Inward pointing trajectories, Lavrentieff phenomenon and normality of maximum principle for Bolza problem under state constraints

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Joint work with Hélène Frankowska

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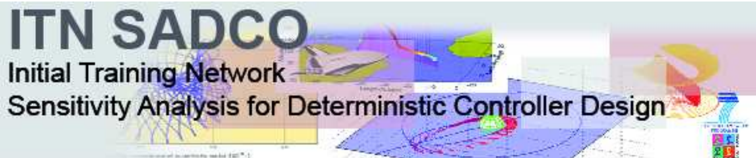

The Bolza Problem

Consider the control system

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{for a.e. } t \in [0,1], \tag{1}$$

$$x(t) \in K \quad \text{for all } t \in [0,1], \quad (x(0), x(1)) \in K_1, \tag{2}$$

- $U(\cdot)$ measurable set-valued map from $[0,1]$ into nonempty closed subsets of a complete separable metric space $\mathcal{Z}$,
- $f : [0,1] \times \mathbb{R}^n \times \mathcal{Z} \to \mathbb{R}^n$, $f(\cdot, x, \cdot)$ is $\mathcal{L} \times \mathcal{B}$-measurable and $f(t, \cdot, u)$ is locally Lipschitz continuous,
- $K \subset \mathbb{R}^n$ and $K_1 \subset \mathbb{R}^n \times \mathbb{R}^n$ are closed subsets

$S^K_{[0,1]} := \{x(\cdot) \in W^{1,1}([0,1]) | \ x(\cdot) \text{ is a solution to (1) satisfying (2)}\}$

A pair $(x(\cdot), u(\cdot))$, with $x(\cdot)$ absolutely continuous and $u(\cdot)$ measurable, is called a *viable* trajectory/control pair if it satisfies (1) and (2)

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The Bolza optimal control problem:

$$(MIN) \qquad \inf\left\{\varphi(x(0),x(1)) + \int_0^1 L(t,x(t),u(t))dt \,\middle|\, x(\cdot)\in S^K_{[0,1]}\right\},$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $L : [0,1]\times\mathbb{R}^n\times\mathcal{Z}\to\mathbb{R}$

For $\lambda \in \{0,1\}$ define $H_\lambda : [0,1]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$

$$H_\lambda(t,x,p) := \sup_{u\in U(t)} \{\langle p, f(t,x,u)\rangle - \lambda L(t,x,u)\}$$

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The Pontryagin's Maximum Principle (PMP)

For $(\bar{x}(\cdot), \bar{u}(\cdot))$ optimal for (MIN), $\exists(\lambda, p(\cdot), \psi(\cdot)) \neq 0$ where $\lambda \in \{0,1\}$, $p(\cdot) \in W^{1,1}$ and $\psi(\cdot) \in NBV$, integrable mappings $A : [0,1] \to M(n \times n)$, $\pi : [0,1] \to \mathbb{R}^n$ and vectors $\pi_0, \pi_1 \in \mathbb{R}^n$ s.t.

i) $\psi(0) = 0$, $\psi(t) = \int_{[0,t]} \nu(s) d\mu(s)$, for all $t \in (0,1]$ for a positive finite Borel measure μ on $[0,1]$ and a Borel measurable selection $\nu(s) \in N_K(\bar{x}(s)) \cap B$ μ-a.e.

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Under suitable hypotheses, in the classical PMP
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In general the adjoint system could be expressed as a Hamiltonian inclusion

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where $\partial_x H_\lambda$ denotes the generalized gradient of H_λ with respect to x, or as an Euler-Lagrange inclusion

However, if $\lambda = 0$ and $p(\cdot)$ and $\psi(\cdot)$ are s.t.

$$|p'(t)| \leq k(t)|p(t) + \psi(t)| \quad \text{a.e.}$$

for some $k(\cdot) \in L^1$, then we can find an integrable matrix valued mapping $A(\cdot)$ such that

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Non-degeneracy and normality

$$\langle p(t)+\psi(t),\bar{x}'(t)\rangle - \lambda L(t,\bar{x}(t),\bar{u}(t)) = \sup_{u\in U(t)} \{\langle p(t)+\psi(t), f(t,\bar{x}(t),u)\rangle - \lambda L(t,\bar{x}(t),u)\}$$

If $\lambda + \sup_{t\in(0,1]}|p(t)+\psi(t)| = 0$ the PMP gives no useful information about optimal controls because the maximum is then satisfied by every $u \in U(t)$

When $\lambda + \sup_{t\in(0,1]}|p(t)+\psi(t)| \neq 0,$

a triple $(\lambda, p(\cdot), \psi(\cdot))$ is called *non-degenerate*

If $\lambda = 0$, the PMP does not depend on the cost functions L and φ and expresses some relations between the control system and state constraints

When $\lambda = 1$ the PMP is called normal

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Preliminary definitions

Let $\emptyset \neq K \subset \mathbb{R}^n$.
Distance $d_K(x) := \inf_{y \in K} |x - y| \ \forall x \in \mathbb{R}^n$

Oriented distance $d(x) := d_K(x) - d_{\mathbb{R}^n \setminus K}(x) \ \forall x \in \mathbb{R}^n$

Contingent cone to K at $x \in K$

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$$C_K(x) := \operatorname*{Lim\,inf}_{h \to 0^+,\, K \ni y \to x} \frac{K - y}{h} \qquad N_K(x) := [C_K(x)]^-$$

Reachable gradient $\partial^* f(x) := \text{Lim sup}_{y \to x} \{\nabla f(y)\}$ for $f \in W^{1,\infty}(\mathbb{R}^n)$

Preliminary definitions

Let $\emptyset \neq K \subset \mathbb{R}^n$.
Distance $d_K(x) := \inf_{y \in K} |x - y| \ \ \forall x \in \mathbb{R}^n$

Oriented distance $d(x) := d_K(x) - d_{\mathbb{R}^n \setminus K}(x) \ \ \forall x \in \mathbb{R}^n$

Contingent cone to K at $x \in K$

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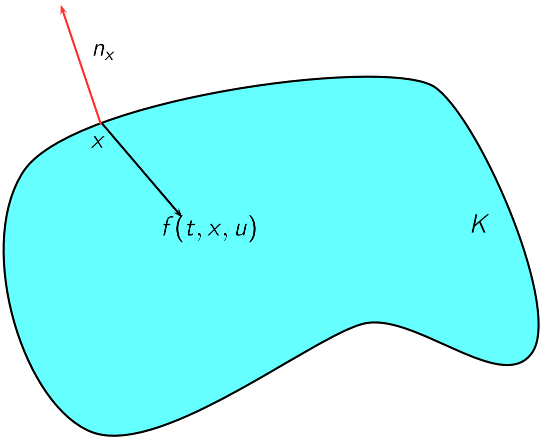
Inward Pointing Condition

Classical : $\forall x \in \partial K, t \in [0,1]$ there exists $u \in U(t)$ such that

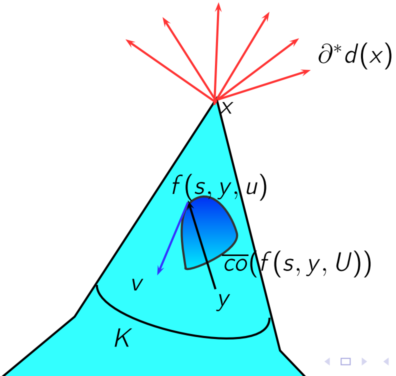
$$\langle n_x, f(t,x,u)\rangle < 0 \quad n_x \text{ is the unit outer normal to } K \text{ at } x$$



$$G^+(t,x) := \{f(t,x,u) |\ u \in U(t),\ \max_{p \in \partial^* d(x)} \langle p, f(t,x,u)\rangle \geq 0\}$$

We shall use the following inward pointing condition (IP)

$$\begin{cases} \exists M, \rho > 0 \text{ s.t. } \forall (t,x) \in [0,1] \times \partial K,\ \exists \delta > 0 \text{ s.t.} \\ \text{for a.e. } s \in [0,1], \forall y \in K \text{ with } |(s,y)-(t,x)| < \delta,\ \forall f(s,y,u) \in G^+(s,y) \\ \exists\ v \in T_{\overline{co}(f(s,y,U(s)))}(f(s,y,u)),\ |v| \leq M \\ \text{ satisfying } \max_{p \in \partial^* d(x)} \langle p, v\rangle \leq -\rho. \end{cases}$$



Inward Pointing Trajectories

Let $K_1 = Q_0 \times Q_1$, where Q_i is a closed subset of K, for $i \in \{0, 1\}$
Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a viable trajectory/control pair.

Inward pointing trajectories are solution of

$$\begin{cases} w'(t) = A(t)w(t) + v(t), & v(t) \in \mathcal{T}(t) \text{ for a.e. } t \in [0,1] \\ w(t) \in \operatorname{Int}(C_K(\bar{x}(t))) & \forall t \in (0,1] \end{cases} \quad (3)$$

$$w(0) = 0 \quad \text{or} \quad w_0 \in \operatorname{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0))$$

where

$$\mathcal{T}(t) := \begin{cases} T_{\overline{co}(f(t,\bar{x}(t),U(t)))}(\bar{x}'(t)) & \text{if } \bar{x}'(t) \in f(t, \bar{x}(t), U(t)) \\ \{0\} & \text{otherwise} \end{cases}$$

Since $\operatorname{Int}(C_K(\bar{x}(t)))$ is open and $x \rightsquigarrow C_K(x)$ is not upper semicontinuous in general $\Rightarrow$ we cannot use results from Viability Theory

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Inward pointing trajectory Theorem-1

Let $K_1 = Q_0 \times \mathbb{R}^n$, $Q_0 \subset \mathbb{R}^n$ closed. Assume that (IP) holds. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a viable trajectory/control pair. Assume that

$$\mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0)) \neq \emptyset,$$

Then, for any integrable $(n \times n)$-matrix valued map $A : [0,1] \to M(n \times n)$ and any $w_0 \in \mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0))$, there exists a solution $w(\cdot)$ of (3) which satisfies $w(0) = w_0$.

Inward pointing trajectory Theorem-2

Assume that (IP) holds. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a viable trajectory/control pair. Then, for any integrable $(n \times n)$-matrix valued map $A : [0,1] \to M(n \times n)$, there exists a solution $w(\cdot)$ of (3), with $w(0) = 0$.

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Lemma

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be extremal for an abnormal triple* $(0, p(\cdot), \psi(\cdot))$ *and let* $A(\cdot)$ *be the corresponding matrix valued map. Then for every solution* $w(\cdot)$ *of the viability problem*

$$\left\{ \begin{array}{rcl} w'(t) & = & A(t)w(t) + v(t), \quad v(t) \in \mathcal{T}(t) \text{ for a.e. } t \in [0,1] \\ w(t) & \in & C_K(\bar{x}(t)) \quad \forall t \in [0,1] \\ (w(0), w(1)) & \in & C_{K_1}((\bar{x}(0), \bar{x}(1))), \end{array} \right.$$

we have

$$\int_0^1 \langle p(s) + \psi(s), v(s) \rangle ds = 0, \quad \int_0^1 w(s) d\psi(s) = 0,$$

$$-\langle p(1) + \psi(1), w(1) \rangle + \langle p(0), w(0) \rangle = 0.$$

Proposition

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be extremal for a triple $(\lambda, p(\cdot), \psi(\cdot))$ and let $A(\cdot)$ be the corresponding matrix valued map. Then $\lambda = 1$ whenever there exists a solution $\bar{w}(\cdot)$ to the viability problem

$$\begin{cases} w'(t) & \in & A(t)w(t) + \mathcal{T}(t) \quad \text{a.e. in } [0,1] \\ w(t) & \in & \mathrm{Int}(C_K(\bar{x}(t))) \quad \forall t \in (0,1] \\ w(0) & \in & C_K(\bar{x}(0)) \end{cases}$$

satisfying one of the following relations:

i) $\mathrm{Int}(C_K(x)) \neq \emptyset$ for all $x \in \partial K$, $\bar{w}(0) \in \mathrm{Int}(C_K(\bar{x}(0)))$ and for some $\varepsilon > 0$,

$$(\bar{w}(0), \bar{w}(1) + \varepsilon B) \subset C_{K_1}((\bar{x}(0), \bar{x}(1))).$$

ii) $(\lambda, p(\cdot), \psi(\cdot))$ is non-degenerate and for some $\varepsilon > 0$,

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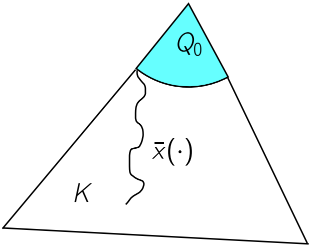

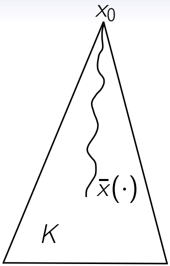

Theorem

Assume that $K_1 = Q_0 \times \mathbb{R}^n$, *where* Q_0 *is a closed subset of* $\mathbb{R}^n$, *(IP) and*

$$\mathrm{Int}(C_K(z)) \cap C_{Q_0}(z) \neq \emptyset, \forall z \in \partial K \cap \partial Q_0.$$

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Assume that $K_1 = \{x_0\} \times \mathbb{R}^n$ *for some* $x_0 \in \mathbb{R}^n$ *and (IP).*
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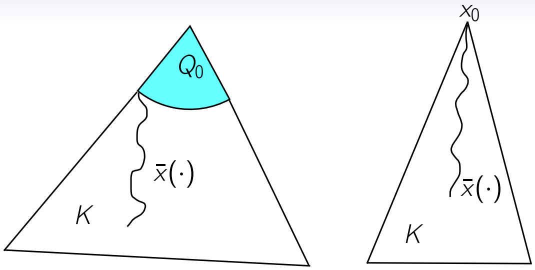

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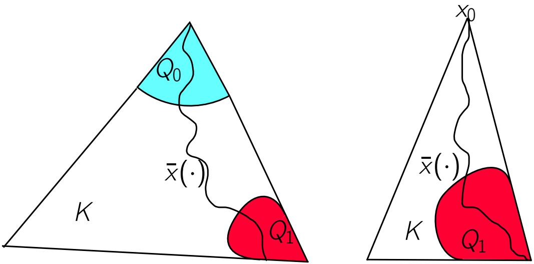

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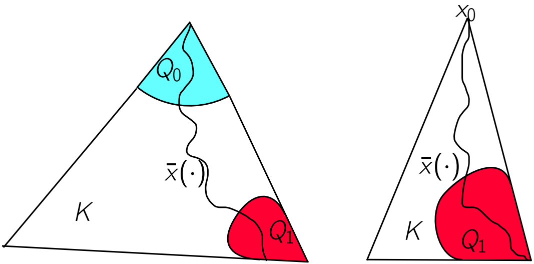

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Lipschitz continuity of optimal trajectories under Tonelli's growth condition on L

Let $K_1 := Q_0 \times \mathbb{R}^n$, where Q_0 is a compact subset of $\mathbb{R}^n$, $L \geq 0, \varphi \geq 0$. Suppose that the infimum in

$$(MIN) \qquad \inf \left\{ \varphi(x(0), x(1)) + \int_0^1 L(t, x(t), u(t))dt \,\middle|\, x(\cdot) \in S^K_{[0,1]} \right\},$$

is finite.

Assumption (G):

there exists a function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{r \to +\infty} \frac{\phi(r)}{r} = +\infty$ and $L(t, x, u) \geq \phi(|f(t, x, u)|)$, for all $(t, x, u) \in [0, 1] \times \mathbb{R}^n \times \mathcal{Z}$.

Assumption (H):

i) f, L and φ are continuous, $U(\cdot)$ is lower semicontinuous;

ii) $\mathrm{Int}(C_K(x)) \cap C_{Q_0}(x) \neq \emptyset, \forall x \in \partial K \cap \partial Q_0$;

iii) for all $t \in [0, 1]$, $x \in K$, the set $F(t, x) := \{(L(t, x, u) + \eta, f(t, x, u)) |\ u \in U(t), \eta \geq 0\}$ is closed and convex;

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Assume (G), (H), (IP) and some Lipschitz regularity of $L(t,\cdot,\cdot), f(t,\cdot,u), \varphi(\cdot,\cdot)$ for all $t \in [0,1]$, $u \in U(t)$. Then the infimum is attained and every optimal trajectory $\bar{x}(\cdot)$ is Lipschitz
Moreover, if $\mathcal{Z}$ is a separable Banach space and $\forall R > 0$

$$\liminf_{||u||_{\mathcal{Z}} \to \infty} \operatorname*{ess\,inf}_{t \in [0,1]} \inf_{x \in RB} |f(t,x,u)| = +\infty$$

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Sketch of the proof

Thanks to (G) and (H) we can use a theorem of Cesari to prove the existence of an optimal solution in $S^K_{[0,1]}$

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal trajectory/control pair, thanks to (H) and (IP) a normal PMP holds (cf. Vinter + our normality results)

$$\langle p(t) + \psi(t), \bar{x}'(t) \rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p(t) + \psi(t)) \quad \text{a.e.}$$

Continuity of f, L and the lower semicontinuity of $U(\cdot)$ and (G) imply that H is lower semicontinuous, hence locally bounded from below

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Variational Equation

Let $K_1 = Q_0 \times \mathbb{R}^n$, where Q_0 is a closed subset of K

Theorem

Assume (IP), f differentiable w.r.t. x and for some integrable $k : [0,1] \to \mathbb{R}_+$, $f(t, \cdot, u)$ is $k(t)$-Lipschitz for all $t \in [0,1]$, $u \in U(t)$. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a viable trajectory/control pair and

$$\mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0)) \neq \emptyset.$$

Then every solution $w(\cdot)$ of the viability problem

$$\begin{cases} w'(t) & \in & \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))w(t) + \mathcal{T}(t) \text{ for a.e. } t \in [0,1] \\ w(t) & \in & C_K(\bar{x}(t)) \quad \forall t \in [0,1] \\ w(0) & \in & C_{Q_0}(\bar{x}(0)), \end{cases}$$

is an element of the contingent cone to $S^K_{[0,1]}$ at $\bar{x}(\cdot)$.

This results can be used to prove the normal PMP in a direct way.

Variational Equation

Let $K_1 = Q_0 \times \mathbb{R}^n$, where Q_0 is a closed subset of K

Theorem

Assume (IP), f differentiable w.r.t. x and for some integrable $k : [0,1] \to \mathbb{R}_+$, $f(t,\cdot,u)$ is $k(t)$-Lipschitz for all $t \in [0,1]$, $u \in U(t)$. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a viable trajectory/control pair and

$$\mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0)) \neq \emptyset.$$

Then every solution $w(\cdot)$ of the viability problem

$$\begin{cases} w'(t) & \in & \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t)) w(t) + \mathcal{T}(t) \text{ for a.e. } t \in [0,1] \\ w(t) & \in & C_K(\bar{x}(t)) \quad \forall t \in [0,1] \\ w(0) & \in & C_{Q_0}(\bar{x}(0)), \end{cases}$$

is an element of the contingent cone to $S^K_{[0,1]}$ at $\bar{x}(\cdot)$.

This results can be used to prove the normal PMP in a direct way.

[1] H. Frankowska. Normality of the maximum principle for absolutely continuous solutions to Bolza problems under state constraints. *Control Cybernet.*, 38(4B):1327–1340, 2009.

[2] H. Frankowska and E. M. Marchini. Lipschitzianity of optimal trajectories for the Bolza optimal control problem. *Calc. Var. Partial Differential Equations*, 27(4):467–492, 2006.

[3] H. Frankowska and D. Tonon. Inward pointing trajectories, normality of the maximum principle and the non occurrence of the Lavrentieff phenomenon in optimal control under state constraints. *Submitted*.

For every $R > 0$, $\exists C_R > 0$ such that, for any $t \in [0, 1]$, $x_1, x_2, y_1, y_2 \in RB \cap K$ and any $u \in U(t)$,

*i*1) $|\varphi(x_1, y_1) - \varphi(x_2, y_2)| \leq C_R(|x_1 - x_2| + |y_1 - y_2|)$,

*i*2) $|L(t, x_1, u) - L(t, x_2, u)| \leq C_R|x_1 - x_2|[1 + L(t, x_1, u) \wedge L(t, x_2, u)]$,

*i*3) $|f(t, x_1, u) - f(t, x_2, u)| \leq C_R|x_1 - x_2|[1 + |f(t, x_1, u)| \wedge |f(t, x_2, u)| + L(t, x_1, u) \wedge L(t, x_2, u)]$;