The vanishing viscosity limit for Hamilton-Jacobi equations on networks

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joint with **Claudio Marchi** (Padova) and **Dirk Schieborn** (Tübingen).

Hamilton-Jacobi equations on networks

Three different approaches to HJ equation on networks

- **Achdou, Camilli, Cutrì,Tchou**: Hamilton-Jacobi equations constrained on networks, hal-00503910 (NoDea)
- **Imbert, Monneau, Zidani**: A Hamilton-Jacobi approach to junction problems and application to traffic flows, arXiv:1107.3250, 2011 (ESAIM Cocv).
- **Camilli, Schieborn**: Viscosity solutions of eikonal equations on topological networks, arXiv:1103.4041v1 (CVPDE)

All the previous papers aim to extend the concept of viscosity solution to networks, but they differ for the assumptions made on the Hamiltonian at the vertices.

Vanishing Viscosity Approximation

In many problems, the vanishing viscosity method arises as a natural selection principle of the physical admissible solution in the class of weak solutions

Goal

Show that viscosity solution of the Hamilton-Jacobi equation

$$H(x, u, \partial u) = 0, \qquad x \in \Gamma,$$

given in **Camilli-Schieborn** can be obtained as limit of solutions to the viscous approximation

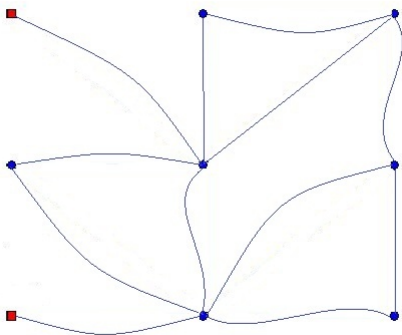$$-\varepsilon \partial^2 u_\varepsilon + H(x, u_\varepsilon, \partial u_\varepsilon) = 0, \qquad x \in \Gamma,$$

letting the viscosity $\varepsilon \to 0^+$

The definition of Network

A network is a connected set Γ consisting of vertices $V := \{v_i\}_{i \in I}$ and edges $E := \{e_j\}_{j \in J}$ connecting the vertices.

We assume that the network is imbedded in the Euclidian space so that any two edges can only have intersection at a vertex.

A coordinate $\pi_j : [0, l_j] \to \mathbb{R}^N$, $j \in J$ is chosen on the edge e_j.

Some Notations

- $Inc_i := \{j \in J : e_j \text{ incident to } v_i\}$ is the set of arcs incident the vertex v_i.

- A vertex v_i is a boundary vertex if it has only one incident edge. We denote by $\partial\Gamma = \{v_i, \ i \in I_B\}$ the set of boundary vertices. A vertex v_i is a transition vertex if it has more than one incident edge. We denote by $\Gamma_T = \{v_i, \ i \in I_T\}$ the set of transition vertices

- The graph is not oriented. The parametrization of the arcs e_j induces an orientation on the edges, expressed by the signed incidence matrix $A = \{a_{ij}\}_{i \in I, j \in J}$

$$a_{ij} := \left\{ \begin{array}{ll} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\ 0 & \text{otherwise.} \end{array} \right.$$

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The network is an imbedded subset of $\mathbb{R}^n$. It follows that the problem is intrinsically 1-dimensional and we can always work with the variable giving the parametrization of the edges

Given $u : \Gamma \to \mathbb{R}$, $u^j : [0, l_j] \to \mathbb{R}$ denotes the restriction of u to $\bar{e}_j$, i.e. $u^j(y) = u(\pi_j(y))$ for $y \in [0, l_j]$

- u is **continuous** ($u \in C^0(\Gamma)$) if $u^j \in C([0, l_j])$ for any $j \in J$ and

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \qquad \text{for any } i \in I, j, k \in Inc_i$$

- For $\alpha \in \mathbb{N}$, differentiation is defined with respect to the parameter variable

$$\partial_j^\alpha u(x) := \frac{d^\alpha}{dy^\alpha} u^j(y), \qquad \text{for } x \in e_j, \ y = \pi_j^{-1}(x)$$

and at a vertex v_i by

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$$\mathbf{H}(\mathbf{x}, \mathbf{u}, \partial \mathbf{u}) = \mathbf{0} \qquad x \in \Gamma$$

where Hamiltonian $H : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a family $\{H^j\}_{j \in J}$ with $H^j : [0, l_j] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

(H1) $H^j \in C^0([0, l_j] \times \mathbb{R} \times \mathbb{R})$ for $j \in J$

(H2) $H^j(x, r, p)$ is non decreasing in r for $p \in \mathbb{R}$, $x \in [0, l_j]$, $j \in J$

(H3) $H^j(x, r, \cdot)$ is convex, coercive in p for any $x \in [0, l_j]$, $j \in J$

(H4) $H^j(\pi_j^{-1}(v_i), r, p) = H^k(\pi_k^{-1}(v_i), r, p)$ for any $p \in \mathbb{R}$, $i \in I$, $j, k \in Inc_i$

(H5) $H^j(\pi_j^{-1}(v_i), r, p) = H^j(\pi_j^{-1}(v_i), r, -p)$ for any $p \in \mathbb{R}$ $i \in I$, $j \in Inc_i$

(H1)- (H3) are standard conditions for HJ equation. Assumptions (H4)-(H5) are continuity at the vertices and independence of the orientation of the incident edge.

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Example

Consider the eikonal equation

$$\lambda u + |\partial u|^2 - f(x) = 0, \qquad x \in \Gamma.$$

where $\lambda \geq 0$, $f \geq 0$ and $f \in C^0(\bar{\Gamma})$, i.e. $f(x) = f^j(\pi_j^{-1}(x))$ for $x \in \bar{e}_j$, $f^j \in C^0([0, l_j])$, and $f^j(\pi_j^{-1}(v_i)) = f^k(\pi_k^{-1}(v_i))$ for any $i \in I$, $j, k \in Inc_i$.

In the theory of viscosity solution, the class of test functions is a crucial point.

Test functions

- φ is a test function at $x \in e_j$, if $\varphi^j := \varphi \circ \pi_j : [0, l_j] \to \mathbb{R}$ is differentiable at $y = \pi_j^{-1}(x)$.

- φ is *(j,k)-test function* at $x = v_i$ if $\partial_j \varphi_j(\pi_j^{-1}(x))$ and $\partial_k \varphi_k(\pi_k^{-1}(x))$ exist. Moreover

$$a_{ij} \partial_j \varphi_j(\pi_j^{-1}(x)) + a_{ik} \partial_k \varphi_k(\pi_k^{-1}(x)) = 0, \qquad (1)$$

where (a_{ij}) is the incidence matrix.

Remark

Condition (1) says that, taking into account the orientation, the function φ is differentiable at v_i along the direction given by the couple of edges e_j and e_k (no condition, except continuity, along the other incident edges)

An u.s.c. function u is called a **viscosity subsolution** if

• for any $x \in e_j$ and any test function φ at x for which $u - \varphi$ attains a local maximum at x, then

$$H^j(y, u^j(y), \partial_j \varphi_j(y)) \leq 0, \qquad y := \pi_j^{-1}(x) \tag{2}$$

• for any $x = v_i$, $i \in I_T$, and for any $j, k \in Inc_i$, any (j, k)-test function φ at x for which $u - \varphi$ attains a local maximum at x, then

$$H^j(y, u^j(y), \partial_j \varphi^j(y)) \leq 0, \qquad y := \pi_j^{-1}(v_i) \tag{3}$$

A l.s.c. function u is called a **viscosity supersolution** if

• for any $x \in e_j$, any test function φ at x for which $u - \varphi$ attains a local minimum at x, then

$$H^j(y, u^j(y), \partial_j \varphi^j(y)) \geq 0, \qquad y := \pi_j^{-1}(x) \tag{4}$$

• for any $x = v_i$, $i \in I_T$, and for any $j \in Inc_i$, there exists $k \in Inc_i$, $k \neq j$ such that for any (j, k)-test function φ at x for which $u - \varphi$ attains a local minimum at x, then

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If supersolutions would be defined similarly to subsolutions, the distance function from the boundary could not be a supersolution (note that the edge e_1 giving the shortest path to the boundary is admissible for the other two edges).
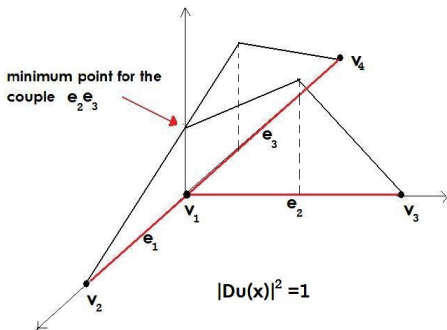

Figure: The distance function

Three basic results for HJ equations

Perron

Let $w \in USC(\Gamma)$ be a subsolution and $W \in LSC(\Gamma)$ a supersolution of (HJ) such that $w \leq W$ and $w_*(x) = W^*(x) = g(x)$ for $x \in \partial\Gamma$. Set
$$u(x) := \sup\{v(x) : v \text{ is a subsol. of (HJ) with } w \leq v \leq W \text{ on } \Gamma\}$$
Then, u^* and u_* are respectively a subsolution and a supersolution to (HJ) in Γ with $u = g$ on $\partial\Gamma$.

Comparison

Let u_1 and u_2 be a subsolution and a supersolution of (HJ) such that $\mathbf{u_1(v_i)} \leq \mathbf{u_2(v_i)}$ for all $v_i \in \partial\Gamma$. Then $\mathbf{u} \leq \mathbf{v}$ in Γ.

Stability

Assume $H_n(x, r, p) \to H(x, r, p)$ locally uniformly for $n \to \infty$. Let u_n be a solution of $H_n(x, u, \partial u) = 0$ for $x \in \Gamma$. Assume $u_n \to u$ uniformly in Γ for $n \to \infty$. Then u is a solution of (HJ).

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Remark

While we do not need to assume convexity of H for the previous results, instead coercivity of H seems to be necessary for this theory. Coercivity gives Lipschitz continuity of the viscosity subsolutions, a regularity that allows to control the behavior of the viscosity solution near the transition vertices when there are several incident edges.

An example

$$\begin{cases} |\partial u|^2 = f(x) & x \in \Gamma \\ u = 0 & x \in \partial\Gamma \end{cases}$$
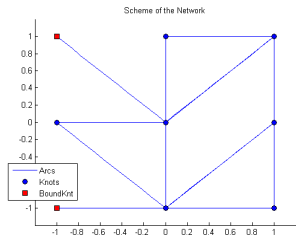
(a) Network
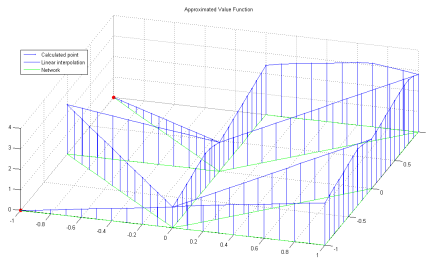
(b) Numerical solution

The solution is the distance from the boundary and it is computed via the numerical scheme in:
F.Camilli, A.Festa e D.Schieborn, Shortest paths and Eikonal equations on a graph, preprint, arXiv:1105.5725, 2011

The viscous HJ equation

Definition

We say that a function $u \in C_K^2(\Gamma)$, if

- $u \in C^0(\Gamma)$
- $u^j \in C^2([0, l_j])$ for any $j \in J$
- for any $i \in I_T$, u satisfies the

    Kirchhoff condition: $\boxed{S(u) := \sum_{j \in Inc_i} a_{ij} \partial_j u(v_i) = 0}$

We look for a solution $u \in C_K^2(\Gamma)$ of the viscous HJ equation

$$\begin{cases} -\varepsilon \partial_j^2 u + H(x, u, \partial u) = 0 & x \in e_j, j \in J \\ u(v_i) = g_i & i \in I_B \end{cases}$$

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For second order problems, we consider the equation inside the edges and the Kirchhoff condition at the nodes. It is known from the theory of linear elliptic PDE on networks that the Kirchhoff condition (or other transition conditions) is necessary for the maximum principle.

The role of the Kirchhoff condition for the maximum principle

Assume that the function w satisfies

$$\partial^2 w(x) \geq 0, \; x \in e_j, \, j \in J, \quad \text{and} \quad S(w) := \sum_{j \in Inc_i} a_{ij} \partial_j w(v_i) \geq 0, \, i \in I_T$$

Then w cannot attain a maximum in $\Gamma \setminus \partial\Gamma$.

Proof.: Assume first $\partial^2 w > 0$ and $S(w) > 0$ and there exists $x_0 \in \Gamma \setminus \partial\Gamma$ such that w attains a maximum at x_0. If $x_0 \in e_j$ for some $j \in J$, then it follows that $\partial_j w(x_0) = 0$ and $\partial_j^2 w(x_0) \leq 0$, a contradiction to $\partial^2 w > 0$.

If $x_0 = v_i$ for some $i \in I_T$, then $a_{ij} \partial_j w(v_i) \leq 0$ for all $j \in Inc_i$, hence $Sw(v_i) \leq 0$, a contradiction to $S(w) > 0$.

For the general case, we prove that there exists $\varphi \in C^2(\Gamma)$ such that $\partial^2 \varphi(x) > 0$ for $x \in e_j, j \in J$ and $S\varphi(v_i) > 0$ for $i \in I_T$ and we consider $w_\delta = w + \delta\varphi, \delta > 0$. Then w_δ cannot attain a maximum in $\Gamma \setminus \partial\Gamma$ and we conclude sending $\delta \to 0^+$.

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The role of the Kirchhoff condition for the maximum principle

Assume that the function w satisfies

$$\partial^2 w(x) \geq 0, \; x \in e_j, \, j \in J, \quad \text{and} \quad S(w) := \sum_{j \in Inc_i} a_{ij} \partial_j w(v_i) \geq 0, \, i \in I_T$$

Then w cannot attain a maximum in $\Gamma \setminus \partial \Gamma$.

Proof.: Assume first $\partial^2 w > 0$ and $S(w) > 0$ and there exists $x_0 \in \Gamma \setminus \partial \Gamma$ such that w attains a maximum at x_0. If $x_0 \in e_j$ for some $j \in J$, then it follows that $\partial_j w(x_0) = 0$ and $\partial_j^2 w(x_0) \leq 0$, a contradiction to $\partial^2 w > 0$.

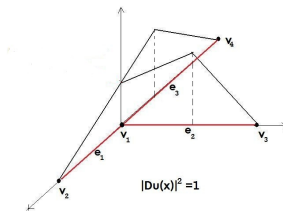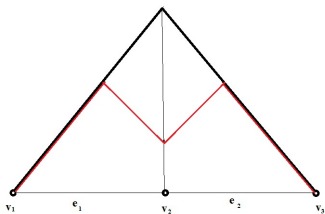If $x_0 = v_i$ for some $i \in I_T$, then $a_{ij} \partial_j w(v_i) \leq 0$ for all $j \in Inc_i$, hence $Sw(v_i) \leq 0$, a contradiction to $S(w) > 0$.

For the general case, we prove that there exists $\varphi \in C^2(\Gamma)$ such that $\partial^2 \varphi(x) > 0$ for $x \in e_j, j \in J$ and $S\varphi(v_i) > 0$ for $i \in I_T$ and we consider $w_\delta = w + \delta\varphi$, $\delta > 0$. Then w_δ cannot attain a maximum in $\Gamma \setminus \partial\Gamma$ and we conclude sending $\delta \to 0^+$.

The Kirchhoff condition is a first order condition for a second order problem. What about Kirchhoff conditions for the HJ equation?

Is it sufficient to assume only the **continuity of the solution at the vertices** to characterize the viscosity solution? No, there are infinite many a.e. solutions satisfying this condition
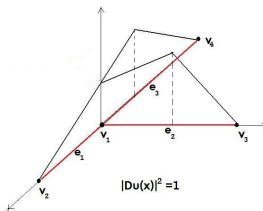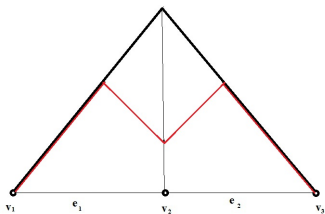
Can we use the Kirchhoff condition to **characterize the viscosity solution** of the HJ equation? No, it is not satisfied by the distance function from the boundary.

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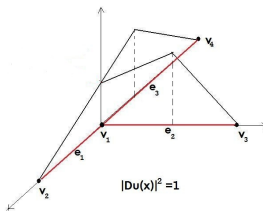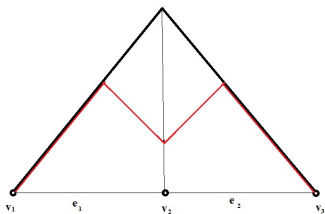
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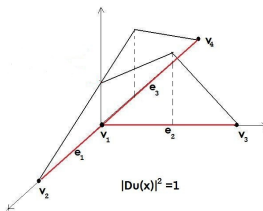

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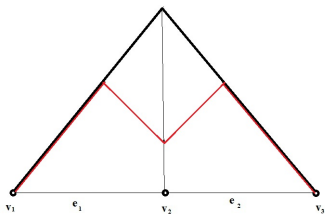
The viscous eikonal equation

Form now on, we consider

$$\begin{cases} -\varepsilon\partial_j^2 u + |\partial u|^2 - f(x) = 0 & x \in e_j, j \in J \\ u(v_i) = g_i & i \in I_B \end{cases}$$

Remark

Only for the **existence** of a solution to the second order problem we are forced to consider the particular case of the eikonal equation. The problem is that a complete theory for nonlinear elliptic PDE on network is still not available. Hence to prove **existence** of a solution to the second order problem we need to resume, via the Hopf-Cole transformation, to the linear case.

For **uniqueness, a priori estimates and vanishing viscosity limit** we can consider the general case $-\varepsilon\partial^2 u_\varepsilon + H(x, u_\varepsilon, \partial u_\varepsilon) = 0$.

The Hopf-Cole transform

Let $u_\varepsilon \in C_K^2(\Gamma)$ be a solution of

$$\begin{cases} -\varepsilon \partial_j^2 u + |\partial u|^2 - f(x) = = 0 & x \in e_j, j \in J \\ u(v_i) = g_i & i \in I_B \end{cases}$$

and set $w_\varepsilon = e^{-\frac{u_\varepsilon}{\varepsilon}} - 1$. Then $w_\varepsilon \in C_K^2(\Gamma)$ (the Hopf-Cole transform preserves the Kirchhoff's cond.) is a solution of the linear problem

$$\begin{cases} \varepsilon \partial_j^2 u + f(x) w_\varepsilon - f(x) = 0 & x \in e_j, j \in J \\ w_\varepsilon(v_i) = e^{-g_i/\varepsilon} - 1 & i \in I_B \end{cases}$$

The linear problem

We consider a linear operator L defined on Γ by

$$L^j w(x) := a^j(x)\partial_j^2 w(x) + b^j(x)\partial_j w(x) - c^j(x)w(x)$$

with $a^j, b^j, c^j, g^j \in C((0, l_j))$, $j \in J$, and

$$a^j(x) \geq \lambda > 0 \text{ and } c^j(x) \geq 0 \; \forall x \in (0, l_j), j \in J$$

Theorem

There exists a unique solution $w \in C_K^2(\Gamma)$ of

$$\begin{cases} L^j w(x) + g^j(x) = 0 & x \in e_j, j \in J \\ w(v_i) = \gamma_i & i \in I_B. \end{cases}$$

Idea of the proof

Uniqueness: Consequence of the maximum principle

Existence: Freidlin-Wentzell (Ann.Prob., 1993):

$$u(x) = \mathbb{E}_x \Big\{ \int_0^\tau e^{-c(Y(s))} g(Y(s)) ds + e^{-c(Y(\tau))} \gamma_{i(\tau)} \Big\}$$

where $Y(s)$ is a Markov process defined on the graph which on each edge e_j solves the stochastic differential equation

$$dY(s) = b^j(Y(s)) ds + a^j(Y(s)) dW(s)$$

$\tau = \inf\{ t > 0 : Y(t) \in \partial\Gamma \}$, $i(\tau) \in I_B$ is such that $Y(\tau) = v_{i(\tau)} \in \partial\Gamma$. In the probabilistic interpretation the Kirchhoff condition implies that the process almost surely spends zero time at each vertex v_i, $i \in I_T$.

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In the probabilistic interpretation the Kirchhoff condition implies that
the process almost surely spends zero time at each vertex v_i, $i \in I_T$.

By the previous result we get via Hopf-Cole transform

Corollary

For any $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in C_K^2(\Gamma)$ of

$$\begin{cases} -\varepsilon \partial_j^2 u + |\partial_j u|^2 - f(x) = 0 & x \in e_j, j \in J \\ u(v_i) = g_i & i \in I_B \end{cases}$$

A comparison result for the semilinear problem

The following comparison result is a key ingredient to get a-priori estimates for the solution of the second order problem

Theorem

Assume that $H(x, r, p)$ is strictly increasing in r. Let $w_1, w_2 \in C^2(\Gamma)$ be such that

$$
\begin{cases}
-\varepsilon \partial_j^2 w_1 + H(x, w_1, \partial_j w_1) \geq -\varepsilon \partial_j^2 w_2 + H(x, w_2, \partial_j w_2) & x \in e_j, \, j \in J, \\
S_\beta^i w_1 \leq S_\beta^i w_2 & i \in I_T \\
w_1(v_i) \geq w_2(v_i) & i \in I_B
\end{cases}
$$

Then $w_1 \geq w_2$ on Γ.

A priori estimates

Theorem

Assume that for each ε, there is a solution $u_\varepsilon \in C_k^2(\Gamma)$ of

$$\begin{cases} -\varepsilon \partial_j^2 w_\varepsilon + H(x, w_\varepsilon, \partial_j w_\varepsilon) = 0 & x \in e_j, j \in J, \\ w_\varepsilon(v_i) = g_i & i \in I_B \end{cases}$$

Then there is $\bar{\varepsilon}$ sufficiently small such that for any $0 < \varepsilon < \bar{\varepsilon}$, the functions u_ε are uniformly bounded and equi-Lipschitz continuous on Γ.

Remark: The estimates are first obtained at the vertices using the boundary condition and the Kirchhoff condition and, then, inside the edges using the estimate at the vertices.

The vanishing viscosity limit

Theorem

Let $u_n := u_{\varepsilon_n} \in C_K^2(\Gamma)$ be a sequence of solutions of

$$\begin{cases} -\varepsilon \partial_j^2 u_\varepsilon + H(x, u_\varepsilon, \partial_j u_\varepsilon) = 0 & x \in e_j, \, j \in J, \\ u_\varepsilon(v_i) = g_i & i \in I_B \end{cases}$$

such that u_n and ∂u_n are uniformly bounded on Γ. If u_n converges uniformly to a function $u \in C(\Gamma)$, then u is a solution of

$$\begin{cases} H(x, u, \partial u) = 0 & x \in e_j, \, j \in J, \\ u(v_i) = g_i & i \in I_B \end{cases}$$

Idea of the proof

To prove the supersolution condition we use the following idea:

- consider $i \in I_T$. The Kirchhoff condition $\sum_{j \in Inc_i} a_{ij} \partial_j u_n(v_i) = 0$ says that we can find a subsequence u_{n_k} and an index $j \in Inc_i$ s.t. (assume $a_{ij} = 1$)

$$\partial_j u_{n_k}(v_i) \leq 0 \qquad \forall n$$

- Formally, at the limit, $\partial_j u(v_i) \leq 0$, hence the function u is decreasing along the edge e_j. This means that e_j is a minimizing edge for the "distance" from the boundary, hence it is admissible for all the other edges e_k, $k \in Inc_i$

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Corollary

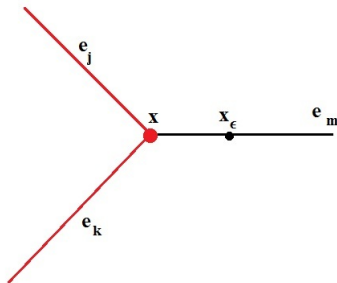Let $u_n := u_{\varepsilon_n} \in C_K^2(\Gamma)$ be the sequence of solutions of

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Then u_n converges uniformly to the solution $u \in C(\Gamma)$ of

$$\begin{cases} |\partial u|^2 - f(x) = 0 & x \in e_j,\, j \in J, \\ u(v_i) = g_i & i \in I_B \end{cases}$$

Remark:

The method of the semi-relaxed limits, which avoids the use of a-priori estimates, does not work with this definition of viscosity solution. Suppose that we want to test $u^*(x) = \limsup^*_{x_\varepsilon \to x, \, \varepsilon \to 0} u_\varepsilon(x_\varepsilon)$ at a vertex x by means of (j, k)-admissible test function φ. We cannot exclude that the possibility that the approximating sequence of points x_ε which defines $u^*(x)$ belongs to a third different edge (e_m in the figure) and we cannot test u_ε at x_ε by means of φ.

Conclusions

- Various transition conditions are introduced in the literature depending on the model problem (linear or nonlinear, dynamical or static, dissipative or non-dissipative, etc). Does the vanishing viscosity limit work with other transition conditions?

- Prove that the various definitions of viscosity solutions for HJ on networks are equivalent, at least under some regularity assumptions.

- Establish a connection between HJ equation and conservation laws on networks (some results are available in Imbert-Monneau-Zidani) .

- Generalize the classical and the viscosity solution theory to fully nonlinear second order problems on networks.

- Various applications: MFG on networks, homogenization, stability and asymptotic behavior of dynamical systems on networks, etc (some of these problems are discussed in the probabilistic literature)

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Thank You!

Everything should be made as simple as possible, but not simpler (A.Einstein)