The Fractional Optimal Control

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SADCO Summer School & Workshop 2012
New Trends in Optimal Control
Ravello, Italy, September 3-7, 2012

The fundamental problem of the Calculus of Variations

The classical variational problem is formulated as follows:

minimize (or maximize) the functional

$$\mathcal{J}(x) = \int_a^b L(t, x(t), x'(t))\, dt$$

on $\mathcal{D} = \{x \in C^1([a, b]) : x(a) = x_a,\ x(b) = x_b\}$,
where $L : [a, b] \times \mathbb{R}^{2n} \to \mathbb{R}$ is twice continuously differentiable.

- In Mechanics:
  - function L is called the *Lagrangian*;
  - functional $\mathcal{J}$ is called the *Action*.

Necessary Optimality Condition

Theorem (The Euler–Lagrange equation)

If x gives a (local) minimum (or maximum) to $\mathcal{J}$ on $\mathcal{D}$, then

$$\frac{d}{dt}\partial_3 L\left(t, x(t), x'(t)\right) = \partial_2 L\left(t, x(t), x'(t)\right)$$

for all $t \in [a, b]$.

- In Mechanics: if the Lagrangian L does not depend explicitly on t, then the *energy*

$$\mathcal{E}(x) := -L(x, x') + \partial_3 L(x, x') \cdot x'$$

  is constant along physical trajectories x (along the solutions of the Euler–Lagrange equations).

Classical Example from Mechanics

- Consider a particle of mass m, and let $x : \mathbb{R} \to \mathbb{R}^3$ denote the trajectory of this particle.
- Define the Lagrangian to be the difference between the kinetic and potential energies:

$$L(t, x, x') := T(x) - V(x) = \frac{1}{2} m \|x'\|^2 - V(x)$$

- The action of the trajectory from time a to b is the integral

$$\mathcal{J}(x) = \int_a^b L(t, x(t), x'(t))\, dt.$$

Hamilton's Principle of Least Action

asserts that particles follow trajectories which minimize the action. Therefore, the solutions of the Euler–Lagrange equations give the physical trajectories.

Discrete embedding of the Euler–Lagrange equations

- The Euler–Lagrange equations give Newton's second law:

$$m\frac{d^2x^i}{dt^2} = -\frac{\partial V}{\partial x^i}$$

- Let us consider the usual discretization of a function

$$f : t \in [a,b] \subset \mathbb{R} \mapsto f(t) \in \mathbb{R}$$

  - denote by $h = (b-a)/N$ the step of discretization;
  - consider the partition $t_k = a + kh$, $k = 0, \ldots, N$, of $[a,b]$;
  - let $\mathbf{F} = \{f_k := f(t_k)\}_{k=0,\ldots,N}$;
  - substitute the differential operator $\frac{d}{dt}$ by Δ_+ or Δ_-:

$$\Delta_+(\mathbf{F}) = \left\{\frac{f_{k+1}-f_k}{h}, 0 \leq k \leq N-1\,,\ 0\right\}$$

$$\Delta_-(\mathbf{F}) = \left\{0, \frac{f_k - f_{k-1}}{h},\ 1 \leq k \leq N\right\}.$$

Numerical scheme

- The discrete version of the Euler–Lagrange equation obtained by the direct embedding is

$$\frac{x_{k+2} - 2x_{k+1} + x_k}{h^2} m + \frac{\partial V}{\partial x}(x_k) = 0, \quad k = 0, \ldots, N-2,$$

where $N = \frac{b-a}{h}$ and $x_k = x(a + kh)$.
- This numerical scheme is of order one: we make an error of order h at each step, which is of course not good.

We can do better!

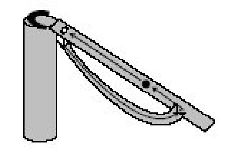
All Lagrangian systems possess a variational structure, i.e., their solutions correspond to critical points of a functional and this characterization does not depend on the system coordinates. This induces strong constraints on solutions, for example the *conservation of energy* for autonomous classical Lagrangian systems: $\mathcal{E}(x) = T(x) + V(x) = const$.

The Search for a Perpetual Motion Machine

Generations of inventors have tried to create a machine, called a perpetual motion machine, that would run forever without fuel:

- such a machine is not forbidden by Newton's laws!
- the principle of inertia seems even to encourage the belief that a cleverly constructed machine might run forever!

$\Longrightarrow$ The magnet draws the ball to the top of the ramp, where it falls through the hole and rolls back to the bottom.



$\Longrightarrow$ As the wheel spins clockwise, the flexible arms sweep around and bend and unbend. By dropping off its ball on the ramp, the arm is supposed to make itself lighter and easier to lift over

Today, a little longer than yesterday

- Innumerable perpetual motion machines have been proposed.
- The reason why these machines don't work is always the same: friction corrupts conservation of energy

Earth: an almost perfect perpetual motion machine

Rotating earth might seem a perfect perpetual motion machine, since it is isolated in the vacuum of outer space with nothing to exert frictional forces on it. But in fact our planet's rotation has slowed drastically since it first formed, and the earth continues to slow its rotation, making today just a little longer than yesterday. The very subtle source of friction is the tides. The moon's gravity raises bulges in earth's oceans, and as the earth rotates the bulges progress around the planet. Where the bulges encounter land, there is friction, which slows earth's rotation very gradually.

Nonconservative systems

- For conservative systems, variational methods are equivalent to the original used by Newton. However, while Newton's equations allow nonconservative forces, the later techniques of Lagrangian and Hamiltonian mechanics have no direct way to dealing with them.
- Let us recall the classical problem of linear friction:

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} - \frac{\partial U}{\partial x} = 0, \quad \gamma > 0.$$

- What is the associated variational problem?

Bauer's theorem

- In 1931, Bauer proved that it is impossible to use a variational principle to derive a single linear dissipative equation of motion with constant coefficients.
- Bauer's theorem expresses the well-known belief that there is no direct method of applying variational principles to nonconservative systems, which are characterized by friction or other dissipative processes.
- Lanczos: "Forces of a frictional nature... are outside the realm of variational principles."

The techniques of Lagrangian and Hamiltonian mechanics, which are derived from variational principles, thus appears to be out of reach.

Mechanics with fractional derivatives

- The proof of Bauer's theorem relies on the tacit assumption that all derivatives are of integer order.
- If a Lagrangian is constructed using fractional (non-integer order) derivatives, then the resulting equation of motion can be nonconservative.

F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, Phys. Rev. E (3) **53** (1996), no. 2, 1890–1899.

F. Riewe, Mechanics with fractional derivatives, Phys. Rev. E (3) **55** (1997), no. 3, part B, 3581–3592.

Fractional Calculus of Variations

Because most processes observed in the physical world are nonconservative, it is important to be able to apply the power of variational methods to such cases.

Riewe's approach

Consider the Lagrangian

$$L = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - U(x) + \frac{1}{2} \gamma \left({}_t D_b^{\frac{1}{2}} x \right)^2 .$$

Using the fractional variational principle we obtain

$$\frac{\partial L}{\partial x} + {}_a D_t^{\frac{1}{2}} \frac{\partial L}{\partial_t D_b^{\frac{1}{2}} x} - \frac{d}{dt} \frac{\partial L}{\partial x'} = 0$$

which becomes

$$m \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} - \frac{\partial U}{\partial x} = 0$$

In order to obtain this casual equation Riewe suggests considering an infinitesimal time interval, that is the limiting case $a \to b$, while keeping $a < b$.

Dreisigmeyer & Young approach

- D.W. Dreisigmeyer and P. M. Young

$$L = \frac{1}{2} m_a D_t^1 x_t D_b^1 x - U(x) + \frac{1}{2} \gamma_a D_t^{\frac{1}{2}} x_t D_b^{\frac{1}{2}} x$$

- D. W. Dreisigmeyer and P. M. Young, Nonconservative Lagrangian mechanics: a generalized function approach, J. Phys. A **36** (2003), no. 30, 8297–8310.
- D. W. Dreisigmeyer and P. M. Young, Extending Bauer's corollary to fractional derivatives, J. Phys. A **37** (2004), no. 11, L117–L121.

Other approaches

- M. Klimek

$$L = 2m(D^2x)^2 - U(x) + \gamma(Dx)^2$$

where $D := \frac{1}{2}\left({}_aD_t^{\frac{1}{2}} + {}_tD_b^{\frac{1}{2}}\right)$.

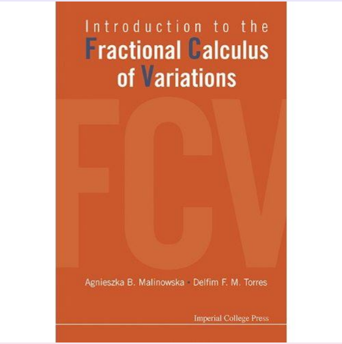
M. Klimek, Fractional sequential mechanics—models with symmetric fractional derivative, Czechoslovak J. Phys. **51** (2001), no. 12, 1348–1354.

- J. Cresson, FVC with

$$D_\mu^{\alpha,\beta} := \frac{1}{2}\left({}_aD_t^\alpha - {}_tD_b^\beta\right) + \frac{i\mu}{2}\left({}_aD_t^\alpha + {}_tD_b^\beta\right)$$

J. Cresson, Fractional embedding of differential operators and Lagrangian systems, J. Math. Phys. **48** (2007), no. 3, 033504, 34 pp.

The Book



Agnieszka B. Malinowska and Delfim F. M. Torres,
Introduction to the Fractional Calculus of Variations,
Imperial College Press, London &
World Scientific Publishing, Singapore (2012).

Fractional Differentiation

The name of the game

fractional derivatives means "derivatives of arbitrary order".

- Its origin goes back more than 300 years, when in 1695 L'Hopital asked Leibniz the meaning of $\frac{d^n y}{dx^n}$ for $n = \frac{1}{2}$.
- After that, many famous mathematicians, like J. Fourier, N. H. Abel, J. Liouville, B. Riemann, among others, contributed to the development of the Fractional Calculus.
- The theory of derivatives and integrals of arbitrary order took more or less finished form by the end of the XIX century.
- The theory is very rich: fractional differentiation may be introduced in several different ways – fractional derivatives of Riemann-Liouville; Grünwald-Letnikov; Caputo; Miller-Ross.

From Theory to Practice

- Fractional Calculus had its origin in the 1600s
- For three centuries the theory of fractional derivatives developed as a pure theoretical field of mathematics, useful only for mathematicians.
- In the last few decades, fractional differentiation proved very useful in various fields: physics (classic and quantum mechanics, thermodynamics, etc), chemistry, biology, economics, engineering, signal and image processing, and control theory.

I recommend

I refer to the web site of Podlubny:

http://people.tuke.sk/igor.podlubny/fc_resources.html

Fractional Differentiation is alive!

Several international conferences on the subject:

- IFAC Workshop on Fractional Derivative and Applications (IFAC FDA2010), held in University of Extremadura, Badajoz, Spain, October 18-20, 2010.
- The 5th Workshop on Fractional Differentiation and its Applications (FDA'2012), May 14-17 2012, Hohai University, Nanjing, China.
- The sixth FDA workshop, FDA'2013 (FDA : Fractional differentiation and its Applications) will be held in Grenoble, France, February 4-6, 2013

Fractional Differentiation is alive!

Several specialized journals on the subject:

1. Fractional Calculus & Applied Analysis
2. Fractional Dynamic Systems
3. Communications in Fractional Calculus
4. Journal of Fractional Calculus and its Applications

J.T. Machado, V. Kiryakova, F. Mainardi, Recent History of Fractional Calculus, Commun. Nonlinear Sci. Numer. Simulat. **16** (2011), no. 3, 1140–1153.

Let us begin from the beginning

We are all familiar with the idea of derivatives. The usual notation:

$$\frac{df(x)}{dx} = D^1 f(x)\,, \quad \frac{d^2 f(x)}{dx^2} = D^2 f(x)\,.$$

We are also familiar with properties like

$$D^1[f(x) + g(x)] = D^1 f(x) + D^1 g(x)\,.$$

But what would be the meaning of notation like

$$\frac{d^{1/2} f(x)}{dx^{1/2}} = D^{1/2} f(x) \ ?$$

Several giants toyed with the idea

- The notion of derivative of order $1/2$ (*fractional* derivative) was discussed briefly as early as the XVIII century by Leibniz.
- Other giants of the past including L'Hopital, Euler, Lagrange, Laplace, Riemann, Fourier, Liouville, and others toyed with the idea.
- Let us toy also a little with it :)

FDs of exponential function (Liouville, 1832)

We are familiar:

$$D^1 e^{\lambda x} = \lambda e^{\lambda x}\,, \quad D^2 e^{\lambda x} = \lambda^2 e^{\lambda x}\,, \ldots \quad D^n e^{\lambda x} = \lambda^n e^{\lambda x}\,,$$

when n is an integer. Why not to replace n by $1/2$ and write

$$D^{1/2} e^{\lambda x} = \lambda^{1/2} e^{\lambda x} \text{ ?}$$

Why not go further and let n be an irrational number like $\sqrt{2}$ or a complex number like $1+i$? We will be bold and write

$$D^\alpha e^{\lambda x} = \lambda^\alpha e^{\lambda x} \qquad \text{(exp)}$$

for any value of α, integer, rational, irrational, or complex.

If a definition of FD is found, we expect (exp) to follow from it. Liouville used this approach.

Meaning of (exp) when $\alpha \in \mathbb{Z}^-$

We naturally want

$$e^{\lambda x} = D^1\left(D^{-1}\left(e^{\lambda x}\right)\right).$$

Since $e^{\lambda x} = D^1\left(\frac{1}{\lambda}e^{\lambda x}\right)$, we have

$$D^{-1}\left(e^{\lambda x}\right) = \frac{1}{\lambda}e^{\lambda x} = \int e^{\lambda x}dx.$$

Similarly,

$$D^{-2}\left(e^{\lambda x}\right) = \int\int e^{\lambda x}dxdx.$$

So it is reasonable to interpret D^α when α is a negative integer $-n$ and the nth iterated integral.

D^α represents a derivative if α is a positive real number, and an integral if α is a negative real number.

Examples

- Let $f(x) = e^{2x}$, $0 < \alpha < 1$. Then $D^\alpha(e^{2x}) = 2^\alpha e^{2x}$. We see that

$$D^0\left(f(x)\right) = f(x) \le D^\alpha\left(f(x)\right) \le D^1\left(f(x)\right)$$

- Let $f(x) = e^{\frac{1}{3}x}$, $0 < \alpha < 1$. Then $D^\alpha(\frac{1}{3}x) = \left(\frac{1}{3}\right)^\alpha e^{\frac{1}{3}x}$. We see that

$$D^1\left(f(x)\right) \le D^\alpha\left(f(x)\right) \le D^0\left(f(x)\right)$$

Trigonometric functions: sine and cosine

We are familiar with the derivatives of the sine function. Each time we differentiate, the graph is shifted $\frac{\pi}{2}$ to the left:

$$D^0 \sin x = \sin x \,,$$

$$D^1 \sin x = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$D^2 \sin x = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right)$$

$$\vdots$$

$$D^n \sin x = \sin\left(x + n\frac{\pi}{2}\right)$$

As before, we will replace the positive integer n with an arbitrary α:

$$D^\alpha \sin x = \sin\left(x + \alpha\frac{\pi}{2}\right) \,, \quad D^\alpha \cos x = \cos\left(x + \alpha\frac{\pi}{2}\right) \,. \qquad \text{(sin)}$$

Derivatives of x^p (Lacroix, 1819)

We now look at derivatives of powers of x:

$$D^0 x^p = x^p\,, \quad D^1 x^p = p x^{p-1}\,, \quad D^2 x^p = p(p-1)x^{p-2}$$

$$\vdots$$

$$D^n x^p = p(p-1)\cdots(p-n+1)x^{p-n} = \frac{p!}{(p-n)!}x^{p-n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)}x^{p-n}\,.$$

The gamma function gives meaning to factorial for real numbers, so for any α we put

$$D^\alpha x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}\,. \qquad \text{(power)}$$

Example

Using equation (power) for the constant function $f(x) \equiv 1$, we have for $0 < \alpha < 1$ that

$$D^\alpha(1) = \frac{\Gamma(1)}{\Gamma(1-\alpha)} x^{-\alpha}$$

The FD of a constant is not equal to zero for $0 < \alpha < 1$. However, we have agreement with classical calculus:

$$\alpha \to 1 \Rightarrow \Gamma(1-\alpha) \to +\infty \Rightarrow D^1(1) = 0$$

Remark

By plotting a discrete number of derivatives $D^{\alpha}(1)$ between $D^{0}(1)$ and $D^{1}(1)$ (use, for example, Excel!) an interesting picture is obtained:

- There is a continuous deformation of a polynomial via fractional derivatives between 0 and 1.

"If we sketch the half derivative its graph will be about half way between the function and its 1st derivative."

Didactics

Illustration of the transition from function to first derivative via fractional derivatives in Excel:

D. A. Miller and S. J. Sugden, Insight into the fractional calculus via a spreadsheet, Spreadsheets in Education, Vol. 3, No. 2, 2009, Article 4.

Our 1st attempt to define fractional derivative

With (power) we can extend the idea of a fractional derivative to a large number of functions. Given any function that can be expanded in a Taylor series in powers of x,

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n \,,$$

and assuming we can differentiate term by term we get

$$D^\alpha f(x) = \sum_{n=0}^{+\infty} a_n D^\alpha x^n = \sum_{n=0}^{+\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \,. \qquad \text{(Liouville)}$$

Expression (Liouville) presents itself as a possible candidate for the definition of the fractional derivative for the wide variety of functions that can be expanded in a Taylor's series in powers of x. However...

A mysterious contradiction

We wrote

$$D^{\alpha} e^{x} = e^{x} \,. \tag{1}$$

Let us now compare this with our definition (Liouville) to see if they agree. From the Taylor series,

$$e^{x} = \sum_{n=0}^{+\infty} \frac{1}{n!} x^{n}$$

and our definition (Liouville) gives

$$D^{\alpha} e^{x} = \sum_{n=0}^{+\infty} \frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)} \tag{2}$$

- The expressions (1) and (2) do not match!
- We have discovered a contradiction that historically has caused great problems (controversy between 1835 and 1850).

Iterated Integrals

We could write $D^{-1}f(x) = \int f(x)dx$, but the right-hand side is indefinite. Instead, we will write

$$D^{-1}f(x) = \int_0^x f(t)dt\,.$$

The second integral will then be

$$D^{-2}f(x) = \int_0^x \int_0^{t_2} f(t_1)dt_1 dt_2\,.$$

The region of integration is a triangle, and if we interchange the order of integration, we can write the iterated integral as a single integral (method of Dirichlet, 1908):

$$D^{-2}f(x) = \int_0^x \int_{t_1}^x f(t_1)dt_2 dt_1 = \int_0^x f(t_1) \int_{t_1}^x dt_2 dt_1 = \int_0^x f(t_1)(x - t_1)dt_1\,.$$

Writing iterated integrals as a single integral (Dirichlet)

Using the same procedure we can show that

$$D^{-3}f(x) = \frac{1}{2}\int_0^x f(t)(x-t)^2 dt\,,$$
$$D^{-4}f(x) = \frac{1}{2\cdot 3}\int_0^x f(t)(x-t)^3 dt$$

and, in general,

$$D^{-n}f(x) = \frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1} dt\,.$$

Now, as we have previously done, let us replace the n with an arbitrary α and the factorial with the gamma function...

Our 2nd attempt to define fractional derivative

... replacing n by α, factorial by gamma function:

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1} dt\,. \qquad \text{(Riemann)}$$

Remark: we have a definition of fractional integral

- As $t \to x$, $x - t \to 0$. The integral diverges for every $\alpha \leq 0$; when $0 < \alpha < 1$ the improper integral converges.
- Since (Riemann) converges only for positive α, it is truly a *fractional integral*.

Lower limit and the Riemann-Liouville integral

- The choice of zero for the lower limit was arbitrary. The lower limit could just as easily have been a.
- Many people who work in the field use the notation ${}_aD_x^\alpha f(x)$ indicating limits of integration going from a to x. With this notation we have:

Riemann-Liouville integral

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt \, . $$

The importance of notation cannot be minimized

- We are not surprised that fractional integrals involve limits, because integrals involve limits.
- Since ordinary derivatives do not involve limits of integration, we were not expecting fractional derivatives to involve such limits!
- We think of derivatives as local properties of functions.
- The fractional derivative symbol D^α incorporates both derivatives (positive α) and integrals (negative α).

Integrals are between limits, it turns out that fractional derivatives are between limits too.

What went wrong before

- The reason for the contradiction is that two different limits of integration were being used!
- We have

$$ {}_aD_x^{-1} e^{\lambda x} = \int_a^x e^{\lambda t} dt = \frac{1}{\lambda} e^{\lambda x} - \frac{1}{\lambda} e^{\lambda a} $$

- We get the form we want when $\frac{1}{\lambda} e^{\lambda a} = 0$, i.e., when $\lambda a = -\infty$.
- If λ is positive, then $a = -\infty$:

Weyl fractional derivative

$$ {}_{-\infty}D_x^{\alpha} e^{\lambda x} = \lambda^{\alpha} e^{\lambda x} $$

What limits will work for the FD of x^p?

We have

$$ {}_aD_x^{-1} x^p = \int_a^x t^p dt = \frac{x^{p+1}}{p+1} - \frac{a^{p+1}}{p+1}. $$

Again, we want $\frac{a^{p+1}}{p+1} = 0$. This will be the case when $a = 0$. We conclude that (power) should be written in the more revealing notation

$$ {}_0D_x^{\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} . $$

The mystery solved

First we calculated ${}_{-\infty}D_x^{\alpha} e^{\lambda x}$; the second time we calculated ${}_0D_x^{\alpha} e^{\lambda x}$.

Fractional Integrals and Derivatives (Riemann-Liouville)

- Fractional integral of f of order α:

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt\,, \quad \alpha > 0\,. $$

- Let $\alpha > 0$ and let m be the smallest integer exceeding α. Then we define the *fractional derivative of* f of order α as

$$ \begin{aligned} {}_aD_x^{\alpha} f(x) &= \frac{d^m}{dx^m}\left[{}_aD_x^{-(m-\alpha)} f(x)\right] \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x f(t)(x-t)^{m-\alpha-1} dt\,. \end{aligned} $$

FD of integer order is the ordinary derivative

If $\alpha = n \in \mathbb{N}$, then $m = n+1$ and ${}_aD_x^n f(x) = \frac{d^n}{dx^n} f(x)$.

Fractional Models and Dynamical Love

- When one wants to include memory effects, i.e., the influence of the past on the behavior of the system at present time, one may use fractional derivatives to describe such effect.
- Fractional models have been shown by many researchers to adequately describe the operation of a variety of physical, engineering, and biological processes and systems.
- Such models are represented by differential equations of non-integer order.

Applications include also psychological and life sciences

W. M. Ahmad and R. El-Khazali, Fractional-order dynamical models of love, Chaos Solitons Fractals **33** (2007), no. 4, 1367–1375.

Fractional Optimal Control

- Fractional Optimal Control is a 16 years old subject: was born in 1996 with the works of Riewe.

My first works on the subject

- G. S. F. Frederico and D. F. M. Torres,
  A formulation of Noether's theorem
  for fractional problems of the calculus of variations,
  J. Math. Anal. Appl. **334** (2007), no. 2, 834–846.
- G. S. F. Frederico and D. F. M. Torres,
  Fractional conservation laws in optimal control theory,
  Nonlinear Dynam. **53** (2008), no. 3, 215–222.

Fractional Noether's Theorem in Lagrangian form

Theorem (Frederico & Torres, 2007)

If the fractional functional

$$I[q(\cdot)] = \int_a^b L\left(t, q(t), {}_aD_t^\alpha q(t)\right) dt$$

is invariant under a symmetry $\bar{t} = t + \varepsilon\tau(t, q) + o(\varepsilon)$, $\bar{q}(\bar{t}) = q(t) + \varepsilon\xi(t, q) + o(\varepsilon)$, then

$$\left[L\left(t, q, {}_aD_t^\alpha q\right) - \alpha\partial_3 L\left(t, q, {}_aD_t^\alpha q\right) \cdot {}_aD_t^\alpha q\right]\tau(t, q) \\ + \partial_3 L\left(t, q, {}_aD_t^\alpha q\right) \cdot \xi(t, q)$$

is a fractional constant of motion.

Hamiltonian form

$$\mathcal{D}^{\alpha}\left\{\left[\mathcal{H}-(1-\alpha)\,p(t)\cdot {}_aD_t^{\alpha}q(t)\right]\tau-p(t)\cdot\xi\right\}=0$$

- For autonomous problems, it follows from our fractional Noether's theorem that

$$\mathcal{H}-(1-\alpha)\,p(t)\cdot {}_aD_t^{\alpha}q(t) \tag{3}$$

is a fractional constant of motion.

Non-fractional case: $\alpha = 1$

In the classical framework of optimal control theory $\alpha = 1$ and we then get from (3) the well known fact that the Hamiltonian is a preserved quantity along any extremal (conservation of energy in Mechanics).

Fractional Optimal Control

- While various fields of application of fractional derivatives are already well established, some are under strong current progress: this is the case of fractional optimal control, even in the particular case of fractional calculus of variations.
- The study of fractional problems of the Calculus of Variations and respective Euler-Lagrange type equations is a subject of current strong research.

The Fractional Optimal Control

has born in 1996-1997 with the work of F. Riewe: he obtained a version of the Euler-Lagrange equations for problems of the Calculus of Variations with fractional derivatives, combining the conservative and non-conservative cases.

Conclusion

- During the last two decades, fractional differential equations have increasingly attracted the attention of many researchers: many mathematical problems in science and engineering are represented by these kinds of equations.
- FC is a useful tool for modeling complex behaviours of physical systems from diverse domains such as mechanics, electricity, chemistry, biology, economics, and many others.
- Science Watch of Thomson Reuters identified this area as an Emerging Research Front.

- The study of fractional variational problems is a subject of strong current study because of its numerous applications.
- The fractional theory of optimal control is in its childhood so that much remains to be done.

References: books on fractional calculus

- I. Podlubny, “Fractional differential equations”, Academic Press, Inc., San Diego, CA, 1999.
- R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
- A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, “Theory and applications of fractional differential equations”, Elsevier, Amsterdam, 2006.
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Thank you very much for your attention!

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