Large time behavior of weakly coupled systems of first-order Hamilton-Jacobi equations

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1 Problem

We study the large time behavior of systems of Hamilton-Jacobi equations

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + H_i(x, Du_i) + \displaystyle\sum_{j=1}^{m} d_{ij} u_j = 0 \quad (x,t) \in \mathbb{T}^N \times (0,+\infty), \\ u_i(x,0) = u_{0i}(x), \end{cases} \tag{1}$$

where $d_{ii} \geq 0,\ d_{ij} \leq 0 \text{ for } i \neq j$ and $\sum_{j=1}^m d_{ij} = 0$ for all $i = 1, \ldots, m$. We are interested in finding an ergodic constant vector $(c_1, \ldots, c_m) \in \mathbb{R}^m$ and a function $(v_1, \ldots, v_m)$ such that

$$H_i(x, Dv_i) + \sum_{j=1}^{m} d_{ij}(x) v_j = c_i, \quad x \in \mathbb{T}^N, \quad i = 1, \ldots, m \tag{2}$$

and, for all $i = 1, \ldots, m$,

$$u_i(x,t) + c_i t \to v_i(x) \ \text{ uniformly as } t \text{ tends to infinity,}$$

2 Hypotheses+main result

We assume for $i = 1, \ldots, m$ that

$$\begin{cases} (i) \ \text{ The function } p \mapsto H_i(x,p) \text{ is differentiable a.e.,} \\ (ii) \ \ (H_i)_p \ \ p - H_i \geq 0 \text{ for a.e. } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N, \\ (iii) \ \text{ There exists a, possibly empty, compact set } K \text{ of } \mathbb{T}^N \text{ such that} \\ \quad (a) \ \ H_i(x,p) \geq 0 \text{ on } K \times \mathbb{R}^N, \\ \quad (b) \ \ \text{If } H_i(x,p) \geq \eta > 0 \text{ and } d(x,K) \geq \eta, \text{ then } (H_i)_p \ \ p - H_i \geq \Psi(\eta) > 0. \end{cases}$$

(Result) Assume that $H_i \in C(\mathbb{T}^N \times \mathbb{R}^N)$ satisfies the above hypothesis. Then, the solution $(u_1, \ldots, u_m) \in W^{1,\infty}(^N \times (0,\infty))^m$ of (1) converges uniformly to a solution $(v_1, \ldots, v_m)$ of (2).

3 Applications

A typical example satisfies our result is

$$\begin{cases} \frac{\partial u_1}{\partial t} + |Du_1 + f_1(x)|^2 - |f_1(x)|^2 + u_1 - u_2 = 0, \\ \frac{\partial u_2}{\partial t} + |Du_2 + f_2(x)|^2 - |f_2(x)|^2 + u_2 - u_1 = 0. \end{cases} \ (x,t) \in^N \times (0,+\infty),$$

where $f_i \in C(\mathbb{T}^N)$. Another example which will be explained through control optimal is given in the next section.

4 Control optimal

Consider the controlled random evolution process (X_t, ν_t) with dynamics

$$\begin{cases} \dot{X}_t = b_{\nu_t}(X_t, a_t), \quad t > 0, \\ (X_0, \nu_0) = (x, i) \in \mathbb{T}^N \times \{1, \ldots, m\}, \end{cases} \tag{3}$$

where the control law $a : [0,\infty) \to A$ is a measurable function (A is a compact subset of some metric space), $b_i \in L^\infty(\mathbb{T}^N \times A; \mathbb{R}^N)$, satisfies

$$|b_i(x,a) - b_i(y,a)| \leq C|x - y|, \qquad x, y \in \mathbb{T}^N, \ a \in A, \ 1 \leq i \leq m. \tag{4}$$

For every a_t and matrix of probability transition $G = (\gamma_{ij})_{i,j}$ satisfying $\sum_{j \neq i} \gamma_{ij} = 1$ for $i \neq j$ and $\gamma_{ii} = -1$, there exists a solution (X_t, ν_t), where $X_t : [0,\infty) \to \mathbb{T}^N$ is piecewise C^1 and $\nu(t)$ is a continuous-time Markov chain with state space $\{1, \ldots, m\}$ and probability transitions given by

$$\mathbb{P}\{\nu_{t+\Delta t} = j \mid \nu_t = i\} = \gamma_{ij} \Delta t + o(\Delta t)$$

for $j \neq i$.

We introduce the value functions of the optimal control problems

$$u_i(x,t) = \inf_{a_t \in L^\infty([0,t],A)} \mathbb{E}_{x,i}\{\int_0^t f_{\nu_s}(X_s) ds + u_{0,\nu_t}(X_t)\}, \quad i = 1, \ldots m, \tag{5}$$

where $\mathbb{E}_{x,i}$ denote the expectation of a trajectory starting at x in the mode i, $f_i, u_{0,i} : \mathbb{T}^N \to \mathbb{R}$ are continuous and $f_i \geq 0$.

It is possible to show that the following dynamic programming principle holds:

$$u_i(x,t) = \inf_{a_t \in L^\infty([0,t],A)} \mathbb{E}_{x,i}\{\int_0^t f_{\nu_s}(X_s) ds + u_{\nu_h}(X_h, t-h)\} \qquad 0 < h \leq t.$$

Then the functions u_i satisfy the system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \sup\limits_{a \in A} -\langle b_i(x,a), Du_i \rangle + \displaystyle\sum_{j \neq i} \gamma_{ij}(u_i - u_j) = f_i \ (x,t) \in \mathbb{T}^N \times (0,+\infty), \\ u_i(x,0) = u_{0,i}(x) \qquad\qquad\qquad\qquad x \in \mathbb{T}^N, \end{cases} \ i = 1, \cdots m,$$

which has the form (1) by setting $H_i(x,p) = \sup_{a \in A} -\langle b_i(x,a), p \rangle - f_i(x)$ and $d_{ii} = \sum_{j \neq i} \gamma_{ij} = 1$ and $d_{ij} = -\gamma_{ij}$ for $j \neq i$.

We assume that

$$\mathcal{F} = \{x_0 \in^N : f_i(x_0) = 0 \ \text{ for all } i = 1, \ldots, m\} \neq \emptyset, \tag{6}$$

And the following controllability assumption is satisfied: for every i, there exists $r > 0$ such that for any $x \in^N$, the ball $B(0,r)$ is contained in $\overline{\text{co}}\{b_i(x,A)\}$.

Then our result applies in this case. Roughly speaking, it means that the optimal strategy is to drive the trajectories towards a point x^* of $\mathcal{F}$ and then not to move anymore (except maybe a small time before t). This is suggested by the fact that all the f_i's have minimum 0 at x^* and, at such point, the running cost is 0.

5 References

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