Noncoercive Hamiltonians, Absolute Minimizers and Aronsson Equation

Pierpaolo Soravia
Università di Padova (Italy)
soravia@math.unipd.it

The problem

For $f : \mathbb{R}^n \times A \to \mathbb{R}^n$, $h : \mathbb{R}^n \times A \to \mathbb{R}$ bounded from below, we have a Hamiltonian

$$H(x,p) = \sup_{a \in A}\{-f(x,a)\cdot p - h(x,a)\}(\geq 0).$$

A may be unbounded. H is continuous but in general not coercive.
Given $\Omega \subset \mathbb{R}^n$ and $g : \partial\Omega \to \mathbb{R}$ we want to minimize

$$\sup_{x \in \Omega} H(x, Du(x)),$$

subject to $u(x) = g(x)$, for $x \in \partial\Omega$.

Question. Choose the correct class of functions.

If $H(x,\cdot)$ is coercive ($H(x,p) \to +\infty$ as $|p| \to +\infty$) then it is natural to consider $u \in W^{1,+\infty}_{loc}(\Omega) \cap C(\overline{\Omega})$ so that

$$\text{ess sup } H(x, Du(x)) \to \min .$$

A classical problem: Lipschitz extension: if $g \in Lip(\partial\Omega)$,

$$|g(x) - g(y)| \leq L|x - y|, \quad x, y \in \partial\Omega,$$

find $u \in Lip(\overline{\Omega})$, $u = g$ on $\partial\Omega$ such that

$$|u(x) - u(y)| \leq L|x - y|, \quad x, y \in \overline{\Omega}.$$

Equivalently, minimize

$$\text{ess sup } |Du(x)|, \quad (or \text{ ess sup } \frac{|Du(x)|^2}{2}).$$

The Lipschitz extension problem has possibly many solutions, and the extremal ones are explicit.
It is a nonstandard variational problem versus

$$\int_\Omega |Du(x)|^p \, dx \to \min$$

(worst case analysis) and its solutions do not have in general the property of being local minima in order to be able to derive an *Euler equation*.

The Aronsson Equation

We will introduce the notion of Absolute minimizer and the Aronsson equation

$$-D\left(H(x, Du(x))\right) \cdot D_p H(x, Du(x)) = 0,\ x \in \Omega \quad (AE)$$
$$u(x) = g(x) \in \ C(\partial\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is open and connected, $H = H(x,p)$, $H : \Omega \times \mathbb{R}^N \to [0, +\infty)$ and $H(x, \cdot)$ is convex and in general not coercive, $u : \overline{\Omega} \to \mathbb{R}$.

We will introduce the notion of Absolute minimizer and the Aronsson equation

$$-D\left(H(x,Du(x))\right)\cdot D_pH(x,Du(x))=0,\ x\in\Omega \quad (AE)$$
$$u(x)=g(x)\in\ C(\partial\Omega),$$

where $\Omega\subset\mathbb{R}^N$ is open and connected, $H=H(x,p)$, $H:\Omega\times\mathbb{R}^N\to[0,+\infty)$ and $H(x,\cdot)$ is convex and in general not coercive, $u:\overline{\Omega}\to\mathbb{R}$.

Classical case: $H\in C^1$, $u\in C^2$. (AE) is a quasilinear degenerate elliptic pde.
Taking derivatives AE becomes

$$-\mathrm{Tr}(D_pH\otimes D_pH\ D^2u)-D_xH\cdot D_pH=0$$

Recall that:

$$H(x,p) = \sup_{a \in A}\{-f(x,a) \cdot p - h(x,a)\},$$

- the problem can be reduced to $H \geq 0$,
- total controllability of the vectogram $f(x, A)$ in every connected and open subdomain is a crucial ingredient.

Included is the case of a Carnot-Carathèodory structure:

$$H(x,p) = \tilde{H}(x, \sigma^t(x)p) \tag{1}$$

Included is the case of a Carnot-Carathèodory structure:

$$H(x,p) = \tilde{H}(x, \sigma^t(x)p) \tag{1}$$

$\sigma : \Omega \to \mathbb{R}^{N\times M}$, $M \leq N$, is a matrix valued function whose columns are a family of vector fields $\{\sigma_j\}_{j=1,\dots,M}$, satisfying Hörmander's finite rank condition and $\tilde{H} : \Omega \times \mathbb{R}^M \to [0, +\infty)$ is coercive.

Infinity Laplace equation

When in particular (for $A = \sigma\sigma^t$, $N \times N$ matrix,)

$$H(x,p) = \frac{1}{2}A(x)p \cdot p = \frac{1}{2}|\sigma^t(x)Du(x)|^2,$$

Infinity Laplace equation

When in particular (for $A = \sigma\sigma^t$, $N \times N$ matrix,)

$$H(x,p) = \frac{1}{2} A(x) p \cdot p = \frac{1}{2} |\sigma^t(x) Du(x)|^2,$$

we obtain the infinity-Laplace equation with respect to the family of vector fields,

$$-\Delta_\infty^\sigma u(x) = -D_\sigma^2 u(x)\; D^\sigma u(x) \cdot D^\sigma u(x) = 0, \quad x \in D,$$

Infinity Laplace equation

When in particular (for $A = \sigma\sigma^t$, $N \times N$ matrix,)

$$H(x,p) = \frac{1}{2}A(x)p \cdot p = \frac{1}{2}|\sigma^t(x)Du(x)|^2,$$

we obtain the infinity-Laplace equation with respect to the family of vector fields,

$$-\Delta_\infty^\sigma u(x) = -D_\sigma^2 u(x)\ D^\sigma u(x) \cdot D^\sigma u(x) = 0, \quad x \in D,$$

where $D^\sigma u(x) = \sigma^t(x)Du(x)$ are the directional derivatives with respect to the family of vector fields (horizontal gradient) and the horizontal hessian $D_\sigma^2 u(x) = \left(\frac{1}{2}(D_j^\sigma(D_i^\sigma u(x)) + D_i^\sigma(D_j^\sigma u(x)))\right)_{i,j}$.

Variational theory

Aronsson's Lipschitz extension problem (1965): given $g \in Lip(\partial\Omega)$, find $u : \overline{\Omega} \to \mathbb{R}$ such that $u|_{\partial\Omega} = g$ with the same best Lipschitz constant.
Equivalently minimize the following functional

$$\|Du\|_{L^\infty(\Omega)} \to \min$$

with prescribed boundary conditions.

Variational theory

Aronsson's Lipschitz extension problem (1965): given $g \in Lip(\partial\Omega)$, find $u : \overline{\Omega} \to \mathbb{R}$ such that $u|_{\partial\Omega} = g$ with the same best Lipschitz constant.
Equivalently minimize the following functional

$$\|Du\|_{L^\infty(\Omega)} \to \min$$

with prescribed boundary conditions.
The problem has minima and they are nonunique. To derive a Euler-Lagrange equation, define Absolute Minimizers, i.e. minima in every relatively compact open subdomain.

Aronsson proves that $u \in C^2$ is an absolute minimizer iff it solves the infinity-Laplace equation. Existence of absolute minimizers is obtained through approximation with more classical variational problems

$$\int_\Omega |Du|^p dx \to \min,$$

as $p \to +\infty$.

Aronsson proves that $u \in C^2$ is an absolute minimizer iff it solves the infinity-Laplace equation. Existence of absolute minimizers is obtained through approximation with more classical variational problems

$$\int_\Omega |Du|^p dx \to \min,$$

as $p \to +\infty$.
In general given a Hamiltonian $H(x,p)$, for $u : \Omega \to \mathbb{R}$, the problem

$$\operatorname{ess\,sup}_{x\in\Omega} H(x, Du(x)) \to \min$$

corresponds to the Aronsson equation.

More references

- Jensen (1993): obtains Aronsson's results for the infinity-Laplacian without the regularity assumption on u;
- Juutinen (1998/2002): extends Jensen to a class of strictly convex functionals and existence of the absolutely minimizing Lipschitz extension in length spaces;
- Barron-Jensen-Wang (2001): derivation of AE for general Hamiltonian and existence of absolute minimizers via approximation with L^p minimization;
- Barles-Busca (2001): uniqueness proof for a class of quasilinear x-independent equations;
- Crandall (2003)/Crandall-Wang-Yu (2009) general derivation of AE for C^1 Hamiltonians.

- Bieske (2005) equivalence of AM and solutions of AE for Riemannian metrics and Grushin;
- Bieske-Capogna (2005)/Wang (2007): derivation of AE in Carnot-Caratheodory spaces, uniqueness in Carnot groups;
- Peres-Schramm-Sheffield-Wilson (2009) Tug of war approach, existence and uniqueness of the absolutely minimizing Lipschitz extension in length spaces;

towards a pde approach:

- Crandall-Evans-Gariepy (2001): for infinity-Laplace, comparison with cones, continuity estimates;
- Aronsson-Crandall-Juutinen (2004): a complete pde approach for the infinity-Laplace equation (euclidean), comparison with cones, Perron's method, Harnack inequality.
- Champion-De Pascale (2007): equivalence of AM and comparison with cones in length spaces.
- Jensen-Wang-Yu (2009) results on uniqueness and non uniqueness, $x-$independent;
- Savin (2000) C^1 regularity of infinitely harmonic functions in dimention 2.
- Evans-Savin (2008) $C^{1,\alpha}$ regularity of infinitely harmonic functions in dimention 2.

AE and optimal control

There are important relationships with solutions of the HJ equations (for us $H(x,\cdot\,)$ is not symmetric)

$$H(x, Du(x)) = k > 0, \quad H(x, -Du(x)) = k, \ x \in \Omega.$$

(recall that $H(x,p) = \sup_{a \in A}\{-f(x,a)\cdot p - h(x,a)\}$.)

AE and optimal control

There are important relationships with solutions of the HJ equations (for us $H(x,\cdot\,)$ is not symmetric)

$$H(x,Du(x))=k>0,\quad H(x,-Du(x))=k,\ x\in\Omega.$$

(recall that $H(x,p)=\sup_{a\in A}\{-f(x,a)\cdot p-h(x,a)\}$.) These are value functions of optimal control problems, thus we consider the control system

$$\begin{cases} \dot y(t)=f(y(t),a(t)), & t>0,\\ y(0)=x_o, \end{cases} \tag{2}$$

or its *backward* version $\dot y(t)=-f(y(t),a(t))$, for suitable control functions $a(\cdot)\in\mathcal{A}$.

Cone functions

We assume Total controllability. For all $x, z \in D \subset\subset \Omega$, D open and connected, the set

$$\mathcal{A}^D_{x,z} = \{a(\cdot) \in \mathcal{A} : y(0,a) = x,\ y(t_{x,z};a) = z,\ t_{x,z} \geq 0 \\ y(t,a) \in D,\ t \in (0, t_{x,z})\} \neq \emptyset. \tag{3}$$

We assume Total controllability. For all $x, z \in D \subset\subset \Omega$, D open and connected, the set

$$\mathcal{A}^D_{x,z} = \{a(\cdot) \in \mathcal{A} : y(0,a) = x,\ y(t_{x,z};a) = z,\ t_{x,z} \geq 0 \\ y(t,a) \in D,\ t \in (0, t_{x,z})\} \neq \emptyset. \quad (3)$$

For any (large) $k > 0$, we therefore define the following function (generalized cone). For convenience $J_1 \equiv J$.

$$J^D_k(x,z) = \inf_{a(\cdot) \in \mathcal{A}_{x,z}} \int_0^{t_{x,z}} (h(y(t), a(t)) + k)dt < +\infty,$$

$$J^D_k(\hat{x}, \hat{z}) = \liminf_{D\times D \ni (x,z) \to (\hat{x},\hat{z})} J^D_k(x,z), \quad (\hat{x}, \hat{z}) \in \overline{D} \times \overline{D}.$$

Basic setting

Each J_k is a semi-distance in D (cfr. PL Lions' book, it lacks symmetry) and satisfies

$$J(x,z) \geq \lambda_K |x-z| (\geq 0) \quad \text{for } J(x,z) \leq K.$$

Basic setting

Each J_k is a semi-distance in D (cfr. PL Lions' book, it lacks symmetry) and satisfies

$$J(x,z) \geq \lambda_K |x-z| (\geq 0) \quad \text{for } J(x,z) \leq K.$$

We assume it is "uniformly continuous" and satisfies local estimates in D of the form

$$J(x,z) \leq \omega(|x-z|),$$

Basic setting

Each J_k is a semi-distance in D (cfr. PL Lions' book, it lacks symmetry) and satisfies

$$J(x,z) \geq \lambda_K |x-z| (\geq 0) \quad \text{for } J(x,z) \leq K.$$

We assume it is "uniformly continuous" and satisfies local estimates in D of the form

$$J(x,z) \leq \omega(|x-z|),$$

Thus

$$H(x, D_x J_k(x,z)) = k, \quad x \in D \backslash \{z\}, J_k(z,z) = 0.$$
$$H(z, -D_z J_k(x,z)) = k, \quad z \in D \backslash \{x\}, J_k(x,x) = 0.$$

Distances and AE equation

Although $V(x) = J_k(x, z)$ satisfies the HJ equation:

$$H(x, DV(x)) = k, \quad x \in D \backslash \{z\}, \quad (HJ),$$

we have that

Theorem. $V(x) = J_k(x, z)$, $k > 0$, is a viscosity supersolution of the Aronsson equation

$$-D_x (H(x, DV(x))) \cdot D_p H(x, DV(x)) = 0, \quad x \in D \backslash \{z\},$$

and it is a viscosity solution iff it is a bilateral solution of (HJ) (see Barron-Jensen, Barles, Pi. Sor.). In this case V is also unique as an absolute minimizer.

A-priori regularity

An important feature of the distances is that they determine the *regularity of subsolutions* of the HJ equation: if u is a viscosity solution of $H(x, Du(x)) \leq k$ in Ω, then locally (C depends on $\sup |u|$)

$$|u(x) - u(y)| \leq C\ J(x, y).$$

A-priori regularity

An important feature of the distances is that they determine the *regularity of subsolutions* of the HJ equation: if u is a viscosity solution of $H(x, Du(x)) \leq k$ in Ω, then locally (C depends on $\sup |u|$)

$$|u(x) - u(y)| \leq C\ J(x, y).$$

In the CC case $\tilde{H}(x, \sigma^t(x)p)$, subsolutions are locally Lipschitz continuous with respect to the CC distance.

$$d^D_{CC}(x, z) = \inf_{a \in \mathcal{A}^D_{x,z}} t_{x,z}.$$

A-priori regularity

An important feature of the distances is that they determine the *regularity of subsolutions* of the HJ equation: if u is a viscosity solution of $H(x, Du(x)) \leq k$ in Ω, then locally (C depends on $\sup |u|$)

$$|u(x) - u(y)| \leq C \; J(x, y).$$

In the CC case $\tilde{H}(x, \sigma^t(x)p)$, subsolutions are locally Lipschitz continuous with respect to the CC distance.

$$d^D_{CC}(x, z) = \inf_{a \in \mathcal{A}^D_{x,z}} t_{x,z}.$$

We can use the well established theory of Sobolev spaces for CC metrics (e.g. Franchi-Serapioni-Serra Cassano, Garofalo-Nhieu, Franchi-Hajlasz-Koskela, Bonfiglioli-Lanconelli-Uguzzoni).

In particular:

Proposition. (Pi. Sor. 2010) For $u : \Omega \to \mathbb{R}$ we have that

$$\tilde{H}(x, \sigma^t(x)Du(x)) \leq k$$

in the viscosity sense if and only if

$$\tilde{H}(x, Xu(x)) \leq k, \quad \text{a.e. } x \in \Omega.$$

Here the differential operator $X = \sigma^t(x)D = D^\sigma$ has to be interpreted in the sense of distributions.

This holds in particular if $\sigma(x) \equiv I_n$.

On J_k we also assume

$$J_k(x,z) \geq \kappa_1^D(J(x,z))\kappa_2^D(k), \quad \text{for all } z \in D, \tag{4}$$

if the set $\{x : J(x,z) \leq r\} \subset \Omega$. Here

$\kappa_1^D,\ \kappa_2^D : (0,+\infty) \to (0,+\infty)$ are increasing and κ_2^D is surjective.
This happens for instance for:

- A bounded ;
- A unbounded if $H(x,\cdot)$ is positively homogeneous.

Generalized cone functions

A (generalized) cone with positive slope $k > 0$ and vertex $z \in \mathbb{R}^n$, possibly constrained in D with $z \in \partial_a D$ is a function ($b \in \mathbb{R}$)

$$C(x) = J_k(x, z) + b.$$

A (generalized) cone with negative slope $k < 0$ and vertex $z \in \mathbb{R}^n$ is a function

$$C(x) = -J_{|k|}(z, x) + b.$$

Viscosity absolute minimizers

Definition. Given a Hamiltonian $H \in C(\Omega \times \mathbb{R}^n)$, $H(x, \cdot)$ convex for all x, we say that $u \in C(\overline{U})$, $U \subset \Omega$ open, is an absolute minimizer (in the viscosity sense) for H if for any open, bounded subset $D \subset\subset U$ we have that whenever $v \in C(\overline{D})$ is such that $u(x) = v(x)$ in ∂D and

$$H(x, Dv(x)) \leq k, \quad \forall x \in D$$

in the viscosity sense, then

$$H(x, Du(x)) \leq k, \quad \forall x \in D, \tag{5}$$

in the viscosity sense.

A delicate property of the HJB

For the forward and backward Hamiltonian it holds that if $u \in C(\Omega)$, then

$$H(x, Du(x)) \leq k, \quad x \in \Omega$$

in the viscosity sense, if and only if

$$H(x, -D(-u)(x)) \leq k, \quad x \in \Omega.$$

Note: u absolute minimizer for H implies that $-u$ is absolute minimizer for $H(x, -p)$.
Also the variational problem may be seen as

$$\sup_{(x,p)\in D\times(D^+u(x)\cup D^-u(x))} H(x, p) \to \min .$$

(Recall here that $D^+(-u) = -D^-u$.)

A locally d_{CC}-Lipschitz continuous function $u : U \to \mathbb{R}$, is an absolute minimizer for H if and only if for any open, bounded subset $D,\ D \subset\subset U$ we have that whenever $v \in C(\overline{D}) \cap W^{1,\infty}_X(D)$ is such that $u(x) = v(x)$ in ∂D and

$$\tilde{H}(x, Xv(x)) \leq k, \quad \text{a.e. } x \in D,$$

then

$$\tilde{H}(x, Xu(x)) \leq k, \quad \text{a.e. } x \in D. \tag{6}$$

This is as saying that u is a local minimizer in U for the variational problem

$$\operatorname{ess\,sup} H(x, Xv(x)) \to \text{min}$$

among all locally d_{CC}-Lipschitz continuous functions v.

Perron’s method

Comparison with cones

We have the following characterization.

Theorem. (Pi. Sor., cfr. Champions-De Pascale)
$u \in C(U)$ is an absolute minimizer if and only if:

- $u \in CCA(U)$: for any open, connected and bounded set $V \subset\subset U$, $k > 0$, $z \notin V$ and cone $C(x)$ (possibly relative to V) with slope k and vertex z we have that

$$u(x) - C(x) \leq \sup_{w \in \partial V} \{u(w) - C(w)\}, \quad \text{for all } x \in V;$$

- $u \in CCB(U)$ i.e. for $k < 0$, $z \notin V$ we have that

$$u(x) - C(x) \geq \inf_{w \in \partial V} \{u(w) - C(w)\}, \quad \text{for all } x \in V.$$

Properties of functions in CCA and CCB

Propositions.

- (local Lipschitz/Hölder continuity) If $u \in CCA(D)$ and $E \subset\subset D$, then we can find a constant L, depending only on $\|u\|_\infty$, κ_1^E, κ_2^E and $\inf_{y\in E} d(y, \partial D)$, such that

$$|u(x) - u(y)| \le L\hat{J}(x,y), \quad \forall z \in E,$$
$$x, y \in B_R(z),\ 3R < \inf_{z\in E} d(z, \partial D).$$

- Uniformly bounded families of functions in $CCA(D)$ are locally equi-Lipschitz continuous with respect to the distance $\hat{J}$ induced by the Hamiltonian;

- (strong maximum principle) if $u \in CCA(D)$ has a local maximum, then it is locally constant;
- (existence of functions in CCA) a cone C with negative slope satisfies $C \in CCA$ (and it is a viscosity subsolution of the (AE));
- (Harnack inequality) If $H(x,\cdot)$ is positivey homogeneous, $u \geq 0$, $u \in CCB(D)$, then for $z \in D$

$$\max_{J(x,z)\leq R} u(x) \leq 3 \min_{J(y,z)\leq R} u(y), \quad R < \frac{d(z,\partial D)}{4};$$

- Let $\emptyset \neq \mathcal{F} \subset CCA(D)$ and

$$h(x) = \sup_{v \in \mathcal{F}} v(x).$$

If h is locally bounded from above then $h \in CCA(D) \cap C(D)$.

- We couple the last property with the more standard (in viscosity solutions theory): Let $\emptyset \neq \mathcal{F} \subset C(D)$ be a family of viscosity subsolutions of (AE),

$$h(x) = \sup_{v \in \mathcal{F}} v(x).$$

If $h \in C(D)$ then h is a viscosity subsolution of the (AE).

Existence

Theorem. (Pi.Sor. 2010) Let $D \subset \Omega$ open, $g \in C(\partial D)$, $b^-, b^+ \in \mathbb{R}$, $k^- < 0$, $k^+ > 0$, $z \in \partial D$:

$$C^-(x) = -J_{|k^-|}(z,x) + b^- \leq g(x) \leq J_{k^+}(x,z) + b^+ = C^+(x),$$

for all $x \in \partial D$. Then there exists $u \in C(\overline{D})$ an absolute minimizer, such that $u = g$ on ∂D and

$$C^-(x) \leq u(x) \leq C^+(x), \quad x \in D.$$

If moreover $H \in C^1(D \times \mathbb{R}^N)$ then there exists an absolute minimizer that is in addition a viscosity solution of the AE.

NB. Perron's method cannot prove that AE is satisfied by all of the absolute minimizers (unless comparison principle holds).