Control problems in stratified structures

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joint work with Z. Rao and H. Zidani

Infinite horizon pb in stratified setting

$\mathbb{R}^d = \Omega_1 \cup \Omega_2 \cup \Gamma$

Ω_1, Ω_2 open sets, interface $\Gamma = \partial\Omega_1 = \partial\Omega_2$ C^2- manifold

Two separate control systems on Ω_i with

dynamics $f_i : \overline{\Omega}_i \times A_i \longrightarrow \mathbb{R}^d$

cost $\ell_i : \overline{\Omega}_1 \times A_i \longrightarrow \mathbb{R}$

discount factor $\lambda > 0$ for both domains

usual assumptions apart global boundedness of f_i

$A_i \subset \mathbb{R}^m$ compact, f_i, ℓ_i bdd continuous, f_i Lip. in

x uniformly in a.

Glue together the two systems to get an integrated one on the whole $\mathbb{R}^d$.

On Ω_i: controlled dynamics equivalent to

$$\boxed{\dot{y} \in F_i(y) = \{ f(x, a) : a \in A_i \}}$$

We define

$$F(x) = \boxed{\begin{cases} F_i(x) & \text{on } \Omega_i \\ \{ \lambda F_1(x) + (1-\lambda) F_2(x) : \lambda \in [0,1] \} & \text{on } \Gamma \end{cases}}$$

F_i Lipschitz continuous, but F is just usc compact, but non convex valued. Need specific controllability-type to guar assumptions on Γ to guarantee existence of solutions to differential inclusion.

Equivalent controlled dynamics (Filippov)

$$\dot{y} = f(y, \alpha)$$

$\alpha \in L^{\infty}(0, +\infty; \mathbb{R}^m)$ adapted to the state

$\alpha(t) \in A_i$ a.e. t with $y(t) \in \Omega_i$

$\alpha(t) = \mu a_1 + (1-\mu) a_2$ a.e. t with $y(t) \in \Gamma$

corresponding convexification of costs

$\ell(y(t), \alpha(t)) = \ell_i(y(t), \alpha(t))$ $\quad y(t) \in \Omega_i$

$\ell(y(t), \alpha(t)) = \lambda \ell_1(y, a_1) + (1-\lambda) \ell_2(y, a_2)$ $\quad y(t) \in \Gamma$

convexification is a natural scalarization in multiobjective optimization. Less standard the connection dynamics - costs.

Barles - Briani - Chasseigne 2011

Value function

$$v(x) = \inf \left\{ \int_0^{+\infty} e^{-\lambda s} \ell(y,\alpha)\, ds : (y,\alpha) \text{ admissible}, \ y(0) = x \right\}$$

Write down an adapted HJB equation admitting v as unique viscosity solution in $\mathbb{R}^d$.

$$\boxed{\begin{array}{l} \lambda u + H(x, Du) = 0 \\ H = \begin{cases} \max\limits_{A_i} \{ -p \cdot f_i(x,\alpha) - \ell_i(x,\alpha) \} & \text{in } \Omega_i \times \mathbb{R}^d \\ ? & \text{on } \Gamma \end{cases} \end{array}}$$

Wide literature on switched and hybrid systems from a dynamic viewpoint (engineering applications).

Dual approach in terms of PDE has so far little contributions.

Soravia (2002) discontinuous coefficients, uniqueness for related boundary pbs

Camilli and Siconolfi (2003) coercive measurable Hamiltonians

works sharing same difficulties

Imbert, Monneau, Zidani (junctions and traffic), Falcone and coauthors (patchy schemes and parallel algorithms), Camilli and coauthors (HJ on graphs and networks

More influential

Bressan, Hong (2007) introduce stratified settings, relevance of tangential equations

Barnard Wolanski (preprint) essential dynamics, Filippov appr. Thm

Barles and al. (2011) coercive Ham., Lip. continuity of subsol's

$$\boxed{\dot{y} = F(y)}$$

$\Theta = \{ t : y(t) \in \Gamma \}$

For a.e. $t \in \Theta$

t is not isolated in Θ (iso. points are countable)

y is diffe at t (a.e. differentiability)

for such a t, take $t_n \to t$, $t_n \to t$, $t_n \in \Theta$

$$\dot{y}(t) = \lim_n \frac{y(t_n) - y(t)}{t_n - t} \in F(y(t)) \cap T_\Gamma(y(t))$$

y integral curve of an essential dynamics

$$\boxed{F^{ess}(x) = \begin{cases} F_i(x) & x \in \Omega_i \\ F(x) \cap T_\Gamma(x) & x \in \Gamma \end{cases}}$$

we expect to find in the Hamiltonian on Γ only essential directions.

In this sense theory not adequate

for subsol'ns

$$\boxed{\begin{array}{ll} \lambda u + \max\{H_1(x, Du), H_2(x, Du)\} \geq 0 & \text{on } \Gamma, \text{ visco sense} \\ \lambda u + \min\{H_1, H_2\} \leq 0 & \end{array}}$$

Supersol'n condition involves ε-optimal directions, $\forall \varepsilon > 0$.

Subsol'n condition implies (assume v diff'le)

$$-Dv(x) \cdot f_i(x, a) - \ell_i(x, a) \leq 0$$

for a choice of i, any $a \in A_i$, involves directions <u>not</u> <u>effective</u> for the dynamics

1- dimensional example

$\Omega_1 = (0, +\infty)$, $\Omega_2 = (-\infty, 0)$, $\Gamma = \{0\}$

$A_1 = A_2 = [-1, 1]$, $\lambda = 1$

$$\boxed{\begin{array}{lll} f_1(x, a) = a & & [0, +\infty) \times [-1, 1] \\ f_2(x, a) = a & & (-\infty, 0] \times [-1, 1] \end{array}}$$

state dependent switching on costs

$$\boxed{\begin{array}{l} \ell_1(x, a) = -2a \\ \ell_2(x, a) = -a \end{array}}$$

optimal strategy is to go right with velocity +1

optimal curve $x+t$ for any initial curve

$$\boxed{v(x) = \begin{cases} -2 & x \geq 0 \\ -1 - e^{x} & x < 0 \end{cases}}$$

Hamiltonians

$$H_1(x,p) = \max_{[-1,1]} \{(-p+2)a\} = |2-p| \qquad x \geq 0,\ p \in \mathbb{R}$$

$$H_2(x,p) = |1-p| \qquad x < 0,\ p \in \mathbb{R}$$

HJB equations

$$u + |2-u'| = 0 \qquad (0,+\infty)$$

$$u + |1-u'| = 0 \qquad (-\infty,0)$$

v is classical solution of both eqns + continuous at 0

Same property holds for

$$u_\mu = \begin{cases} -2+\mu e^{x} & x \geq 0 \\ -1+(\mu-1)e^{x} & x < 0 \end{cases} \qquad \mu \leq 0$$

v recovered for $\mu = 0$

Introduce a tangential Hamiltonian

$$H_\Gamma(x,p) = \max\{-f(x,a)\cdot p - \ell(x,a) : f(x,a) \in F^{ess}_{(x)}\}$$

and a tangential equation

$$\lambda u + H_\Gamma(x, Du) \le 0$$

test function ϕ st x loc. max of $u-\phi$ in Γ

$$(HJB) \quad \boxed{\begin{cases} \lambda u + H_i(x,Du) = 0 & \text{in } \Omega_i \\ \lambda u + \quad\quad \ge 0 & \text{on } \Gamma \quad (\text{usual test fcts}) \\ \lambda u + H_\Gamma(x,Du) \le 0 & \text{on } \Gamma \quad (\text{tangential test fcts}) \end{cases}}$$

under suitable transmission ass.

$\boxed{\text{Fact 1}}$ $\boxed{\text{Value fuct. } v \text{ is bdd continuous sol. to (HJB).}}$

Fact 2

If u, w bdd usc subsol'n and lsc supersol to (HJB) and u continuous at any point of Γ, then $u \leq w$

Fact 3

Value function is unique bdd continuous sol to (HJB).

Assumptions at any $x \in \Gamma$

$i = 1, 2$

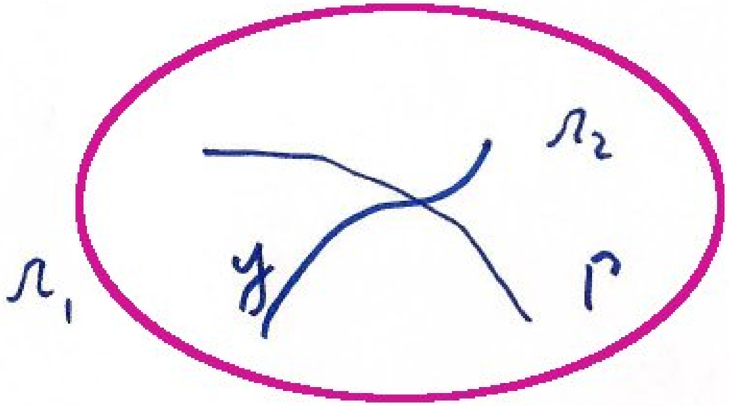
$\exists\, a, b \in A_i$ such that $f_i(x, a)$ points strictly inward Ω_1, $f_i(x, b)$ points strictly inward Ω_2

$\exists\, R > 0$ such that $F^{ess}(x) \supset B(0, R) \cap T_\Gamma(x)$

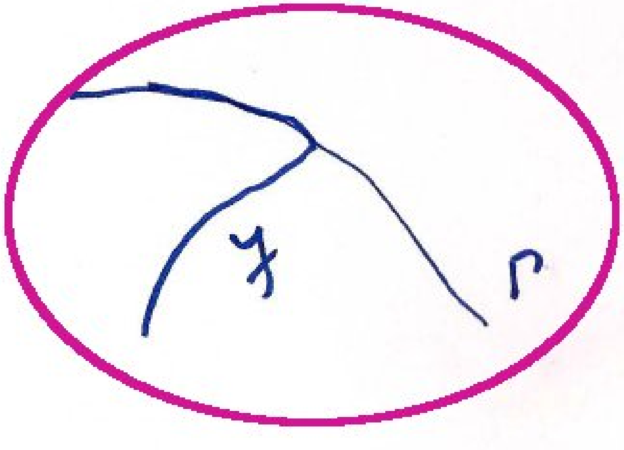
controllability on the interface.

First condition implies : $\forall x \in \Gamma$ $\exists$ admissible traj. starting at x and living in Ω_i $(i=1,2)$ for a locally uniformly estimated time.

When an admissible trajectory reach Γ two options



cross the interface and pass to the other region



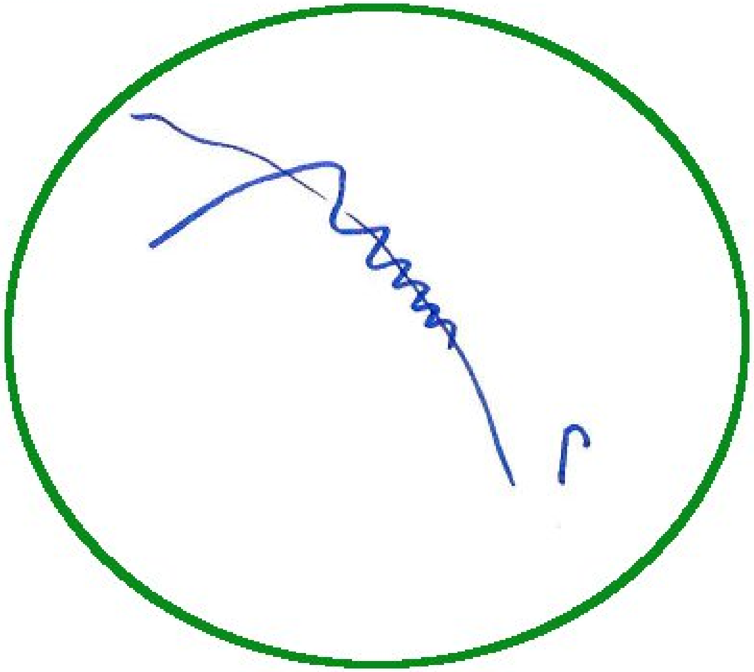
take a sliding mode and follow the interface for a while.

Crucial difficulty is that trajectories with a more complicated behavior are also possible.

Event times for a traj. y

$$\boxed{\partial \{ t : y(t) \in \Omega_1 \cup \Omega_2 \}}$$

Event times set can have accumulation points



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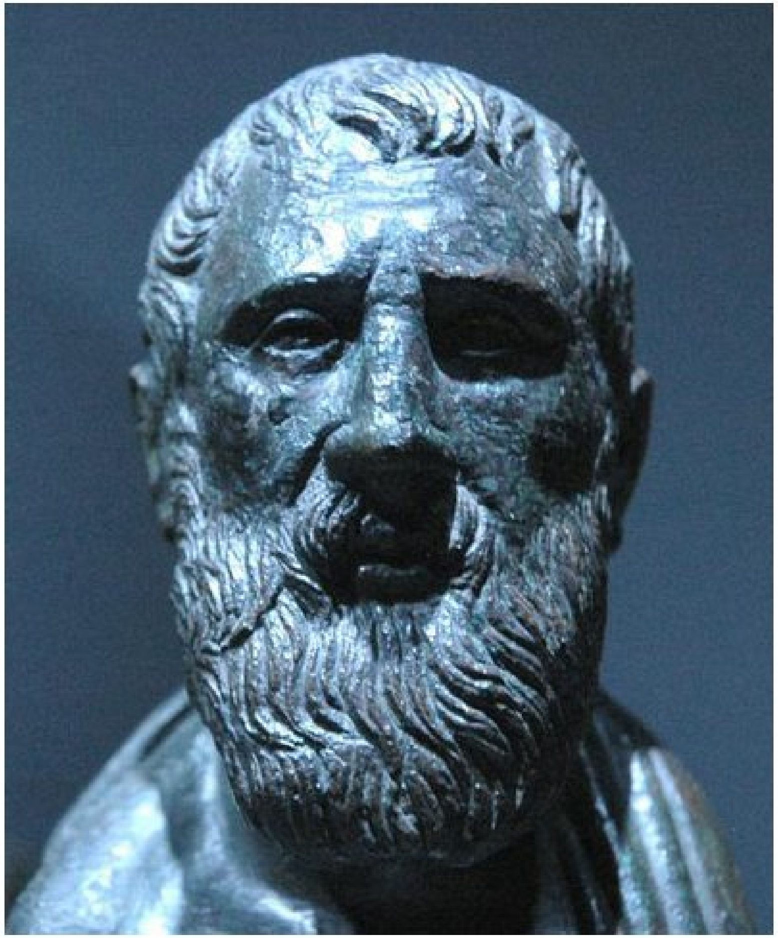

Zeno of Elea (ca 490 bc - ca 430 bc)

Zeno behavior - Zenonnes -