d \end{aligned}$$

and $m_\varepsilon^{\rho,h}[\mu](x,t)$ is defined as we did with μ above. The key property for our main result, which we are able to prove only in dimension 1, is

Lemma 3.1 *Suppose that* $d = 1$. *Then, there exists* $C > 0$ *(independent of* $(\rho, h, \varepsilon, \mu)$*) such that for any* $i \in \mathbb{Z}^d$ *and* $k = 0, \ldots, N-1$, *we have that*

$$\sum_{j \in \mathbb{Z}^d} \beta_i\left(\Phi_\varepsilon^{\rho,h}(x_j, t_k, t_{k+1})\right) \leq 1 + Ch.$$

Consequently, $m_\varepsilon^{\rho,h}[\mu](\cdot,\cdot)$ *is bounded in* L^∞ *independently of* $(\rho, h, \varepsilon, \mu)$.

We consider the following fully-discretization of (1):

$$\text{Find } \mu \in \mathcal{S}^{N+1} \text{ such that } \ \mu_i^k = (m_\varepsilon^{\rho,h}[\mu])_i^k \ \text{ for all } i \in \mathbb{Z}^d \text{ and } k = 0, \ldots, N.$$

We have the following existence result:

Theorem 3 *The fully-discrete problem admits at least one solution.*

Our main result is that we can prove convergence in dimension 1.

Theorem 4 *Suppose that* $d = 1$ *and consider a sequence of positive numbers* $\rho_n, h_n, \varepsilon_n$ *satisfying that* $\rho_n = o(h_n\varepsilon_n)$, $h_n = o(\varepsilon_n)$ *and* $\varepsilon_n \to 0$. *Then every limit point of* $m_{\varepsilon_n}^{\rho_n,h_n}$ *(there exists at least one) is a solution of* (1). *In particular, if* **(M)** *holds we have that* $m_{\varepsilon_n}^{\rho_n,h_n} \to m$ *(the unique solution of* (1)*) in* $C([0,T]; \mathcal{P}_1)$ *and in* $L^\infty(\mathbb{R} \times [0,T])$*-weak-*$*$.

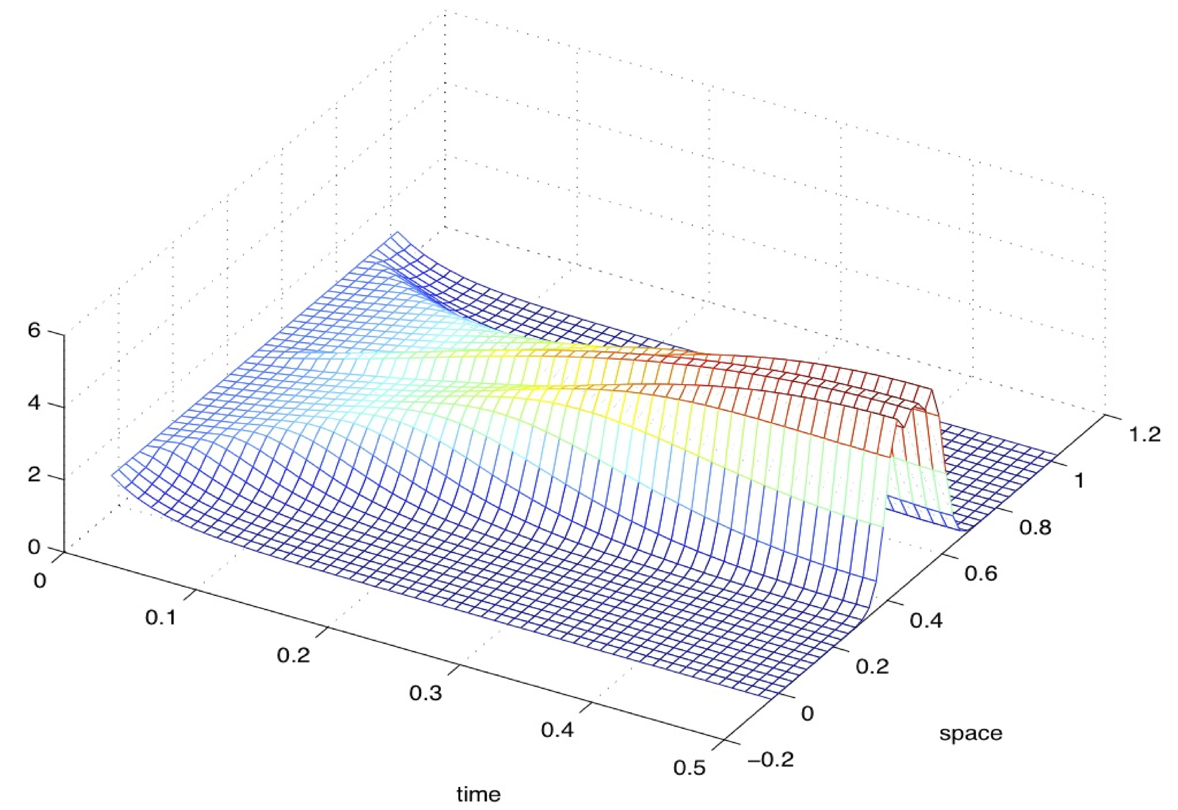
4 A numerical example

Example: [People willing to go to the center but not together]

- Space-time domain: $\Omega \times [0,T] = [0,1] \times [0, 0.05]$
- $F(x,m) = (x - 0.5)^2 + h * (h * m)$, where

$$h(x) = \frac{\hat{h}(x)}{\int_0^1 \hat{h}(y)\mathrm{d}y} \quad \text{and} \quad \hat{h}(x) = e^{-x^2/8}\mathbb{I}_{[-\frac{1}{4}, \frac{1}{4}]}.$$

- $G(x,m) = 0$.
- $m_0 \equiv 1$ in $[0,1]$.
- toll$= 10^{-3}$, $\rho = 2.5 \cdot 10^{-2}$ and $h = 0.01$.



References

[1] F. Camilli and F.J. Silva A semi-discrete in time approximation for a first order-finite mean field game problem Network and Heterogeneous Media 7-2: 263–277, 2012.

[2] E. Carlini and F.J. Silva A fully-discrete semi-Lagrangian scheme for a first order-finite Mean Field Game problem Preprint, to appear, 2012.

[3] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen II. Horizon fini et contrôle optimal *C. R. Math. Acad. Sci. Paris*, 343:679–684, 2006.

[4] J.-M. Lasry and P.-L. Lions. Mean field games *Jpn. J. Math.*, 2: 229–260, 2007.