On exit-time optimal control problems with a vanishing Lagrangian

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- For any x, α, let $y(\cdot) \doteq y_x(\cdot, \alpha)$ denote the solution

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$$t_x(\alpha) = \inf\{t \geq 0 : \quad y(t) \in \mathbf{C}\}$$

- **Payoff**

$$\mathcal{J}(x,\alpha) = \int_0^{t_x(\alpha)} l(y(t),\alpha(t))\,dt$$

Degeneracy

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- We may have instead finite cost even if $t_x(\alpha) = +\infty$

- WE CONSIDER the asymptotic value function

$$\mathcal{V}(x) \doteq \inf_{\{\alpha:\ \liminf_{t\to t_x^-(\alpha)} \mathbf{d}(y(t))=0\}} \mathcal{J}(t_x(\alpha),\alpha)$$

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- **Remark.** The asymptotic exit-time problem is strictly related to the infinite horizon problem. For instance, in the INFINITE HORIZON LINEAR QUADRATIC PROBLEM :

$$\begin{cases} \mathcal{V}^\infty = \inf_\alpha \int_0^{+\infty} [yPy + \alpha Q\alpha]\, dt, \\ y' = f_0(y) + \sum_{i=1}^m f_i(y)\alpha_i, \quad y(0) = x \quad (\alpha(t) \in \mathbb{R}^m) \end{cases}$$

where P, Q are symmetric, positive definite matrices and $f_0(0) = 0$:

$$\mathcal{V}^\infty \equiv \mathcal{V} \ \text{ if } \ \mathbf{C} = \{0\}.$$

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- "Lavrentiev" phenomenon: regularized problems (e.g., $l_n > 0 \to l$), may fail to converge to the original minimization problem.
- "ill posedness" for impulsive problems (if A is unbounded) [Guerra, Sarychev,09]

Goals

GIVE general and explicit sufficient conditions in order to have

- $\mathcal{V}$ continuous in its domain (when continuous on $\partial\mathbf{C}$);
- penalized problems converging to $\mathcal{V}$ (including coercive approximations of non coercive problems);
- characterization of $\mathcal{V}$ as unique non negative viscosity solution of a suitable boundary value problem;
- well-posedness for impulsive problems.

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- $\exists p \geq 1,\ M > 0$ and $\forall R > 0\ \exists L_R > 0$:

$$|f(x_1, a) - f(x_2, a)| \leq L_R(1 + |a|^p)|x_1 - x_2| \quad \text{if } |x_1|, |x_2| \leq R,$$
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- **Remark.** We consider both

(C) $l(x, a) \geq C_2|a|^q - C_1 \qquad C_1, C_2 > 0$
for $q > p$, coercive case; and for $q = p$, weakly coercive case;

(NC) $l \equiv l(x)$, cheap control (in general, l non coercive)

$\forall x$, we consider the set of asymptotic controls

$$\mathcal{A}(x) \doteq \{\alpha \in L^q_{loc}([0,+\infty[,A) : \liminf_{t \to t_x^-(\alpha)} \mathbf{d}(y(t)) = 0\},$$

where $\mathbf{d}(x) \doteq dist(x, \mathbf{C})$.

Continuity propagation

- **Propagation Theorem.** Assume (H1) below. If $\mathcal{V}$ is continuous on $\partial\mathbf{C}$, then it is continuous on its domain.

(H1) *$\forall M$, $\delta>0$, $\exists T$, $K>0$ such that $\forall x\in\mathbb{R}^n\setminus\overline{B(\mathbf{C},\delta)}$ and $\forall\alpha\in\mathcal{A}(x)$ verifying $\int_0^{t_x(\alpha)} l(y_x(t,\alpha),\alpha(t))\,dt\le M$, one has*

$$t_x^\delta(\alpha)\doteq\inf\left\{t>0:\ y_x(t,\alpha)\in\overline{B(\mathbf{C},\delta)}\right\}\le T$$

and

$$\int_0^{t_x^\delta(\alpha)}|\alpha(t)|^q\,dt\le K.$$

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and

$$\int_0^{t_x^\delta(\alpha)}|\alpha(t)|^q\,dt\le K.$$

- **Remark.** If l is satisfies the coercivity condition (C) both for $q>p$ or $q=p$, the last control constraint can be dropped.

Penalized problems

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$$\mathcal{V}_\varepsilon(x) \doteq \inf_{\alpha \in \mathcal{A}(x)} \int_0^{t_x(\alpha)} [l(y(t), \alpha(t)) + \varepsilon \rho(y(t), \alpha(t))] \, dt;$$

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- MOST USED PENALIZATIONS: $\rho \equiv 1$, $\rho \equiv |a|^r$ $(r > q)$

Approximation Theorem. We have

(i) $\lim_{\varepsilon\to 0^+} \mathcal{V}_\varepsilon(x) = \mathcal{U}(x) \doteq \inf_{\{\alpha\in\mathcal{A}(x):\ \int_0^{t_x(\alpha)} \rho(y(t),\alpha(t))\,dt<+\infty\}} \mathcal{J}(t_x(\alpha),\alpha).$

(ii) If $\mathcal{U}$ is continuous on $\partial\mathbf{C}$, then $\mathcal{V}\equiv\mathcal{U}$.

(iii) If $\mathcal{V}_{\bar\varepsilon}$ is continuous on $\partial\mathbf{C}$ for some $\bar\varepsilon>0$ and $\mathcal{V}$ is continuous in its domain,

$$\lim_{\varepsilon\to 0^+} \mathcal{V}_\varepsilon = \mathcal{V}$$

uniformly on compact sets $Q\subset Dom(\mathcal{V})$.

Characterization of $\mathcal{V}$

Consider

$$\mathcal{V}^f(x) \doteq \inf_{\{\alpha:\ t_x(\alpha)<+\infty\}} \mathcal{J}(t_x(\alpha),\alpha), \quad \mathcal{V}^e(x) \doteq \inf_{\alpha} \mathcal{J}(t_x(\alpha),\alpha)$$

IN GENERAL: $\mathcal{V}^e(x) < \mathcal{V}(x) < \mathcal{V}^f(x)$ for some x

$$\Downarrow$$

NO UNIQUE solution to the associated HJ equation

Coercive case

Uniqueness Teorem 1. Assume (C) (i.e. $l \geq C_2|a|^q - C_1$, $C_1, C_2 > 0$) for $q > p$ and "f, l-convexity", in short, (CV). If $\mathcal{V} = \mathcal{V}^e$ and $\mathcal{V}$ is continuous on $\partial\mathbf{C}$, then $\mathcal{V}$ is continuous and it is the unique nonnegative viscosity solution to

$$- \inf_{a \in A} \{\langle f(x,a), Du \rangle + l(x,a)\} = 0 \quad \text{in} \quad Dom(\mathcal{V}) \setminus \mathbf{C}$$

such that $u = 0$ on $\partial\mathbf{C}$.

Remark. We assume $\mathcal{V} = \mathcal{V}^e$, but $\mathcal{V}^f$ may be different...

Weakly coercive and non coercive problems

Following the graph-completion approach, we should introduce generalized or impulsive controls and trajectories.
[Bressan, Rampazzo,88], [Rampazzo, Sartori,00], [Motta, 04]

WE DENOTE by W the generalized asymptotic value function obtained by minimizing over asymptotic generalized controls.

In an analogous way, we introduce also W^f, W^e.

Remark. IN GENERAL, $W^e \leq W \leq W^f$ and

$$W \leq \mathcal{V}, \quad W^f \leq \mathcal{V}^f, \quad W^e \leq \mathcal{V}^e.$$

For UNIQUENESS IN THE NON COERCIVE CASE, consider

MODEL CASE: f, l polynomials in the control variable. E.g. $q = p = 2$ and

$$f(x,a) = f_0(x) + \sum_{i=1}^m f_i(x)a_i + \sum_{i,j=1}^m F_{i,j}(x)a_i\,a_j,$$
$$l(x,a) = l_0(x) + \sum_{i=1}^m l_i(x)a_i + \sum_{i,j=1}^m L_{i,j}(x)a_i\,a_j$$

The recession function of f, l are defined as:

$$f^\infty(x,a) = \sum_{i,j=1}^m F_{ij}(x)a_i\,a_j, \quad l^\infty(x,a) = \sum_{i,j=1}^m L_{ij}(x)a_i\,a_j.$$

Assume

- the traversality condition:

(T) $\forall x \notin \mathbf{C}$, $\forall a \in A$, $a \neq 0$: $|f^{\infty}(x,a)| + l^{\infty}(x,a) \neq 0$.

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- **Remark.** l weakly coercive, i.e. $l \geq C_2|a|^p - C_1 \implies$ (T);

- l cheap: if $l = l(x)$ and $f(x,a) = f_0(x) + \sum_{i=1}^m f_i(x)a_i$,

$$(T) \iff \sum_{i=1}^{m} f_i(x)w_i \neq 0 \quad \forall w \in A,\ w \neq 0.$$

Uniqueness Teorem, 2. Let $q = p$ and $\mathcal{V}$ continuous on $\partial\mathbf{C}$. Then

(i) $\mathcal{V} \equiv W$;

(ii) Assume (CV), (T) (i.e., convexity plus trasversality) and $W = W^e$. Then $\mathcal{V}$ is continuous and it is the unique nonnegative viscosity solution to

$$-\inf_{a\in A}\{\langle f(x,a), Du\rangle + l(x,a)\} = 0 \quad \text{in} \quad Dom(\mathcal{V}) \setminus \mathbf{C}$$

such that $u = 0$ on $\partial\mathbf{C}$.

Remark. Hypothesis $W = W^e$ is stronger than $\mathcal{V} \equiv \mathcal{V}^e$ (assumed in Uniqueness Theorem 1)

The remaining time will be used to state

(i) the result of [Motta, Rampazzo, 12] IMPLYING the continuity of $\mathcal{V}$ on $\partial\mathbf{C}$

(ii) some EXPLICIT conditions IMPLYING both propagation of continuity and $\mathcal{V} \equiv \mathcal{V}^e$, $W \equiv W^e$

Asymptotic Controllability with a cost

Definition. [Motta, Rampazzo, 12] A continuous function $U : \mathbb{R}^n \setminus \mathbf{C} \to \mathbb{R}$ is a **Minimum Restraint Function** , **MRF**, for $A' \subset A$ if

- U is locally semiconcave, positive definite, proper on $\mathbb{R}^n \setminus \mathbf{C}$;
- $\exists k > 0$ such that

$$H(x, k, D^*U(x)) \doteq \inf_{a \in A'} \{\langle D^*U(x) \, , \, f(x, a)\rangle + k\, l(x, a)\} < 0.$$

Sufficient condition, 1

- By [Motta-Rampazzo, '12] it follows that

Theorem 1. If there exists a (local) MRF U for $A' \subset A$, A' compact, then $\mathcal{V}$ is continuous on $\partial \mathbf{C}$.

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> **Theorem 1.** If there exists a (local) MRF U for $A' \subset A$, A' compact, then $\mathcal{V}$ is continuous on $\partial\mathbf{C}$.

- **Remark.** We can obtain similar sufficient conditions for the continuity of $\mathcal{V}_\varepsilon$, $\mathcal{U}$ on $\partial\mathbf{C}$ and $\rho = 1$, $\rho = |a|^r$ $(r > q)$, ...

Sufficient condition, 2

Theorem 2. Assume either coercivity, i.e. (C) for $q \geq p$ or (CV)+(T) (convexity plus trasversality). If

$$l(x,a) \geq m(\mathbf{d}(x)) \quad (\mathbf{d}(x) \doteq dist(x, \mathbf{C}))$$

for some continuous, increasing function $m :]0,+\infty[\to]0,+\infty[$, then

(i) the continuity of $\mathcal{V}$ on $\partial\mathbf{C}$ propagates to its domain;

(ii) $\mathcal{V} \equiv W \equiv W^e \equiv \mathcal{V}^e$.

Sufficient condition, 3

Theorem 3. If there exists a MRF U for A such that

$$\sup_{a\in A}\{\langle D^*U(x)\,,\,f(x,a)\rangle + k\,l(x,a) + (1+|a|^q)m(U(x))\}\leq 0$$

for some continuous, increasing function $m:]0,+\infty[\to]0,+\infty[$, then

(i) $\mathcal{V}$ is continuous in $\mathbb{R}^n$;

(ii) $\mathcal{V}\equiv W\equiv W^e\equiv\mathcal{V}^e$.