Semi-Lagrangian schemes for curvature-related filtering models

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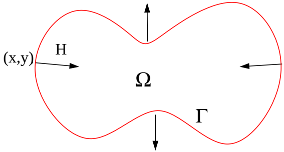
Outline

The Mean Curvature Motion of manifolds

- Ω bounded domain
- Γ boundary of Ω
- $\mathcal{V} = \mathcal{V}(x, y, t)$ smooth unit normal vector at (x, y, t)
- $H = -\text{div}(\mathcal{V})\mathcal{V}$ mean curvature vector at (x, y, t)

$(x, y) \in \Gamma_t$ evolves according to the ODE

$$\begin{cases} \dot{z}(s) = -[\text{div}(\mathcal{V})\mathcal{V}](z(s), s) & s > t \\ z(t) = (x, y). \end{cases}$$



The level set equation for MCM of curve

The initial curve $\Gamma_0 = \partial\Omega$ is represented by the 0-level set of an auxiliary function u_0:

$$u_0(x,y) \begin{cases} > 0 & \text{if } (x,y) \notin \Omega \\ < 0 & \text{if } (x,y) \in \text{int } \Omega, \\ = 0 & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

The time-dependent curve $\Gamma_t = \{(x,y) \in \mathbb{R}^2 : u(x,y,t) = 0\}$ is obtained as the solution u of

$$(MCM) \begin{cases} u_t(x,y,t) = \text{div}\left(\frac{Du(x,y,t)}{|Du(x,y,t)|}\right)|Du(x,y,t)| \\ u(x,y,0) = u_0(x,y) \end{cases}$$

This equation projects the diffusion orthogonally with respect to the gradient (see Osher & Sethian)

Analytic features:

- Degenerate parabolic
- Singular (undefined if $Du = 0$)
- Interest in nonsmooth solutions

Applications:

- Image processing: denoising
- Image processing: active contours
- Phase transitions
- Mathematical biology

Generalized characteristics in $\mathbb{R}^2$

The trajectories satisfying the following s.d.e.

$$\begin{cases} dy_{x,t}(s) = \sqrt{2}P(Du(y_{x,t}(s), t-s))dW(s) \\ y_{x,t}(t) = x \end{cases}$$

play the role of *generalized characteristics* .
Here, dW is the differential of a standard Wiener process and

$$P(Du) = I - \frac{Du \bigotimes Du}{|Du|^2} = \frac{1}{|Du|^2} \begin{pmatrix} u_{x_2}^2 & -u_{x_1}u_{x_2} \\ -u_{x_1}u_{x_2} & u_{x_1}^2 \end{pmatrix}$$

which projects the diffusion on the space *orthogonal to the gradient* of the solution u

Representation formula for MCM in $\mathbb{R}^2$

If u is a smooth solution of (MCM) by the Ito-Taylor expansion it turn out that,if $Du \neq 0$:

$$u(x,t) = \mathbb{E}\{u_0(y_{x,t}(t))\}. \tag{1}$$

The general representation formula reads

$$u(x,t) = \inf_{\nu \in \mathcal{A}} \operatorname{ess\,sup}_\Omega \{u_0(y^\nu_{x,t}(t)\}, \tag{2}$$

where $\mathcal{A}$ is the set of admissible controls and y^ν satisfies

$$\begin{cases} dy^\nu_{x,t}(s) = \sqrt{2}P(\nu(s))dW(s) \\ y^\nu_{x,t}(0) = x \end{cases}$$

References: *Soner - Touzi, Buckdahn - Cardaliaguet - Quincampoix*

Representation formula for MCM in $\mathbb{R}^2$

Representation formula (1) on a single time step $t_n \to t_{n+1} = t_n + \Delta t$:

$$u(x, t_{n+1}) = \mathbb{E}\{u(y_{x,t_{n+1}}(\Delta t), t_n)\}. \quad (3)$$

Brownian dimension reduction (from $\mathbb{R}^2$ to $\mathbb{R}$):

$$\begin{aligned} \sqrt{2}P(Du)dW &= \frac{\sqrt{2}}{|Du|}\begin{pmatrix} u_{x_2} \\ -u_{x_1} \end{pmatrix}\left(\frac{u_{x_2}dW_1}{|Du|} - \frac{u_{x_1}dW_2}{|Du|}\right) = \\ &= \frac{\sqrt{2}}{|Du|}\begin{pmatrix} u_{x_2} \\ -u_{x_1} \end{pmatrix} d\hat{W} = \sigma(Du)d\hat{W} \end{aligned}$$

we can replace the s.d.e. by

$$\begin{cases} dy_{x,t}(s) = \sigma(Du(y_{x,t}(s), t-s))d\hat{W}(s) \\ y_{x,t}(0) = x. \end{cases}$$

Main steps for the numerical discretization

In order to set up

$$u(x, t_{n+1}) = \mathbb{E}\{u(y_{x,t_{n+1}}(\Delta t), t_n)\}$$

in a fully discrete form:

- The computation of $u(\cdot, t_n)$ is replaced by a numerical reconstruction $I[u^n](\cdot)$ (Lagrange, ENO, WENO,...)
- Partial derivatives u_{x_i} are replaced by finite differences
- An approximation of the expectation $\mathbb{E}\{u(y_{x,t_{n+1}}(\Delta t), t_n)\}$ is computed by weak convergence scheme for SDEs

Weak Euler scheme

Assume that $y(t)$ satisfy the scalar (for simplicity) SDE

$$\begin{cases} dy_{x,t}(s) = \sigma(s, y_{x,t}(s))dW(s) \\ y_{x,t}(t) = x. \end{cases}$$

Weak Euler scheme with $t_k = t_0 + k\Delta t$ and $y_k \simeq y_{x,t}(t_k)$:

$$\begin{cases} y_{k+1} = y_k + \sigma(t_k, y_k)\Delta W_k \\ y_0 = x. \end{cases}$$

with ΔW_k distributed as

$$P(\Delta W_k = \pm\sqrt{\Delta t}) = \frac{1}{2}.$$

Then (if $\sigma(\cdot,\cdot), h(\cdot)$ are smooth enough), $y_1 \simeq y_{x,t}(\Delta t)$ satisfies

$$\mathbb{E}\{h(y_{x,t}(\Delta t))\} = \frac{1}{2}\left(h(y_1(\sqrt{\Delta t})) + h(y_1(-\sqrt{\Delta t}))\right) + O(\Delta t^2)$$

Construction of the SL scheme (in $\mathbb{R}^2$)

Generalized characteristics :

$$\begin{cases} dy_{x,t_{n+1}}(s) = \sigma(Du(y_{x,t_{n+1}}(s), t_{n+1}-s))d\hat{W}(s) \\ y_{x,t_{n+1}}(0) = x \end{cases}$$

Discrete characteristics :

$$\begin{cases} y_1 = x + \sigma(Du(x,t_{n+1}))\Delta\hat{W} \\ y_0 = x \end{cases}$$

with

$$P(\Delta\hat{W}_k = \pm\sqrt{\Delta t}) = \frac{1}{2}$$

Time-discretization:

$$\begin{aligned} u_{\Delta t}(x,t_{n+1}) &= \frac{1}{2}u_{\Delta t}(x+\sigma(Du(x,t_n))\sqrt{\Delta t}, t_n) + \\ &\quad + \frac{1}{2}u_{\Delta t}(x-\sigma(Du(x,t_n))\sqrt{\Delta t}, t_n). \end{aligned}$$

Construction of the SL scheme (in $\mathbb{R}^2$):

Fully discrete scheme for $Du \neq 0$

- $I[\cdot]$ bilinear interpolation
- $D_j^n \simeq Du(x_i, t_n)$ central differences
- $\sigma_j^n = \sigma(D_j^n)$

$$u_j^{n+1} = \frac{1}{2}\left(I[u^n](x_j + \sigma_j^n\sqrt{\Delta t}) + I[u^n](x_j - \sigma_j^n\sqrt{\Delta t})\right)$$

- needs a suitable treatment of singularity
- convergence analysis via Barles–Souganidis theory
  - consistency
  - monotonicity
  - L^∞ stability

Weak notion of consistency

Let $\phi \in C(\mathbb{R}^2 \times [0,T])$ and
$(\Delta x_m, \Delta t_m) \to 0$ and $(x_{j_m}, t_{n_m}) \to (x,t)$. Then, the scheme S_j is said to be consistent with

$$\phi_t(x,t) + F(D\phi, D^2\phi)(x,t) = 0$$

if

$$\begin{cases} \liminf\limits_{m\to\infty} \dfrac{\phi(x_{j_m}, t_{n_m+1}) - S_{j_m}(\phi^{n_m})}{\Delta t_m} \geq \phi_t + \underline{F}(D\phi, D^2\phi)(x,t) \\ \limsup\limits_{m\to\infty} \dfrac{\phi(x_{j_m}, t_{n_m+1}) - S_{j_m}(\phi^{n_m})}{\Delta t_m} \leq \phi_t + \overline{F}(D\phi, D^2\phi)(x,t). \end{cases} \tag{4}$$

Treatment of singularities

The MCM equation is undefined at points such that $Du = 0$. Therefore, in general

$$\underline{F}(D\phi, D^2\phi) \neq \overline{F}(D\phi, D^2\phi)$$

- from the analytical viewpoint, suitable conditions ensure existence and uniqueness
- from the numerical viewpoint, it suffices for the scheme to be consistent with the (suitably scaled) heat equation when $Du = 0$:
  - without a threshold: min-max technique
  - with threshold: $\begin{cases} \text{explicit treatment of the heat equation} \\ \text{implicit treatment of the heat equation} \end{cases}$

Treatment with threshold

When $|D_j^n| \leq C\Delta x^s$,the scheme switches to an approximation of the heat equation

$$u_t = \frac{1}{2}\Delta u.$$

In this case, the evolution operator under the threshold satisfies the condition

$$\underline{F}(Du, D^2u) \leq -\frac{1}{2}\Delta u \leq \overline{F}(Du, D^2u)$$

and a consistent numerical approximation allows to recover the weak consistency condition.

- explicit treatment: the discrete laplacian is computed on a "large" $(O(\sqrt{\Delta t}))$ stencil:

$$u_j^{n+1} = \frac{1}{4} \sum_i I[u^n](x_j + \delta_i),$$

with $\delta_i = (\pm\sqrt{\Delta t}, \pm\sqrt{\Delta t})$.

- implicit treatment:

$$u_j^{n+1} = u_j^n + \Delta t \Delta_h u^{n+1},$$

in which the part of the solution above the threshold is used as a boundary condition.

Treatment by a Min-Max scheme

$$u_j^{n+1} = \min_{\mu \in S^1} \Big(\max(I[u^n](x_j + \sqrt{2\Delta t}\mu), I[u^n](x_j - \sqrt{2\Delta t}\mu) \Big)$$

The minmax operation basically selects the direction orthogonal to Du, but does not require a special handling of stationary points.

- **Advantages:**
  - defined also at singular points
  - monotone by construction
- **Drawbacks**: more expensive and less accurate

References: *Catté - Dibos - Koepfler, Kohn - Serfaty (semi-discrete versions)*.

Convergence

All the versions of the scheme are consistent (for a suitable $\Delta t/\Delta v$ relationship), but only the minmax scheme is also monotone.
For the basic scheme, following Crandall & Lions, we introduce an additional discretization parameter ρ and rewrite the scheme as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{1}{2\rho^2}\left(I[u^n](x_j + \sigma_j^n \rho) + I[u^n](x_j - \sigma_j^n \rho) - u_j^n\right).$$

- convergence is proved for this scheme with a further vanishing viscosity term (for monotonicity)
- three discretization parameters:
  - Δx space step
  - Δt time step
  - ρ step for the second directional derivative

Affine Morphological Scale Space

This model is a derivation of the MCM equation:

$$\begin{cases} u_t(x,t) = \operatorname{div}\left(\frac{Du(x,t)}{|Du(x,t)|}\right)^{1/3} |Du(x,t)| \\ u(x,0) = u_0(x). \end{cases} \tag{5}$$

- the collection of images $(x \to u(x,t))_{t \geq 0}$ satisfying (5) represents the Affine Morfological Scale Space
- existence and uniqueness in the class of viscosity solution.

References: Alvarez - Guichard - Lions - Morel, Sapiro - Tannenbaum

Affine Morphological Scale Space

The AMSS is the only semigroup $T_t : u_0 \to u(\cdot, t)$ s.t.

Monotonicity if $u \leq v$, then $T_t(u) \leq T_t(v)$ (no enhancement of the original image, just smoothing)

References: Alvarez - Guichard - Lions - Morel

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Grey scale invariance $T_t(g \circ u) = g \circ T_t(u)$, g monotone scalar function (independence from the grey-level scale)

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Translation invariance $T_t(\tau_h \circ u) = \tau_h \circ T_t(u)$, $h \in \mathbb{R}^2$ and $\tau_h f(x) = f(x+h)$ (independence of image analysis from change of position of objects)

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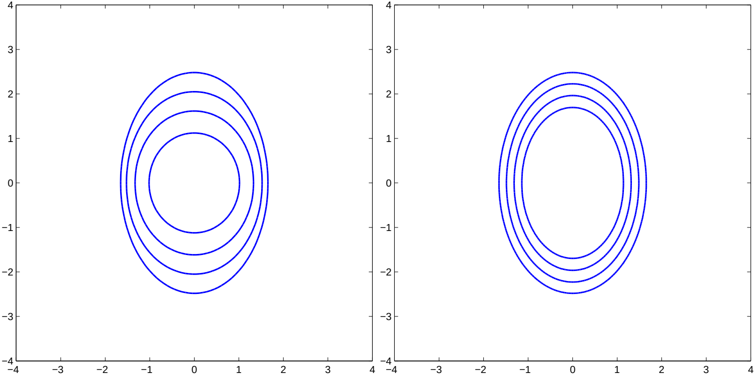
Affine invariance $T_t(u \circ \phi) = T_{t \cdot \det|\phi|} u \circ \phi$, ϕ affine map (invariance of image analysis under any planar projection of a planar shape)

References: Alvarez - Guichard - Lions - Morel

Affine invariance



MCM AMSS

Some references

- Finite Difference scheme (FDS)
  (Guichard - Morel)
- Level Lines Affine Shortening (LLAS)
  The algorithm has three steps:
  - extraction of the level lines of the bilinear interpolation of the initial image (Monasse - Guichard);
  - independent evolution of each level line by affine curve shortening (Moisan - Koepfler - Cao);
  - reconstruction of a new image from the evolved level lines.

  (Ciomaga - Monasse - Morel)

Properties of the $MCM^{1/3}$ operator

Define $\mathrm{curv}(u) = \mathrm{div}\left(\frac{Du(x,t)}{|Du(x,t)|}\right)$ and observe

$$|Du|\mathrm{curv}(u)^{\frac{1}{3}} = (|Du|^3\mathrm{curv}(u))^{\frac{1}{3}},$$

and

$$|Du|^3\mathrm{curv}(u) = |Du|^2(|Du|\mathrm{curv}(u)) = \\ |Du|^2\left(\hat{\sigma}(Du)^t D^2 u \hat{\sigma}(Du)\frac{1}{|Du|^2}\right) = \hat{\sigma}(Du)^t D^2 u \hat{\sigma}(Du),$$

where $\hat{\sigma}(Du) := (Du)^{\perp}$.
Then (5) can be rewritten as

$$u_t = (\hat{\sigma}(Du)^t D^2 u \hat{\sigma}(Du))^{1/3}$$

Reference: Guichard - Morel, “Image Analysis and PDEs”

Construction of the SL scheme

- Δx-Central Finite Difference
  $D_j^n \simeq Du(x_j, t_n)$ and $\hat{\sigma}_j^n \equiv \hat{\sigma}(D_j^n) = (D_j^n)^\perp$

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  $D_j^n \simeq Du(x_j, t_n)$ and $\hat{\sigma}_j^n \equiv \hat{\sigma}(D_j^n) = (D_j^n)^{\perp}$
- ρ-Discretization of directional derivative

$$\hat{\sigma}(Du)^t D^2 u \hat{\sigma}(Du) \simeq \frac{u(x_j + \rho\hat{\sigma}_j^n, t) + u^n(x_j - \rho\hat{\sigma}_j^n, t) - 2u_j^n}{\rho^2}$$

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- Δt-Discretization of time derivative

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \left(\frac{u^n(x_j + \rho\hat{\sigma}_j^n, t) + u^n(x_j - \rho\hat{\sigma}_j^n, t) - 2u_j^n}{\rho^2}\right)^{\frac{1}{3}}$$

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- Interpolation on characteristics feet

$$u_j^{n+1} = u_j^n + \Delta t \left(\frac{I[u^n](x_j + \rho\hat{\sigma}_j^n) + I[u^n](x_j - \rho\hat{\sigma}_j^n) - 2u_j^n}{\rho^2}\right)^{\frac{1}{3}}$$

Convergence

- consistency (in the weak sense) is checked under suitable relationship between Δx, Δt and ρ
- monotonicity is enforced for the version with a vanishing viscosity term
- convergence follows from Barles–Souganidis theorem

References: Carlini - Ferretti, Mengucci (Tesi di Laurea)

Filtering a noisy image – MCM

Noise 50%

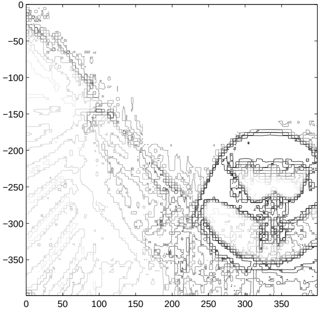
MCM

Filtering a noisy image – MCM vs. $\mathrm{MCM}^{1/3}$

MCM

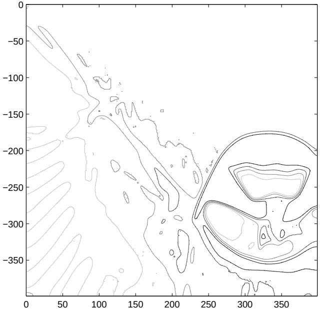
AMSS

Filtering a 'pixelled' image

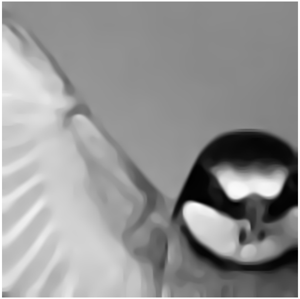


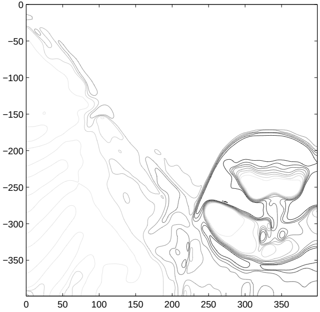
http://www.ipol.im/pub/algo/cmmm_image_curvature_microscope/

Comparison: Level Lines Affine Shortening



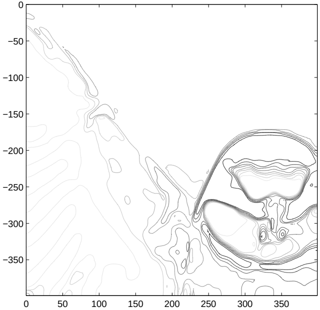
MCM





$$\Delta t = 0.2, \quad C = 0.005, \quad n_{iter} = 80$$

SL



$$\Delta t = 0.2, \quad C = 0.005, \quad n_{iter} = 150$$

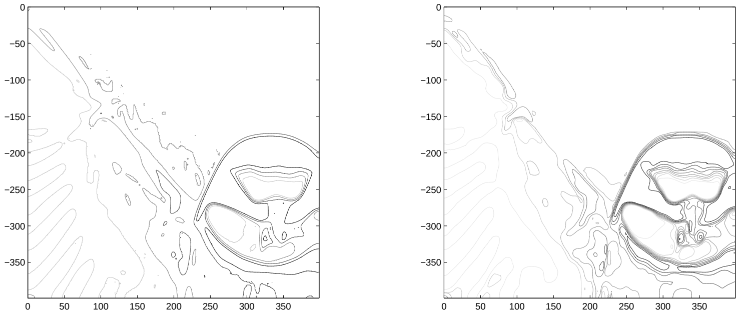
Comparison: SL scheme vs Level Lines Affine Shortening

Comparison: SL scheme vs Level Lines Affine Shortening

LLAS

SL

Comparison: SL scheme vs Level Lines Affine Shortening



LLAS SL

Semi-Lagrangian scheme for Area Preserving Flows

To preserve the enclosed area while smoothing the curve the following level set model has been proposed

$$\begin{cases} u_t = \operatorname{div}\left(\frac{Du(x,t)}{|Du(x,t)|}\right)|Du| - \frac{\pi}{A_0}x \cdot Du \\ u(x,0) = u_0(x) \end{cases} \tag{6}$$

- A_0 is the area of the initial set Ω_0.
- the new term $-\frac{\pi}{A_0}x \cdot Du$ represents a transport along a vector field with unit divergence which, assuming the origin is contained in Ω_0, has the effect to push outwards the interface so that the area is preserved.

References: Sapiro - Tannenbaum

Semi-Lagrangian scheme for Area Preserving Flows

- Δx-Central Finite Difference
  $D_j^n \simeq Du(x_j, t_n)$ and $\sigma_j^n \equiv \sigma(D_j^n) = \left(\frac{D_j^n}{|D_j^n|}\right)^{\perp}$

Semi-Lagrangian scheme for Area Preserving Flows

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  $D_j^n \simeq Du(x_j, t_n)$ and $\sigma_j^n \equiv \sigma(D_j^n) = \left(\frac{D_j^n}{|D_j^n|}\right)^{\perp}$
- Interpolation on characteristics feet by a $I[\cdot]$ bi-linear interpolation

Semi-Lagrangian scheme for Area Preserving Flows

- Δx-Central Finite Difference
  $D_j^n \simeq Du(x_j, t_n)$ and $\sigma_j^n \equiv \sigma(D_j^n) = \left(\frac{D_j^n}{|D_j^n|}\right)^{\perp}$
- Interpolation on characteristics feet by a $I[\cdot]$ bi-linear interpolation
- Fully-discrete

$$
\begin{aligned}
u_j^{n+1} \quad = \quad & \frac{1}{2}\Big[I[u^n]\Big(\Big(1-\Delta t\frac{\pi}{A_0}\Big)x_j+\sqrt{2\Delta t}\,\sigma_j^n\Big) \\
& +I[u^n]\Big(\Big(1-\Delta t\frac{\pi}{A_0}\Big)x_j-\sqrt{2\Delta t}\,\sigma_j^n\Big)\Big]
\end{aligned}
$$

References: Carlini - Ferretti, Balzerani (Tesi di Laurea)

Numerical Tests

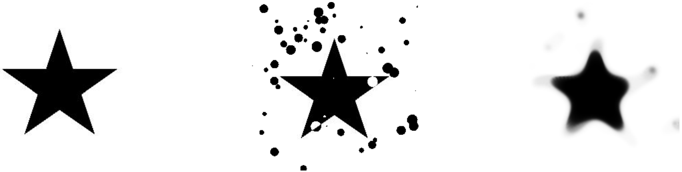


Figure: Original star shape (top), star shape with random droplets (center) and filtered by APMCM (bottom)

Numerical Tests

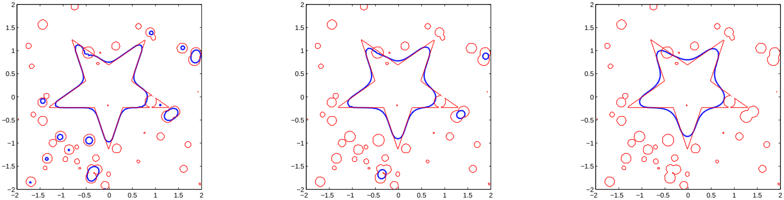
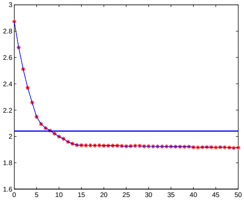


Figure: APMCM flow (blue line) and Initial shape (red line) corresponding to the value $u = 0.5$

Area Comparison APMCM vs MCM



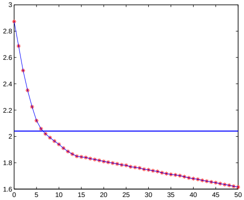


Figure: Area evolution $\mathcal{A}_n$, $n = 0, \ldots, 50$ for APMCM (left) MCM(right)

Numerical Tests

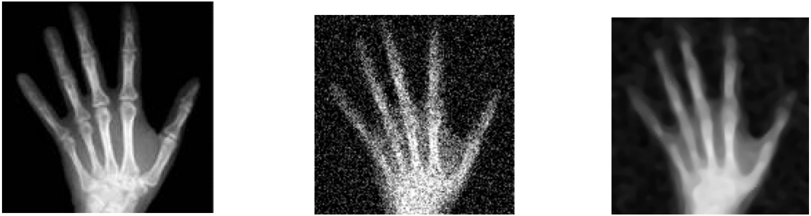


Figure: Noisy image, obtained Gaussian noise, and filtered image

Numerical Tests

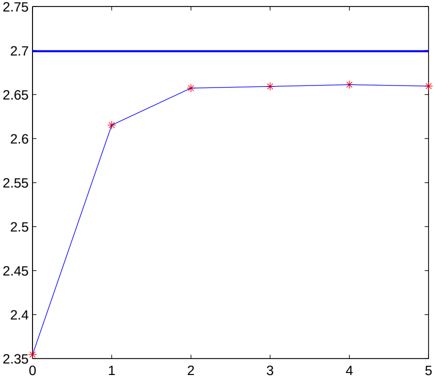


Figure: Area evolution: $\mathcal{A}_n$, $n = 0, \ldots, 5$. Real image with Gaussian noise

Numerical Tests

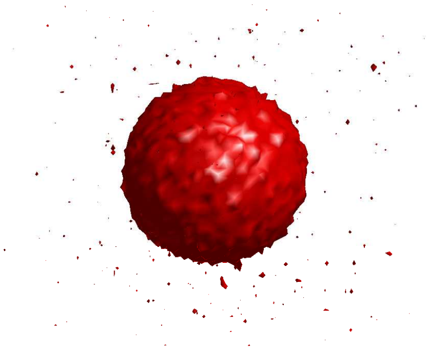


Figure: Noisy 3d shape, APMCM filtering

Numerical Tests

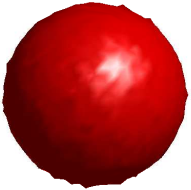


Figure: Noisy 3d shape, APMCM filtering

Numerical Tests

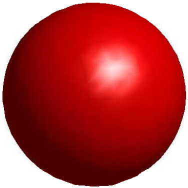


Figure: Noisy 3d shape, APMCM filtering

Numerical Tests

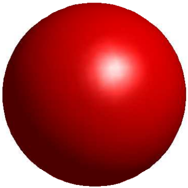


Figure: Noisy 3d shape, APMCM filtering

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